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K-theory Classifications for Symmetry-Protected Topological Phases of Free Fermions

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Abstract

A mathematically rigorous K-theory classification scheme is developed for SPT phases of free fermions in a perfect crystal inspired by the work of Freed & Moore. Although we restrict to systems without (long range) topological order, we allow general space-time symmetries that preserve the crystal. Twisting phenomena such as spinful particles, quantum anomalies and nonsymmorphic crystals are also implemented. Computational techniques are developed and presented by example to deal with the K-theory groups in physically relevant settings. The main focus is on topological phases in the classes A,AI and AII, i.e. without particle-hole and chiral symmetries but possibly with time reversal symmetry. Moreover, these calculations reproduce phase classifications of condensed matter physics literature, such as the periodic table of Kitaev and Kruthoff et al. The most important machinery is the identification of the relevant Atiyah-Hirzebruch type spectral sequence and an explicit model of its second page given by a twisted version of Bredon equivariant cohomology.

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Introduction

Topology has long played a role in mathematical physics, but never before has physics seen such an uprise of topological phenomena as with the discovery of topological phases of matter. Topological phases of matter shatter the classical theory of Ginzburg-Landau on phases of matter and symmetry-breaking. Zero-temperature topological quantum effects include long-range entangled systems, so-called topological order. But even quantum systems that admit only short-range entanglement can admit interesting topological phases if their dynamics are bound to preserve a certain symmetry. Such systems are called symmetry-protected and even symmetry-protected topological phases of free fermions can have surprisingly complicated behaviour. In recent years, topological phases of matter have slowly oozed from the field of condensed matter to a broader audience of popular science, but also into theoretical and mathematical physics. The Nobel prize in physics of 2016 was won by David Thouless, Duncan Haldane and Michael Kosterlitz for their work in the theory of topological phases, mostly conducted in the 20th century. However, only in recent years has it become clear that more sophisticated algebraic topology naturally appears in classification questions of topological phases: K-theory.

In his pioneering work [42], Kitaev classifies symmetry-protected topological phases of free fermions for some important symmetry groups, including the possibility of chiral symmetry, particle-hole symmetry and time reversal symmetry in his theory. In the resulting ‘periodic table’ or ‘ten-fold way’, a stunning connection with representations of Clifford algebras and Bott periodicity can be found by comparing with the mathematical literature. He thereby simultaneously generalized the TKNN-invariant [65] and the Fu-Kane-Mele invariant [28] by putting them in a systematic framework using Clifford algebras and K-theory. Although Kitaev uses mainly local computations and classifying spaces, he emphasizes how the real and complex K-theory of the Brillouin zone appear in the picture. From this perspective, the translational symmetry of the crystal is also included and weak topological invariants can be shown to arise from this extra lattice symmetry. Mainly inspired by these ideas, Freed & Moore generalized these arguments to a more formal framework in their phenomenal work [27], including also for example point group symmetries and nonsymmorphic crystals. Starting only from first principles, they define K-theory groups classifying topological phases of noninteracting fermions moving in a lattice protected by a general symmetry group. They have thereby put the problem of classifying topological phases protected by crystal symmetries on a firm mathematical footing by translating it to computing this so-called twisted equivariant K-theory of the Brillouin zone.

However, until only recently, the twisted equivariant K-theory defined by Freed & Moore was poorly understood. Proofs of basic cohomological properties of well-known types of K-theories, such as homotopy invariance, suspension theorems and induced long exact sequences, were not well-established. Moreover, most computations of these K-theory groups relied on such cohomological properties and were rarely conducted using rigorous techniques. Last few years, Gomi [31] changed this by formulating K-theories very similar to Freed & Moore, providing proofs of various desirable properties. Although computations in explicit examples remained sparse, the work of Gomi recently also resulted in classifications of crystalline topological insulators by Shiozaki, Sato and Gomi [62], mainly using Atiyah-Hirzebruch type spectral sequences. From a mathematical viewpoint, this powerful technique from algebraic topology is expected to exist for any reasonable generalized cohomology theory.

However, mathematically sound proofs that these techniques can be applied specifically to Freed & Moore's K-theory are still lacking. Also, the theory is not yet far enough developed for a full algorithmic approach to arise that classifies topological phases protected by a general symmetry group. This thesis aims to develop the mathematical theory of Freed & Moore's K-theory with the purpose in mind of computing the K-theory of the Brillouin zone. As the existing literature may be confusing, the focus is not only on abstract rigorous proofs, but also on specific, physically relevant examples that illustrate the large spectrum of subtleties that appear. The author hopes that these expositions give new researchers the right intuitions to further develop the theory.

It should be noted here that although this thesis focuses on the approach to free fermion phases of Freed & Moore, in recent years multiple different approaches have been developed. First of all, some mathematically rigorous K-theory classifications of free fermions include the usage of C*-algebras and KK-theory, see for example Thiang [64] and Kubota [45]. These formulations have the advantage of also being useful for disordered systems, since the Brillouin zone becomes noncommutative for quasicrystals, see for example the classical work of Belissard [10]. This also allows mathematical investigations of bulk-boundary correspondences, since a boundary by definition ruins translational symmetry of the crystal. Since KK-theory also forms a natural setting to study index theorems, these formulations are also useful to study the relation between the bulk-boundary correspondence and index theory [13] [52] [46] [48]. However, C*-algebras are less flexible in computations than topological spaces, as they do not admit CW-structures.

A different approach to topological phases, which in certain cases should be related to the K-theory classifications, sets off from the observation that at zero temperature they admit an effective topological quantum field theory description. Although this effective description by definition does not preserve distances and therefore does not allow a satisfactory description of crystal symmetries, it does allow for interactions. Kapustin has advocated this approach and shown that this results in classifications by character groups of bordism groups, which appear using the modern mathematical formulation of topological field theories using functorial quantum field theory. For sufficiently low-dimensional time reversal symmetric fermions, Kapustin et al. [38] have shown that these bordism classifications agree with the free case. However, in general this interacting picture can be completely different; new topological phases can occur, but also topological phases that are different in the free case can now be connected without a phase transition using a non-free Hamiltonian. A far-reaching generalization of this idea is given by Freed and Hopkins [24], who incorporate general symmetries in a relativistic setting. A thorough, explicit comparison of the K-theory approach with such effective topological quantum field theories using explicit examples - perhaps using the bulk-boundary correspondence - would be a desirable future development.

Now a brief outline of the content of this thesis will be given. First of all it should be noted that the reader is assumed to have the following prerequisites: basic abstract algebra as in for example [12], basic algebraic topology as in Hatcher [32], K-theory as in Karoubi [40] and functional analysis as in Rudin [54]. The three appendices at the end of this document provide the reader with enough basic knowledge in the topics of cohomology of topological groups, superalgebras and equivariant algebraic topology to be able to read the main text.

The first chapter reviews the analysis of Freed & Moore on identifying the relevant representation-theoretic structure arising from anomalous behaviour of symmetries in physics. Although most formulations of twisted representations in Freed & Moore are conducted using group extensions, we decide to pursue the equivalent but more computationally attractive notion of group 2-cocycles. Then we define topological phases of free fermions in quantum mechanics. Some time is spent on the development of twisted representation theory and the construction of their (higher) representation rings, because they classify symmetry-protected phases in zero dimensions. For complex and real group algebras of finite groups, techniques to compute these representation rings have long been well-known, using basic complex representation theory of finite groups and the Frobenius-Schur indicator. We do not give an algorithmic approach to compute representation rings of general twisted group algebras, but the literature has generalized the Frobenius-Schur indicator to various contexts, see [62] and [43]. Since these representation rings also form the building blocks for classifying higher-dimensional topological phases, such a test for determining the twisted representation theory of a group would be desirable.

The second chapter sketches how well-established condensed matter theory can be embedded in the abstract quantum mechanical framework of Chapter one. The bundle of Bloch states above the Brillouin zone is identified as the relevant topological data and crystal symmetries are implemented to yield representation-theoretic structures on wave functions. Finally we state the relationship between topological phases of fermions in a crystal with K-theory. In order to agree with our slightly altered definition of K-theory, our definition of what Freed & Moore call 'reduced topological phases' is altered similarly. It would be interesting to review the proofs in Freed & Moore in this subtly changed setting.

Chapter three is fully devoted to the development of twisted equivariant K-theory. We choose to follow Freed & Moore as much as possible, only deviating when strictly necessary. To maximize accessibility and to stay as close to the physics as possible, we avoid the language of topological groupoids, since only the action groupoid is used in our applications. Instead we use the language of equivariant algebraic topology and introduce anomalies and nonsymmorphicity as group 2-cocycles dependending on a point in space, which will be called twists. Relevant analogues of the Eilenberg-Steenrod axioms are identified and proven, resulting in a toolbox from algebraic topology to make computations. A few complications arise, especially because the formulation of K-theory of Freed & Moore does not seem to be well-suited to define higher degree K-theory. Possible solutions are suggested, especially based on the work of Gomi [31].

In the final chapter, computational tools are applied to the Brillouin zone torus in a couple of simple examples of class A, AI and AII topological insulators with crystal symmetries. We primarily make use of an Atiyah-Hirzebruch type spectral sequence, but we also make use of a splitting technique advocated by Royer [53], often reducing computations to the K-theory of spheres. The splitting technique is especially useful for symmorphic class A insulators, since the nontwisted complex equivariant K-theory of real representation spheres has long been known by Karoubi [39]. The possibility of a twisted equivariant Chern character is also mentioned, but not fully pursued, because of the relevance of torsion invariants in class AI and AII. Although this direction is mentioned nowhere else in this work, note that the spectral sequence developed in this thesis in particular applies to equivariant KR-theory. Since such spectral sequences seem to be not well-studied and hard to compute (see for example Dugger [20]), this could be an addition to the mathematical literature. Instead we show in the example of two-dimensional time reversal symmetric spinful fermions protected by the wallpaper group $p2$ how our method applies to the classification of crystalline topological insulators in class AII. We also state the results of our computations in class AI and AII for some other simple wallpaper groups. Our different computational approaches agree and also reproduce classifications of physics literature in [44] [43] [62], but now in a more rigorous setting. In class A, we sketch a comparison between the method made in Kruthoff et al. [44] and the approach made in this document. A comparison of the Atiyah-Hirzebruch spectral sequence approach for class AI and AII with Kruthoff et al. [43] would especially be interesting, since a connection is not a priori clear. In contrary to class A, the author believes that an algorithmic approach for class AI and AII is at moment of writing far from realized. This is not only because the problem of classifying higher representation rings of twisted group algebras and the possibility of nonvanishing higher differentials, but especially because the large amount of torsion results in ambiguity in the construction of K-theory of the spectral sequence alone. Unfortunately, abstract algebraic tools no longer uniquely specify the K-theory in that case and more sophisticated geometric arguments have to be imposed in order to explicitely determine the maps occurring in the spectral sequence. This problem is illustrated with the example of classifying topological phases of time reversal symmetric fermions in a two-dimensional square lattice protected by a parity transformation.

Because of the problems that occur when trying to define higher degree Freed-Moore K-theory groups in the canonical way, it would certainly be of crucial importance to study whether the introduction of a particle-hole symmetry c can lead to similar complications. The crucial question here is whether the definition of Freed & Moore using finite-dimensional vector bundles is the 'right definition' in this context, as it is apparently ill-suited for defining higher-degree K-theories using Clifford actions. Also in the work of Gomi, this problem seems to be unresolved, see Paragraph 3.5 of [31], in particular the remark under Proposition 3.14 and Paragraph 4.4. Recall that in the BCS model of superconductivity, a particle-hole symmetry forces the valence and conduction bands to be mirror images. Because of the theoretical possibility of the existence of topological superconductors, it would also be of interest to theoretical physics to be able to compute K-theory groups for nontrivial homomorphism c. The author anticipates that the Atiyah-Hirzebruch spectral sequence method of this thesis can be readily generalized in this context, yielding classifications of particle-hole symmetric fermions in crystalline topological insulators.

1. Quantum Symmetries

In a condensed matter system, the relevant physical framework is usually nonrelativistic quantum field theory. Quantum theory is needed because of the microscopic scale of the atomic structure and the nonrelativistic approximation is justified by the relatively low energies of the particles. In quantum mechanics, indistinguishability of particles is not naturally built in, so in general second quantization is needed to describe an N-particle system. However, in weakly interacting systems, the system is well-approximated by a one-particle Hamiltonian and therefore usual quantum mechanics suffices.

In this section, the relevant abstract mathematical data corresponding to general symmetries of quantum mechanical systems will be identified following the work of Freed & Moore [27]. These can be summarized compactly using the language of topological groups and their cohomology (for a review on the theory of topological groups, such as group extensions and cohomology, and discussions of mathematical details see Appendix A). Then symmetry-protected topological phases can be defined abstractly as equivalence classes of quantum systems equipped with such a quantum symmetry. In the next section, we will specialize our Hilbert spaces to the more concrete and familiar setting of wave functions of electrons moving in a perfect crystal.

1.1. Quantum Automorphisms

Consider a complex Hilbert space $\mathcal{H}$ of a single particle[1]. The relevant state space for quantum mechanics in this context is the set of pure states, which can be naturally identified with the projective Hilbert space $\mathbb{P}\mathcal{H}$.[2] Observables O are (possibly unbounded) self-adjoint operators on $\mathcal{H}$. The value that can actually be measured in a lab is the expectation value of O in a pure state $l \in \mathbb{P}\mathcal{H}$, which is

$$\operatorname{Tr} P_l O = \frac{\langle \psi, O\psi \rangle}{\langle \psi, \psi \rangle} \quad (\psi \in l),$$

where $P_l \in B(\mathcal{H})$ is the operator projecting on l. Given pure states $l_1, l_2 \in \mathbb{P}\mathcal{H}$, the expectation value of the observable P_{l_2} in state l_1 is called the *transition probability.* A short computation gives the following expression for the transition probability:

$$p(l_1, l_2) := \frac{\langle \psi_1, P_{l_2}\psi_1 \rangle}{\langle \psi_1, \psi_1 \rangle} = \frac{\left\langle \psi_1, \frac{\psi_2 \langle \psi_2, \psi_1 \rangle}{\langle \psi_2, \psi_2 \rangle} \right\rangle}{\langle \psi_1, \psi_1 \rangle} = \frac{|\langle \psi_1, \psi_2 \rangle|^2}{\langle \psi_1, \psi_1 \rangle \langle \psi_2, \psi_2 \rangle},$$

[1] Hilbert spaces will be assumed to be seperable from now on.

[2] We will not talk about mixed states here, since their theory is determined by the behavior of pure states.

where $\psi_1 \in l_1$ and $\psi_2 \in l_2$. Note that the function p is symmetric and maps into $[0, 1]$. Definitely, symmetry operations in quantum mechanics should preserve this probability, motivating the following definition:

Definition 1.1. A bijective mapping $T : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ is called a *projective quantum automorphism* if it preserves the transition probability map p[3]. More precisely, if $T : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$, then

$$p(l_1, l_2) = p(T(l_1), T(l_2)) \quad \forall l_1, l_2 \in \mathbb{P}\mathcal{H}.$$

Let $\mathrm{Aut}_{\mathrm{qtm}}\, \mathbb{P}\mathcal{H}$ denote the group of projective quantum automorphisms.

1.2. Wigner's Theorem

Time evolution of states in physics is usually governed by unitary operators. Therefore it is not surprising that given a unitary map $U : \mathcal{H} \to \mathcal{H}$, the induced map $\mathbb{P}U : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$ is a projective quantum automorphism. There are however other projective quantum automorphisms that may at first sight seem less natural from a physical viewpoint.

Definition 1.2. A map $S : \mathcal{H} \to \mathcal{H}$ is called *anti-unitary* if

1. it is *anti-linear*: $S(\lambda\psi_1) = \bar{\lambda}S(\psi_1)$ and $S(\psi_1 + \psi_2) = S(\psi_1) + S(\psi_2)$ for all $\lambda \in \mathbb{C}$ and $\psi_1, \psi_2 \in \mathcal{H}$;

2. and satisfies $\langle S\psi_1, S\psi_2\rangle = \langle \psi_2, \psi_1\rangle$.

We say a map $T : \mathcal{H} \to \mathcal{H}$ is a *linear quantum automorphism* if it is either unitary or anti-unitary. Write $\mathrm{Aut}_{\mathrm{qtm}}\, \mathcal{H}$ for the set of linear quantum automorphisms.

Note that the composition of two anti-unitary maps is unitary and the composition of a unitary and an anti-unitary map is anti-unitary. Therefore $\mathrm{Aut}_{\mathrm{qtm}}\, \mathcal{H}$ becomes a group. There are many reasonable topologies on this space, but in order to agree with the results of Freed & Moore, we give it the topology they define in Appendix D[4] [27] , which has been taken from Appendix 1 of Atiyah and Segal's work on twisted K-theory [60]. There is a continuous homomorphism $\phi_{\mathcal{H}} : \mathrm{Aut}_{\mathrm{qtm}}\, \mathcal{H} \to \mathbb{Z}_2$ given by

$$\phi_{\mathcal{H}}(T) = \begin{cases} -1 & \text{if } T \text{ is anti-unitary,} \\ 1 & \text{if } T \text{ is unitary.} \end{cases}$$

[3]Such maps will automatically be smooth in a reasonable Hilbert manifold structure, see [23].

[4]This topology is defined as follows. We start by giving $\mathrm{Aut}_{\mathrm{qtm}}\, \mathcal{H}$ the compact-open topology, i.e. the topology of compact convergence (which is slightly stronger than the strong operator topology, but agrees on compact sets). Then it turns out that the inverse map $T \mapsto T^{-1}$ is not continuous and therefore we take the coarsest topology to include the compact-open topology in which this map is also continuous. This construction is for the purpose of firstly having an equivalence between continuous homomorphisms $\mathbb{R} \to \mathrm{Aut}_{\mathrm{qtm}}\, \mathcal{H}$ and strongly continuous one-parameter unitary subgroups and secondly having an equivalence between continuous linear actions $G \times \mathcal{H} \to \mathcal{H}$ and continuous representations $G \to \mathrm{Aut}_{\mathrm{qtm}}\, \mathcal{H}$ for reasonably 'nice' groups (compactly generated is sufficient). These nice properties however come at the cost that the product map is still not continuous and hence $\mathrm{Aut}_{\mathrm{qtm}}\, \mathcal{H}$ is not a topological group.

Introducing the notation

$$^{\epsilon}\overline{z} = \begin{cases} \overline{z} & \text{if } \epsilon = -1, \\ z & \text{if } \epsilon = 1, \end{cases}$$

note that $T(z\psi) = {}^{\phi_{\mathcal{H}}(T)}\overline{z}T(\psi)$ for all $T \in \mathrm{Aut_{qtm}}\,\mathcal{H}, z \in \mathbb{C}$ and $\psi \in \mathcal{H}$. Note also that by anti-linearity an anti-unitary map S induces a map $\mathbb{P}S : \mathbb{P}\mathcal{H} \to \mathbb{P}\mathcal{H}$. Moreover, by an easy computation, it follows that $\mathbb{P}S \in \mathrm{Aut_{qtm}}\,\mathbb{P}\mathcal{H}$. Interestingly, according to Wigner, these are the only possibilities.

Theorem 1.3 (Wigner)**.** *The following sequence of groups, called the* Wigner group extension*, is exact*

$$1 \to U(1) \to \mathrm{Aut_{qtm}}\,\mathcal{H} \to \mathrm{Aut_{qtm}}\,\mathbb{P}\mathcal{H} \to 1.$$

Proof. The nontrivial part is the surjectivity of which a modern proof and discussion can be found in [23]. □

Let $T \in \mathrm{Aut_{qtm}}\,\mathcal{H}$ be a projective quantum operator lifting $[T] \in \mathrm{Aut_{qtm}}\,\mathbb{P}\mathcal{H}$. Then the conjugation action of $\mathrm{Aut_{qtm}}\,\mathbb{P}\mathcal{H}$ on $U(1)$ induced by the Wigner sequence is

$$TzT^{-1} = {}^{\phi_{\mathcal{H}}(T)}\overline{z} = z^{\phi_{\mathcal{H}}(T)},$$

where $z \in U(1)$. So we consider $U(1)$ as a $\mathrm{Aut_{qtm}}\,\mathbb{P}\mathcal{H}$-module via $\phi_{\mathcal{H}}$. Note indeed that the homomorphism $\phi_{\mathcal{H}}$ factors through the projection $\mathbb{P} : \mathrm{Aut_{qtm}}\,\mathcal{H} \to \mathrm{Aut_{qtm}}\,\mathbb{P}\mathcal{H}$ to a group homomorphism $\phi_{\mathbb{P}\mathcal{H}} : \mathrm{Aut_{qtm}}\,\mathbb{P}\mathcal{H} \to \mathbb{Z}_2$. In particular, we see that the Wigner extension is not central.

1.3. Quantum Symmetries

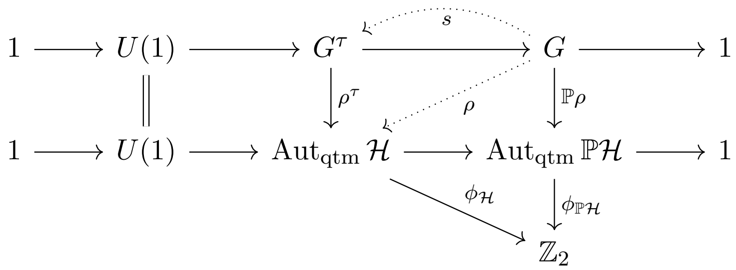
Now that the right notion of automorphisms is known, one can start speaking of how topological groups act as quantum symmetries on quantum systems, which should be as follows:

Definition 1.4. A *projective quantum symmetry* is a continuous[5] homomorphism $\mathbb{P}\rho : G \to \mathrm{Aut_{qtm}}\,\mathbb{P}\mathcal{H}$ from a topological group G to projective automorphisms on a Hilbert space $\mathcal{H}$.

A *linear quantum symmetry* is a continuous homomorphism $\rho : G \to \mathrm{Aut_{qtm}}\,\mathcal{H}$ from a topological group G to linear quantum automorphisms on a Hilbert space $\mathcal{H}$.

Linear quantum symmetries are related to representation theory and therefore relatively well-studied. However, since pure states are the physically important content of the theory, one should consider a projective quantum symmetry $\mathbb{P}\rho : G \to \mathrm{Aut_{qtm}}\,\mathbb{P}\mathcal{H}$ instead and see in what way it can be lifted to linear quantum symmetries. In order to identify the structure of abstract symmetry classes of such groups, note that a quantum symmetry induces some relevant structure on the group as follows. The pull-back of the Wigner group extension gives a group extension of G, which will be called G^{τ}.

[5]We give $\mathrm{Aut_{qtm}}\,\mathbb{P}\mathcal{H}$ the quotient topology under the projection $\mathrm{Aut_{qtm}}\,\mathcal{H} \to \mathrm{Aut_{qtm}}\,\mathbb{P}\mathcal{H}$.



In other words, there is an induced real linear representation of G^τ on $\mathcal{H}$. In order to make the connection with continuous group cohomology, assume this extension has a continuous section[6] $s : G \to G^\tau$ (but s is not necessarily a group homomorphism). By multiplication with $s(1)^{-1}$ we can assume without loss of generality that $s(1) = 1$. This gives a 2-cocycle $\tau \in Z^2(G, U(1)_\phi)$ defined by

$$\tau(g, h) = s(g)s(h)s(gh)^{-1},$$

which measures how far the exact sequence is from being split (if s is a homomorphism, the sequence is split). Note that the 2-cocycle is unital in the sense that $\tau(g, 1) = \tau(1, g) = 1$ for all $g \in G$. If s is the pull-back of a section of the Wigner extension, τ is the pull-back of the cocycle corresponding to the Wigner extension[7] under $\mathbb{P}\rho$. We also get a continuous map $\rho : G \to \mathcal{H}$ defined by $\rho(g) := \rho^\tau(s(g))$. Note that the failure of ρ to be a linear representation is measured by τ (i.e. the pull-back of the class corresponding to the Wigner extension). Indeed

$$\rho(g)\rho(h)\rho(gh)^{-1} = \rho^\tau(\tau(g, h))$$

is just scalar multiplication by $\tau(g, h)$ by commutativity of the diagram above, so that ρ is a *projective representation* or a representation *twisted by* τ. The fact that G itself in general does not admit a linear representation in the quantum system, is the source of many complications, which are called called *quantum anomalies.* There is a natural $\mathbb{Z}_2$-grading on G given by $\phi := \phi_{\mathbb{P}\mathcal{H}}\rho$ and similarly on G^τ. It determines whether (a lift of $g \in G$ to) G^τ acts by unitaries or anti-unitaries under the representation ρ^τ. The action of G on $U(1)$ inherited from the action of the Wigner extension is complex conjugation if $\phi(g) = -1$ and trivial otherwise.

Example 1.5. Consider $G = \mathbb{Z}_2$, which we imagine to be acting by a time reversing operation T. Since commuting T with e^{-itH} should change the sign of i, it would be reasonable to assume that T is anti-unitary. In other words $\phi(T) = -1$. What possibilities are there for T to act projectively on a Hilbert space of states $\mathcal{H}$? Certainly it can just work by complex conjugation $T\psi = \bar{\psi}$. This happens for bosons, and we see that $T(T\psi) = \psi$, so in that case this is a representation of $\mathbb{Z}_2$.

[6] In this document, G is typically discrete, such that this section always exists. In general this section exists whenever G^τ is topologically the product of G and $U(1)$, i.e. when $G^\tau \to G$ is a trivial $U(1)$-bundle. For extensions that are nontrivial fiber bundles, we assume a measurable section to exist, so that we can at least work with measurable cocycles.

[7] Unfortunately, $\mathrm{Aut}_{\mathrm{qtm}}\,\mathcal{H}$ is not a topological group since the product map is not continuous. Therefore strictly speaking, the usual theory of group cohomology cannot be used, but we ignore this fact here.

However, in the case of spin 1/2 particles, things are more complicated since $T^2 = -1$. Can we accommodate this fact mathematically in the above framework? For this we have to consider what 2-cocycles $Z^2(\mathbb{Z}_2, U(1)_\phi)$ can occur[8]. Isomorphism classes of these cocycles are captured in the group cohomology $H^2(\mathbb{Z}_2, U(1)_\phi)$, which can be computed to be $\mathbb{Z}_2$. The trivial class corresponds to the case in which a projective action of T lifts to a linear action with $T^2 = 1$ and the nontrivial class corresponds to the case with $T^2 = -1$. More precisely, in this case we have to consider the quaternions $\mathbb{H}$ as a twisted group algebra of $\mathbb{Z}_2$, which will be the topic of Section 1.6. See Appendix A.1 (in particular Remark A.19, Remark A.22 and the lemmas above it) for details.

Example 1.6. We can go one step further for spin 1/2 particles and include the fact that rotations by 2π quantum mechanically results in a minus sign. As a first example, consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ in which the first $\mathbb{Z}_2$ acts by time reversal T as above and the second $\mathbb{Z}$ acts by rotation R with π. Since spatial transformations should give unitary maps, ϕ is projection onto the first $\mathbb{Z}_2$. In this case the relevant cohomology group $H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)_\phi)$ turns out to be $\mathbb{Z}_2 \times \mathbb{Z}_2$.[9] These four possibilities correspond exactly to the cases $R^2 = \pm 1$ and $T^2 = \pm 1$. This can be generalized to (for example) a symmetry group of a $2n$-gon. Indeed, we get a group cocycle on the dihedral group $D_{2n} = \langle r, s : r^{2n} = 1, s^2 = 1, srs = 2\rangle$ of order $4n$ cross time reversal by pulling back the above cocycle under the homomorphism $D_{2n} \times \mathbb{Z}_2 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ given by $r \mapsto r^n$ and $s \mapsto 1$. It should be noted at this point that if the symmetry group does not contain rotations by π or there is no time reversal, then there is no mathematical distinction in this framework between rotation by 2π acting by ± 1 on the Hilbert space. See Lemma A.23 and Proposition A.24 in Appendix A.1 for details.

Now things will be turned around; projective quantum symmetries will be realized as certain linear representations of abstract groups with extra structure. By abstracting the situation above and regarding the equivalence of extensions and cohomology classes, one is naturally led to the following definition:

Definition 1.7. A *quantum symmetry group* is a triple (G, ϕ, τ), where G is a topological group, $\phi : G \to \mathbb{Z}_2$ a continuous $\mathbb{Z}_2$-grading on G and τ is a unital continuous group 2-cocycle of G with values in the G-module $U(1)_\phi$ given by $g \cdot z = z^{\phi(g)}$. We call τ the *(quantum) anomaly* associated to the symmetry.[10]

Using the equivalence between extensions and cohomology classes, a quantum symmetry group is equivalently a triple (G, ϕ, G^τ), where G is a topological group, $\phi : G \to \mathbb{Z}_2$ a continuous $\mathbb{Z}_2$-grading on G and G^τ is an continuous group extension of G by $U(1)$

$$1 \to U(1) \xrightarrow{i} G^\tau \xrightarrow{\pi} G \to 1$$

[8] A more conceptual way to work with these types of facts would be to consider time reversal inside a relevant spacetime symmetry group, such as the Lorentz group. Then for some (Spin(d)-type) universal covering of this, one should make a choice whether lifts of elements in $SO(d)$ are in $\mathrm{Spin}_\pm(d)$. In this work however, it was decided to use cocycles of finite groups for simplicity and computational purposes.

[9] The reader may start to form the hypothesis that the second cohomology of a group always equals itself, but this is far from true.

[10] In the literature the classes in cohomology corresponding to these cocycles are usually called *discrete torsion classes*, but we think quantum anomaly represents the important content of the subject better, especially because for infinite groups the cohomology group is not necessarily pure torsion.

such that the following commutation relations hold for all $g \in G^\tau$ and $\lambda \in U(1)$:

$$i(\lambda)g \begin{cases} gi(\lambda) & \text{if } \phi(g) = 1, \\ gi(\bar{\lambda}) & \text{if } \phi(g) = -1. \end{cases}$$

Hence what we call a quantum symmetry group is a group together with a ϕ-twisted extension of it in the language of Freed & Moore [27] definition 1.7. However, one should be careful in the distinction between cocyles and cohomology classes; cocycles represent explicit examples of extensions and cohomology classes isomorphism classes. In order to properly incorperate this fact we have to work modulo these isomorphism classes.

Definition 1.8. A *morphism of quantum symmetry groups* (G, ϕ, τ) and (G', ϕ', τ') is a pair (f, λ), consisting of a group homomorphism $f : G \to G'$ and a 1-cochain $\lambda \in C^1(G, U(1)_\phi)$, such that $f^*\tau' = \lambda \cdot \tau$ and $\phi = \phi' \circ f$. Composition of (f_1, λ_1) and (f_2, λ_2) is defined by $(f_2 \circ f_1, \lambda_1 \cdot f_1^*\lambda_2)$.

An isomorphism class of quantum symmetry groups can therefore be seen as isomorphism classes of a topological group G, a continuous homomorphism $\phi : G \to \mathbb{Z}_2$ and a cohomology class $\tau \in H^2(G, U(1)_\phi)$[11]. A morphism of quantum symmetry groups corresponds exactly to a morphism of group extensions. In other words, an isomorphism class of a quantum symmetry group is equivalent to what is called a QM-symmetry class in definition 1.13(ii) of Freed & Moore [27].

Coming back to our original realization of projective quantum symmetries, we want to define actions of quantum symmetry groups (G, ϕ, τ) on Hilbert spaces. This should be a projective representation which lifts to a linear representation of the corresponding G^τ, in such a way that $g \in G$ lifts to an anti-unitary acting $\tilde{g} \in G^\tau$ if and only if $\phi(g) = -1$. Making this precise in the language of cocycles gives the following:

Definition 1.9. Let (G, ϕ, τ) be a quantum symmetry group. A *(ϕ, τ)-twisted representation of G* on a Hilbert space $\mathcal{H}$ is a continuous map $\rho : G \to \operatorname{Aut}_{\mathrm{qtm}} \mathcal{H}$[12] such that

- $\rho(g)$ is unitary if $\phi(g) = 1$ and anti-unitary if $\phi(g) = -1$;
- $\rho(g)\rho(h) = \tau(g, h)\rho(gh)$ for all $g, h \in G$.

Note that a (ϕ, τ)-twisted representation induces a linear quantum symmetry ρ^τ of G^τ such that the subgroup $U(1) \subseteq G^\tau$ acts by scalar multiplication. Also, by a simple diagram-chasing argument, a (ϕ, τ)-twisted representation induces a projective quantum symmetry $\mathbb{P}\rho : G \to \operatorname{Aut}_{\mathrm{qtm}} \mathbb{P}\mathcal{H}$ such that the following diagram commutes:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U(1) & \longrightarrow & G^\tau & \longrightarrow & G & \longrightarrow & 1 \\ & & \| & & \downarrow{\scriptstyle \rho^\tau} & & \downarrow{\scriptstyle \mathbb{P}\rho} & & \\ 1 & \longrightarrow & U(1) & \longrightarrow & \operatorname{Aut}_{\mathrm{qtm}} \mathcal{H} & \longrightarrow & \operatorname{Aut}_{\mathrm{qtm}} \mathbb{P}\mathcal{H} & \longrightarrow & 1 \end{array}$$

and $\phi(g) = \phi_{\mathbb{P}\mathcal{H}}(\rho(g))$. Hence τ is the pull-back of the Wigner extension.

[11] Every cohomology class has a unital representative, see Lemma A.16.

[12] Note that ρ is in general not a group homomorphism.

1.4. Quantum Dynamics and Extended Symmetry

In quantum mechanics the dynamics are required to be governed by unitary operators. In the most general sense, this means that there is a family of projective quantum automorphisms $\mathbb{P}U(t_1, t_2)$ with $t_1, t_2 \in \mathbb{R}$ such that $\mathbb{P}U(t_1, t_2)$ maps a pure state at time t_2 to the state that it will be in at time t_1. Assuming time-translation invariance of the dynamics of the system, one can write $\mathbb{P}U(t_1, t_2) = \mathbb{P}U(t_1 - t_2)$ and we should definitely demand that $\mathbb{P}U(t_1 + t_2) = \mathbb{P}U(t_1)\mathbb{P}U(t_2)$. Since the dynamics are important in quantum mechanics, we want to include them in our abstract structure of symmetry.

Suppose we define a *quantum mechanical dynamical system* to be a projective quantum symmetry of the topological group of additive real numbers, i.e. a continuous group homomorphism $\mathbb{P}U : \mathbb{R}_t \to \operatorname{Aut}_{\text{qtm}} \mathbb{P}\mathcal{H}$.[13] By picking a lift of $\mathbb{P}U(0)$ in $\operatorname{Aut}_{\text{qtm}} \mathcal{H}$, we get a continuous homomorphism $U : \mathbb{R}_t \to \operatorname{Aut}_{\text{qtm}} \mathcal{H}$. Note that because $\mathbb{R}_t$ is connected and $t = 0$ is mapped to the identity operator, $\operatorname{Im} U \subseteq U(\mathcal{H})$. Applying Stone's theorem (see Rudin, Theorem 3.38)[14], there exists a densely defined self-adjoint operator $H : D(H) \to \mathcal{H}$ (that will be called the *Hamiltonian*) such that

$$U(t) = e^{-iHt}$$

for all $t \in \mathbb{R}$. Note that a different choice of lift U will give an extra phase $e^{i\phi}$. Conversely, a self-adjoint operator $H : D(H) \to \mathcal{H}$ will give a quantum mechanical dynamical system. Concretely this means that in our convention quantum mechanical dynamical systems are in one-to-one correspondence with classes of Hamiltonians, where two Hamiltonians give the same dynamical system if and only if they differ by a real constant.

In the setting of condensed matter physics we are usually interested in the behavior of the system relative to the *Fermi energy* E_F. This is a number in the spectrum of the Hamiltonian H such that at zero temperature all states below E_F are filled by electrons. Because of the important role of the Fermi energy in condensed matter, we pick the representative Hamiltonian so that the Fermi energy E_F equals zero.

We will now generalize this argument to show how to implement quantum symmetries within this framework of dynamics. To motivate how this should be done, suppose a quantum symmetry group consists of certain classical isometries of a spacetime manifold M in which a distinguished time-direction and a foliation by spatial slices has been selected. For explanatory purposes, we will now make some simplifying assumptions. Since we want to include time-translational invariance in order to get a Hamiltonian H, assume the group contains the group $\mathbb{R}_t$ of time translations. As our final interest will be solely in spacetime symmetries preserving a lattice of atoms fixed in space, it will also be assumed that there are no symmetries mixing space and time. This means that time reversal is still allowed; this is given by a homomorphism into $\mathbb{Z}_2$, which maps a symmetry to -1 if it reverses the order of time and $+1$ if it preserves it. Considering the structure of spacetime symmetry groups, the above assumptions imply that the symmetry group is

[13] The subscript t is just there to remind the reader that this is the additive group representing time translation.

[14] The topology on $\operatorname{Aut}_{\text{qtm}} \mathcal{H}$ has been chosen such that continuous homomorphisms $U : \mathbb{R}_t \to \operatorname{Aut}_{\text{qtm}} \mathcal{H}$ are exactly strongly continuous one-parameter subgroups (see Appendix D of Freed & Moore).

of the form $G \ltimes_\theta \mathbb{R}_t$, where $\theta : G \to \mathbb{Z}_2$ is the time reversal homomorphism $t \mapsto -t$. A consequence of the following proposition is that the structure of symmetries in quantum mechanical dynamical systems can be compactly algebraically summarized by a quantum symmmetry group (G, ϕ, τ) and such a homomorphism $\theta : G \to \mathbb{Z}_2$.

Proposition 1.10. *Let G be a topological group, $\theta : G \to \mathbb{Z}_2$ a homomorphism and suppose $(G \ltimes_\theta \mathbb{R}_t, \tilde{\phi}, \tilde{\tau})$ is a quantum symmetry group. Then $(G \ltimes_\theta \mathbb{R}_t, \tilde{\phi}, \tilde{\tau})$ is isomorphic to the pull-back of a quantum symmetry group (G, ϕ, τ) (i.e. $\pi : (G \ltimes_\theta \mathbb{R}_t, \tilde{\phi}, \tilde{\tau}) \to (G, \phi, \tau)$ is a morphism of quantum symmetry groups), which is determined uniquely up to isomorphism. Moreover, if $\mathcal{H}$ is a Hilbert space, this construction induces a 1-1 correspondence between $(\tilde{\phi}, \tilde{\tau})$-twisted representations $\tilde{\rho}$ of $G \ltimes_\theta \mathbb{R}_t$ on $\mathcal{H}$ and (ϕ, τ)-twisted representations ρ of G on $\mathcal{H}$ together with a self-adjoint operator H on $\mathcal{H}$ such that*

$$H\rho(g) = c(g)\rho(g)H,$$

where $c = t \cdot \phi$.

Proof. Let $\pi : G \ltimes_\theta \mathbb{R}_t \to G$ denote the projection and $j : \mathbb{R}_t \to G \ltimes_\theta \mathbb{R}_t$ the inclusion homomorphism. Since $\tilde{\phi}(1, 0) = 1$,[15] $j(\mathbb{R}_t)$ is connected and $\tilde{\phi}$ is continuous, it follows that $\ker \pi \subseteq \ker \tilde{\phi}$. Hence there is a unique homomorphism $\phi : G \to \mathbb{Z}_2$ such that the following commutes:

$$\begin{array}{ccc} G & \xleftarrow{\ \pi\ } & G \ltimes_\theta \mathbb{R}_t \\ & \overset{\phi}{\searrow} & \downarrow \tilde{\phi} \\ & & \mathbb{Z}_2 \end{array}$$

To finish the first part, we have to show that $\pi^* : H^2(G, U(1)_\phi) \to H^2(G \ltimes_\theta \mathbb{R}_t, U(1)_{\tilde{\phi}})$ is an isomorphism. For this it is sufficient to show that π is a homotopy equivalence within the category of topological groups. For this, note that the homomorphism $\psi : G \ltimes_\theta \mathbb{R}_t \to G \ltimes_\theta \mathbb{R}_t$ given by $\psi(g, t) = (g, 0)$ is homotopic to the identity via the continuous family of homomorphisms $H_s : G \ltimes_\theta \mathbb{R}_t \to G \ltimes_\theta \mathbb{R}_t$ given by $H_s(g, t) = (g, st)$ for $s \in [0, 1]$.

For the second part, let $\tilde{\rho} : G \ltimes_\theta \mathbb{R}_t \to \mathcal{H}$ be a $(\phi, \pi^*\tau)$-twisted representation. Then $\tilde{\rho} \circ j : \mathbb{R}_t \to \mathrm{Aut}_{\mathrm{qtm}} \mathcal{H}$ is a one parameter unitary subgroup; j is a homomorphism, $\tilde{\rho}|_{j(\mathbb{R}_t)}$ is a homomorphism and $\rho \circ j$ maps into the identity component of $\mathrm{Aut}_{\mathrm{qtm}} \mathcal{H}$. Let $H : D(H) \to \mathcal{H}$ be the self-adjoint operator such that $\tilde{\rho} \circ j(t) = e^{-itH}$. If we set $\rho : G \to \mathcal{H}$ to be $\rho(g) := \tilde{\rho}(g, 0)$, then

$$e^{-iHt}\rho(g) = \tilde{\rho}(1, t)\tilde{\rho}(g, 0) = \tilde{\rho}(g, \theta(g)t) = \tilde{\rho}(g, 0)\tilde{\rho}(1, \theta(g)t) = \rho(g)e^{-iH\theta(g)t}.$$

Hence

$$-iHt\rho(g) = -\rho(g)iH\theta(g) \implies H\rho(g) = \phi(g)\theta(g)\rho(g)H.$$

□

[15] The unit of G and $\mathbb{Z}_2$ is denoted 1, but the unit of $\mathbb{R}_t$ is 0.

Because the role of c on the quantum mechanical level (it determines whether symmetries commute or anti-commute with the Hamiltonian), we promote c to the important data instead of θ. Note that $\theta = c \cdot \phi$ is still contained in this information. We now get to our definition for a relevant set of symmetry data in a general quantum mechanical dynamical system.

Definition 1.11. An *extended quantum symmetry group* is a quadruple (G, ϕ, τ, c) consisting of a quantum symmetry group (G, ϕ, τ) and a homomorphism $c : G \to \mathbb{Z}_2$. A morphism of extended quantum symmetry groups is a morphism of quantum symmetry groups that preserves c. An *extended quantum system* $(\mathcal{H}, H, \rho)$ with extended quantum symmetry group (G, ϕ, τ, c) consists of a self-adjoint operator $H : D(H) \to \mathcal{H}$ and a (ϕ, τ)-twisted representation ρ of G on $\mathcal{H}$ such that $H\rho(g) = c(g)\rho(g)H$.

By the correspondence between cocycles and extensions, extended quantum symmetry classes as in Definition 3.7(i) of Freed & Moore's paper [27] are exactly isomorphism classes of extended quantum symmetry groups.

1.5. Abstract SPT Phases of Free Fermions

Now that we have abstracted the setting of a Hamiltonian with symmetries into the mathematical setting of group theory, we want to implement the idea of topological phases in this setting. Firstly, given a Hamiltonian H, we should be able to talk about the conduction band, the valence band and gapped systems. As we have made choices such that the Fermi energy is zero, this should be the following:

Definition 1.12. We say a Hamiltonian $H : D(H) \to \mathcal{H}$ is *gapped* if $0 \notin \operatorname{Spec} \mathcal{H}$. Using the spectral theorem for unbounded self-adjoint operators, let μ be a projection-valued measure for a gapped H. The *valence band* $\mathcal{H}^-$ is defined to be the image of $\mu(-\infty, 0)$ and the *conduction band* $\mathcal{H}^+$ is the image of $\mu(0, \infty)$.

Note that indeed an eigenstate $\psi \in \mathcal{H}$ of H is in the valence band exactly when it has eigenvalue strictly smaller than E_F, and similarly for the conduction band. In order to agree with the definition of a gapped system in the sense of Freed & Moore Definition 3.8 [27], we have to slightly generalize the notion of a valence band and a conduction band to a general direct sum decomposition. This is motivated by the idea of spectral flattening:

Theorem 1.13 (Spectral flattening)**.** *Let (G, ϕ, τ, c) be an extended quantum symmetry group. For all gapped extended quantum systems $(\mathcal{H}_0, H_0, \rho_0)$ with extended quantum symmetry group (G, ϕ, τ, c), there exists a gapped extended quantum system $(\mathcal{H}_1, H_1, \rho_1)$ homotopic to $(\mathcal{H}_0, H_0, \rho_0)$ such that $H_1^2 = 1$.*

Proof. The idea is to define the one-parameter family of functions $\mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ given by

$$\lambda \mapsto (1-t)\lambda + t\frac{\lambda}{|\lambda|}.$$

The spectral theorem then gives a one-parameter family H_t of Hamiltonians such that $H_0 = H$ and $H_1^2 = 1$. □

In other words, the relevant structure of a topological phase contained in the Hamiltonian is the splitting of the Hilbert space into a valence band and a conduction band. At least in the finite-dimensional context, the data of a Hamiltonian can therefore be replaced by a direct sum decomposition of the representation $\mathcal{H}$ in an abstract valence band and an abstract conduction band. This motivates the next definition, which is Definition 3.8 in Freed & Moore.

Definition 1.14. A *gapped extended quantum system* with extended quantum symmetry group (G, ϕ, τ, c) is an extended quantum system $(\mathcal{H}, H, \rho)$ with H gapped together with a direct sum decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ such that if $c(g) = 1$, then $\rho(g)$ is even with respect to the decomposition and if $c(g) = -1$, then $\rho(g)$ is odd.

Now, given an extended quantum symmetry group (G, ϕ, τ, c), symmetry-protected topological phases should be certain classes of extended quantum systems with this symmetry group. These classes must be invariant under continuous deformations of H that preserve the gap as well as the symmetry. Ideally, we would like to construct a moduli space of gapped extended quantum systems such that the path components correspond to topological phases, but we take a more pedestrian approach.

Definition 1.15. Two gapped extended quantum systems $(\mathcal{H}_0, H_0, \rho_0)$ and $(\mathcal{H}_1, H_1, \rho_1)$ with extended quantum symmetry group (G, ϕ, τ, c) are called *homotopic* if there exists a bundle $p : \mathcal{H}_\bullet \to [0, 1]$ of $\mathbb{Z}_2$-graded Hilbert spaces[16] together with a continuous family of real linear maps of Hilbert bundles $\rho_\bullet : G \ltimes_{\phi \cdot c} \mathbb{R}_t \to \operatorname{Aut}_{\mathrm{qtm}} \mathcal{H}_\bullet$ such that for every $s \in [0, 1]$ the restrictions

$$\rho_s : G \ltimes_{\phi \cdot c} \mathbb{R}_t \to \operatorname{Aut}_{\mathrm{qtm}} \mathcal{H}_s$$

make $\mathcal{H}_s$ into a gapped extended quantum system and the restrictions to $s = 0$ and $s = 1$ recover $(\mathcal{H}_0, H_0, \rho_0)$ and $(\mathcal{H}_1, H_1, \rho_1)$. A *topological phase* is a homotopy class of gapped systems. The set of topological phases of gapped systems with extended quantum symmetry group (G, ϕ, τ, c) is written $\mathcal{TP}(G, \phi, \tau, c)$.

The definition above is motivated by the intuitive description of a topological phase as deformation classes of Hamiltonians. However, since unbounded operators are hard to work with, we want to convert this data into something more algebraic, which will be the topic of the next section.

The set of topological phases can be made into a monoid by taking direct sums, getting one step closer to K-theory. For a short motivation on why this structure is relevant, recall that for non-interacting fermions, we describe the physics by a one-particle Hamiltonian H on a quantum mechanical Hilbert space $\mathcal{H}$ of one-particle states. The associated multiple particle quantum system of indistinguishable fermions is the Fock space

$$F(\mathcal{H}) := \bigwedge \mathcal{H}.$$

The subspace of forms of degree n are the n-particle states. The collection of physical systems of a certain kind in general have a natural 'sum operation' given by adjoining

[16] A bundle of Hilbert spaces here is a locally trivial fiber bundle with Hilbert spaces as fibers. It should be noted that all such bundles are globally trivial, but not canonically.

the systems and consider them to be noninteracting. In quantum physics this operation corresponds (to the student somewhat surpisingly) not to the direct sum of Fock spaces, but to the tensor product, since even noninteracting systems in quantum mechanics can be entangled. For one particle theories, the identities

$$\bigwedge(\mathcal{H}_1 \oplus \mathcal{H}_2) \cong \bigwedge \mathcal{H}_1 \otimes \bigwedge \mathcal{H}_2$$

imply that the natural operation on the one particle Hilbert space is given by the direct sum. Therefore we will consider $\mathcal{TP}(G, \phi, \tau, c)$ as a (commutative) monoid under direct sum.

In general it is very hard to compute $\mathcal{TP}(G, \phi, \tau, c)$, but there is a good approximation of this set which is in general computable, motivated by K-theory.

Definition 1.16. The group of *reduced topological phases* is the quotient of $\mathcal{TP}(G, \phi, \tau, c)$ by the submonoid of phases that admit an odd automorphism squaring to one.

The slight difference from the definition of Freed & Moore [27] (Paragraph 5, Subsection 'reduced topological phases') is indeed subtle and will be motivated in Section 1.7. The author is not aware of a physical interpretation of the set of reduced topological phases.

1.6. Representation Theory of Extended Quantum Symmetry Groups

We now embark on the study of representations of extended quantum symmetry groups, which are finite-dimensional analogues of gapped extended quantum systems. The language and theory of superalgebras Clifford algebras will be used freely (see Appendix B for a review). This leads to following definition, which agrees with Definition 3.7 of Freed & Moore [27].

Definition 1.17. Let (G, ϕ, τ, c) be an extended quantum symmetry group. A c-graded (ϕ, τ)-twisted representation of G on a Hilbert space $\mathcal{H}$ is a $\mathbb{Z}_2$-graded (ϕ, τ)-twisted representation $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ such that g acts by an even operator if $c(g) = 1$ and by an odd operator if $c(g) = -1$. A morphism of c-graded (ϕ, τ)-twisted representations of degree $d \in \mathbb{Z}_2$ is a complex linear map of degree d that graded commutes with the action of G.

Example 1.18. Consider the case in which $G = \mathbb{Z}_2$ and c, ϕ are both nontrivial. Denote the generator of G by C. As discussed in Remark A.22 in the appendix, we have $H^2(G, U(1)_\phi) = \mathbb{Z}_2$. Hence there are two nonisomorphic choices for τ; trivial or nontrivial. A representative of the nontrivial element is given by $\tau(C, C) = -1$ and the rest equal to one (see Remark A.19). For τ trivial, C lifts to an element squaring to 1 in the extension and for nontrivial τ it squares to -1. In other words, we are considering topological phases of Cartan type D and C respectively [55]. Considering Definition 1.11 and Theorem 1.13, our postulate is that (even) isomorphism classes of (ϕ, τ)-twisted c-graded representations of G correspond to quantum systems protected by (G, ϕ, τ, c).

It will now be shown what these representations are assuming a finite dimensional state space, i.e. for zero-dimensional quantum systems. We are given a complex supervector space (i.e. a $\mathbb{Z}_2$-graded vector space) $V = V_0 \oplus V_1$ and an odd complex anti-linear map $\epsilon : V \to V$ such that $\epsilon^2 = \pm 1$. Here the sign is plus if τ is trivial and minus if τ is nontrivial. Writing this out in components, we get two complex anti-linear maps $\epsilon_0 : V_0 \to V_1$ and $\epsilon_1 : V_1 \to V_0$ such that $\epsilon_0\epsilon_1 = \pm id_{V_1}$ and $\epsilon_1\epsilon_0 = \pm id_{V_0}$. Since ϵ_0 is an anti-linear isomorphism, we can use it to identify V_0 with V_1. It is now easy to show that this representation is equivalent to the representation $V = V_0 \oplus V_0$ with

$$\epsilon(v_0, v_1) = (\pm\bar{v}_1, \bar{v}_0),$$

where we have fixed some real structure $v \mapsto \bar{v}$ on V. We see that the isomorphism classes of c-graded (ϕ, τ)-twisted representations of G are classified by $\mathbb{N}$, the complex dimension of V.

To unravel the representation-theoretic characteristics of extended quantum symmetry groups (G, ϕ, τ, c), we identify a real superalgebra of which the supermodules are exactly the c-graded (ϕ, τ)-twisted representations of (G, ϕ, τ, c).

Definition 1.19. The *(ϕ, τ)-twisted c-graded group algebra* ${}^\phi\mathbb{C}^{\tau,c}(G)$ is the superalgebra over $\mathbb{R}$, which as a complex vector space is $|G|$-dimensional with $\mathbb{C}$-basis $\{x_g : g \in G\}$ together with defining relations

$$\begin{aligned} ix_g &= \phi(g)x_g i \\ x_g x_h &= \tau(g, h)x_{gh}. \end{aligned}$$

It is graded by the $\mathbb{C}$-linear map induced by $x_g \mapsto c(g)$. If c is trivial, we write ${}^\phi\mathbb{C}^\tau(G)$ for the (nongraded) real algebra with the same generators and relations as ${}^\phi\mathbb{C}^{\tau,c}(G)$.

Remark 1.20. If ϕ is nontrivial, ${}^\phi\mathbb{C}^{\tau,c}(G)$ is not naturally a complex algebra, but it does have a canonical real subalgebra isomorphic to the complex numbers. Hence, if M is a supermodule over ${}^\phi\mathbb{C}^{\tau,c}(G)$, it has a natural complex structure given by the action of the element $i \in {}^\phi\mathbb{C}^{\tau,c}(G)$. Therefore supermodules are always complex vector spaces and maps of supermodules are complex linear, but elements of ${}^\phi\mathbb{C}^{\tau,c}(G)$ can act anti-linearly.

Remark 1.21. Because of our assumption that $\tau(g, 1) = \tau(1, g) = 1$, the algebra has x_1 as a unit.

Remark 1.22. Associativity is equivalent to τ being a cocycle:

$$\begin{aligned} (z_1x_{g_1}z_2x_{g_2})z_3x_{g_3} &= (z_1^{\phi(g_1)}\bar{z}_2\tau(g_1, g_2)x_{g_1g_2})z_3x_{g_3} \\ &= z_1^{\phi(g_1)}\bar{z}_2^{\phi(g_1g_2)}\bar{z}_3\tau(g_1, g_2)\tau(g_1g_2, g_3)x_{g_1g_2g_3} \\ z_1x_{g_1}(z_2x_{g_2}z_3x_{g_3}) &= z_1x_{g_1}(z_2^{\phi(g_2)}\bar{z}_3\tau(g_2, g_3)x_{g_2g_3}) \\ &= z_1^{\phi(g_1)}\overline{z_2^{\phi(g_2)}\bar{z}_3\tau(g_2, g_3)}\tau(g_1, g_2g_3)x_{g_1g_2g_3} \\ &= z_1^{\phi(g_1)}\bar{z}_2^{\phi(g_1g_2)}\bar{z}_3^{\phi(g_1)}\overline{\tau(g_2, g_3)}\tau(g_1, g_2g_3)x_{g_1g_2g_3}. \end{aligned}$$

Example 1.23. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ with ϕ projection onto the second factor. Denote the generator of the first factor by R and the second one by T. By Proposition A.24, we have four possible anomaly twists in $H^2(G, U(1)_\phi)$. Pick the anomaly in which both R and T square to -1 in the corresponding extension. Then ${}^\phi\mathbb{C}^\tau G$ is generated by i, T and R with relations

$$T^2 = R^2 = i^2 = -1, \quad TR = RT, \quad iR = Ri, \quad iT = -Ti.$$

This algebra is isomorphic to $\mathbb{C} \otimes_\mathbb{R} \mathbb{H}$ (which itself can easily seen to be isomorphic to $M_2(\mathbb{C})$). Indeed, it is readily verified that the map ${}^\phi\mathbb{C}^\tau G \to \mathbb{C} \otimes_\mathbb{R} \mathbb{H}$ given by

$$T \mapsto 1 \otimes j, \quad i \mapsto 1 \otimes i, \quad R \mapsto i \otimes 1$$

is an isomorphism of real algebras.

It is easy to show that the category of (ϕ, τ)-twisted c-graded representations of G is equivalent to the category of supermodules over ${}^\phi\mathbb{C}^{\tau,c}(G)$[17]. Moreover, this correspondence preserves direct sums. Hence we embark on a study of supermodules over the twisted group algebra. The twisted group algebra turns out to be supersemisimple, which makes it isomorphic to a direct sum of super matrix rings over superdivision rings by a super-version of Wedderburn-Artin's theorem, see Propositions B.6 and B.8. According to Proposition B.14, there are exactly ten isomorphism classes of superdivision rings over the real numbers. Moreover, supermodules over superdivision algebras are free in a super-sense, so that supermodules over the twisted group algebra are easily classified once the decomposition into matrix rings is known.

To give some idea of the structure of the twisted group algebra in some simple cases, we state the following lemma:

Lemma 1.24. *Let $p : G \times \mathbb{Z}_2 \to \mathbb{Z}_2$ be projection onto the second factor.*

1. *If ϕ and τ are trivial, then ${}^\phi\mathbb{C}^\tau(G)$ is the complex group algebra, seen as a real superalgebra with a trivial odd component;*
2. *If $\phi = p$ and τ are trivial, then ${}^\phi\mathbb{C}^\tau(G \times \mathbb{Z}_2) \cong M_2(\mathbb{R}) \otimes_\mathbb{R} \mathbb{R}G$ is 2 by 2 matrices over the real group algebra;*
3. *If $\phi = p$ is trivial and τ is the pull-back of an element of $Z^2(\mathbb{Z}_2, U(1)_\phi)$ under ϕ that is not a coboundary, then ${}^\phi\mathbb{C}^\tau(G \times \mathbb{Z}_2) \cong \mathbb{H} \otimes_\mathbb{R} \mathbb{R}G$ is the quaternionic group algebra;*

Proof. The first point is trivial. For the other points, note that ${}^\phi\mathbb{C}^\tau(G \times \mathbb{Z}_2)$ is the real algebra generated by elements x_g for $g \in G$ and elements i and T with relations

$$x_g x_h = x_{gh}, \quad x_g T = T x_g, \quad iT = -Ti, \quad i^2 = -1, \quad T^2 = \pm 1.$$

Here $T^2 = 1$ for the second point and $T^2 = -1$ for the third point. We see that

$${}^\phi\mathbb{C}^\tau(G \times \mathbb{Z}_2) \cong \begin{cases} |Cl_{1,1}| \otimes_\mathbb{R} \mathbb{R}G \\ |Cl_{0,2}| \otimes_\mathbb{R} \mathbb{R}G, \end{cases}$$

where $|Cl_{p,q}|$ is the underlying nongraded algebra of the real Clifford algebra $Cl_{p,q}$ (see Appendix B.2). It is easy to show that $|Cl_{0,2}| \cong \mathbb{H}$ and $|Cl_{1,1}| \cong M_2(\mathbb{R})$. □

[17]The idea is the same as for ordinary complex group algebras as in Serre's book [61] Section II.6.1.

1.7. Supertrivial Supermodules and Twisted Representation Rings

There has been some debate in the literature about the question what topological phases should be called trivial. It has been suggested that we need to quotient out by certain trivial 'product states' in order to get the right definition. We choose to follow the mathematically reasonable route, i.e. the one that produces the relevant K-theory group. This leads us to a notion of 'supertrivial' bundles in order to classify reduced topological phases in a sense similar to Freed & Moore [27]. The ideas developed in earlier chapters imply that the data of a zero-dimensional topological phase (in the sense that we assume a finite dimensional state space) which is protected by an extended quantum symmetry group (G, ϕ, τ, c) is equivalent to the choice of a finite-dimensional c-graded (ϕ, τ)-twisted representation of G, i.e. a supermodule over ${}^{\phi}\mathbb{C}^{\tau,c}G$. First of all, what would mathematically speaking be the most reasonable subset of representations that we can quotient out to get a nice algebraic invariant?

One could ask the topological phases modulo the trivial phases to form something analogous to a representation ring of a group algebra. In case there is no particle-hole symmetry, i.e. the grading homomorphism c is trivial, the form of such a representation ring is well-known; according to a Grothendieck completion procedure it should consist of formal differences of representations. Suppose now that we are given a c-graded (ϕ, τ)-twisted representation $V = V^0 \oplus V^1$. We would like to see this representation as being similar to a formal difference $V^0 - V^1$. If c is trivial, we simply have a pair V^0 and V^1 of (ϕ, τ)-twisted representations. To get the Grothendieck completion back in this case, we should demand that $V^0 \oplus V^1$ corresponds to a trivial topological phase whenever $[V_0] - [V_1] = 0$ in the representation ring, i.e. whenever V_0 and V_1 are isomorphic as (non-graded) twisted representations. This leads Freed & Moore to a reasonable suggestion for the c-graded representation rings; demand that a c-graded (ϕ, τ)-twisted representation $V = V^0 \oplus V^1$ gives a trivial phase if it admits an odd automorphism. Then define the representation ring of an extended quantum symmetry group (and hence the reduced topological phases with that symmetry) as the monoid of c-graded τ -twisted representations modulo the ones that correspond to a trivial phase as above. Here however this definition will be slightly altered, for there is a subtle catch inherent to the fact that we allow anti-unitary operators, which will now be illustrated.

For this we compare our formulation with the ten-fold way of condensed matter and Corollary 8.9 of Freed & Moore. Consider the extended quantum symmetry group of Example 1.23 for which $G = \mathbb{Z}_2$ and c, ϕ are both nontrivial. Denote the generator of G by C. As (ϕ, τ)-twisted c-graded representations of G correspond to zero-dimensional quantum systems protected by G, we can decide when to call such a representation supertrivial by comparing with the well-known classification of topological phases given by the ten-fold way (as in for example [55], table 2). From the example in Section 1.4, it is known that isomorphism classes of representations are given by

$$V = V_0 \oplus V_1 = \mathbb{C}^n \oplus \mathbb{C}^n \quad C(v_0, v_1) = (\pm \bar{v}_1, \bar{v}_0),$$

for every $n \geq 0$, where the sign is plus if τ is trivial (Cartan type D) and minus if τ

is nontrivial (Cartan type C). The ten-fold way gives that in zero dimensions type D topological phases are classified by $\mathbb{Z}_2$ and for type C there is only the trivial topological phase. In order to reproduce this fact, we need V to always correspond to a trivial topological phase in case τ is trivial and V to correspond with a trivial topological phase in case τ is nontrivial if and only if its complex dimension is divisible by four. How should this be achieved?

Since Freed & Moore suggest that (ϕ, τ)-twisted c-graded representations should be called trivial if they admit an odd automorphism, the odd automorphisms of this example will now be determined. Let $A : V \to V$ be an odd automorphism, consisting of complex linear maps $A_0 : V_0 \to V_1$ and $A_1 : V_1 \to V_0$. In order for it to be an automorphism, A and C should *supercommute*. Since they are both of odd degree, this means that they should anti-commute in the ordinary sense (this agrees with Freed & Moore, definition 7.1). We work out this condition:

$$\begin{aligned} CA(v_0, v_1) &= C(A_1(v_1), A_0(v_0)) = (\pm\overline{A_0(v_0)}, \overline{A_1(v_1)}) \\ AC(v_0, v_1) &= A(\pm\bar{v}_1, \bar{v}_0) = (A_1(\bar{v}_0), \pm A_0(\bar{v}_1)). \end{aligned}$$

Hence $CA + AC = 0$ is equivalent to $A_1(v) = \mp\overline{A_0(\bar{v})}$ for all $v \in \mathbb{C}^n$. In particular, A_1 is uniquely determined by A_0. However, for A_0 we can pick any $\mathbb{C}$-linear automorphism of $\mathbb{C}^n$ and using the above formulas, we get an odd automorphism A of c-graded (ϕ, τ)-twisted representations. Hence according to the definitions of Freed & Moore, there are no nontrivial topological phases of type C and D, which contradicts the ten-fold way.

The natural way to solve this problem is to look more precisely at the role that real Clifford algebras play in grading real K-theory. One important mathematical motivation results from comparing the following proposition of Atiyah & Segal [7], restated in the language of this thesis, to the ten-fold way. Note that it is a generalization of the results of Atiyah, Bott and Shapiro [4] to the case of a nontrivial group G.

Proposition 1.25. *Let G be a finite group. Let $M^q(G)$ denote the set of isomorphism classes of $\mathbb{Z}_2$-graded real representations V of G that also come equipped with an action of the real Clifford algebra $Cl_{q,0}$ such that G acts by even maps and the usual generators of the Clifford algebra act by odd maps. In other words, $M^q(G)$ denotes all (even) isomorphism classes of supermodules over the real superalgebra $\mathbb{R}G\hat{\otimes}_{\mathbb{R}}Cl_{0,p}$, where $\hat{\otimes}_{\mathbb{R}}$ denotes the graded tensor product over $\mathbb{R}$. There is a map $r_q : M^{q+1}(G) \to M^q(G)$ given by forgetting the action of one Clifford algebra generator and*

$$KO_G^{-q}(\text{pt}) \cong \operatorname{coker} r_q.$$

In other words, from the perspective of K-theory it seems reasonable to assume that a representation corresponds to a trivial product phase if it admits an extra Clifford action intertwining the G-action. Moreover, it is well-known that the complex and real K-theory of a point is directly related to the ten-fold way. More precisely, one should ask not only for an odd isomorphism, but for an odd isomorphism squaring to ± 1.[18] This exactly

[18]Although the paper of Atiyah & Segal [7], from which the above proposition is taken, makes use of Clifford elements with negative squares, the author notes that one can also make use of positive squares. However, in that case the grading will move backwards.

means that a representation is in the trivial phase if it is also a supermodule over $Cl_{0,1}$ (in case we demand the isomorphism to be squaring to one) or $Cl_{1,0}$ (in case the square is minus one), which moreover supercommutes with the action of G. We now apply this suggestion to the example we were considering.

So when does A square to one and when to -1? We just compute:

$$A^2(v_1, v_2) = A(\mp\overline{A_1(\bar{v}_2)}, A_1(v_1)) = (\mp\overline{A_1(\overline{A_1(v_1)}}, \mp A_1(\overline{A_1(\bar{v}_2)})).$$

If $n = 1$, then A_1 is multiplication by some number $z \in \mathbb{C}$. Hence

$$A^2(v_1, v_2) = \mp(\bar{z}zv_1, z\bar{z}v_2) = \mp|z|^2(v_1, v_2).$$

So in this case A^2 cannot square to a positive number if τ is trivial and it cannot square to a negative number if τ is nontrivial. It can be concluded that for type D, the two-dimensional representation admits an odd isomorphism squaring to -1, but not to 1, while for type C it admits an odd isomorphism squaring to 1 and not to -1. In order to agree with the ten-fold way we see that *if* we want to use the mathematically natural definitions of K-theory and corresponding gradings using Clifford algebras, we *must* ask our bundles to correspond to trivial topological phases if they admit an odd automorphism squaring to 1. Note how this condition is slightly stronger than just asking for an odd automorphism to exist.

We therefore can now finally introduce the relevant representation ring, which will classify reduced symmetry-protected topological phases of free fermions in zero dimensions. For later reference we immediately define higher representation rings as higher dimensional generalizations of the ordinary representation ring, which in the definition below is the case $p = q = 0$.

Definition 1.26. let $p, q \geq 0$ be integers. Define the *(higher) twisted representation ring* ${}^{\phi}R^{\tau,c+(p,q)}(G)$ *of degree* (p, q) as the following abelian group:[19] consider the monoid of c-graded (ϕ, τ)-twisted representations of G equipped with a graded action of the Clifford algebra $Cl_{p,q}$ with direct sum as operation. Then quotient by the submonoid consisting of modules that admit an extra graded Clifford action, i.e. an extension to $Cl_{p+1,q}$. In the notation of Definition B.15 in Appendix B

$${}^{\phi}R^{\tau,c+(p,q)}(G) = R^{p,q}({}^{\phi}\mathbb{C}^{\tau,c}G) = \frac{\mathrm{Mod}_s({}^{\phi}\mathbb{C}^{\tau,c}G\hat{\otimes}_{\mathbb{R}}Cl_{p,q})}{\mathrm{Mod}_s({}^{\phi}\mathbb{C}^{\tau,c}G\hat{\otimes}_{\mathbb{R}}Cl_{p+1,q})}.$$

Note that it agrees with Definition 7.1(iii) in Freed & Moore for the case that p, q and c are trivial, but otherwise it can be different. This follows as this representation ring is the Grothendieck completion of the monoid of (ϕ, τ)-twisted representations of G. See Appendix B.3 for details on the theory of such representation rings.

Example 1.27. Let us consider a physically relevant example not covered by Lemma 1.24 and generalizing Example 1.23. Suppose $G = \mathbb{Z}_n \times \mathbb{Z}_2$, ϕ is projection onto the second

[19] Unlike its name suggests, the twisted representation ring is actually not a ring, since the tensor product of a τ_1-twisted and a τ_2-twisted representation is $\tau_1 \cdot \tau_2$-twisted.

factor and c is trivial. We write R for a generator of the first factor (thought of as rotation) and T for the generator of the second factor (thought of as time reversal). It is easy to see from Lemma A.3 and Lemma A.21 from Appendix A.1 that $H^2(G, U(1)_\phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. In a similar fashion to Proposition A.24 it can be shown that the four possibilities correspond to the choices of signs in $R^n = \pm 1$ and $T^2 = \pm 1$. In order to reproduce the theory of time reversal symmetric fermions, we assume that both signs are negative; rotation by 2π gives a minus sign and time reversal squares to -1 for particles with half-integer spin. The twisted group algebra is

$$\frac{\mathbb{R}[i, T, R]}{(Ti + iT, Ri - iR, TR - RT, T^2 + 1, i^2 + 1, R^n + 1)} \cong \mathbb{H} \otimes_{\mathbb{R}} \frac{\mathbb{R}[R]}{(R^n + 1)},$$

where the quaternion algebra is the subalgebra generated by i and T. Note that we can factor

$$R^n + 1 = \prod_{k=0}^{n-1} \left(T - e^{\frac{\pi i}{n}(2k+1)} \right)$$

over $\mathbb{C}$. The roots are all different and can only be real when n is odd and k is such that $2k + 1 = n$. Hence by the Chinese remainder theorem,

$$\frac{\mathbb{R}[R]}{(R^n + 1)} \cong \begin{cases} \mathbb{C}^{n/2} & n \text{ even}, \\ \mathbb{R} \oplus \mathbb{C}^{(n-1)/2} & n \text{ odd}. \end{cases}$$

Hence the twisted group algebra is

$$\mathbb{H} \otimes_{\mathbb{R}} \frac{\mathbb{R}[R]}{(R^n + 1)} \cong \begin{cases} M_2(\mathbb{C})^{n/2} & n \text{ even}, \\ \mathbb{H} \oplus M_2(\mathbb{C})^{(n-1)/2} & n \text{ odd}. \end{cases}$$

Therefore by Corollary B.24 we see that the representation ring equals

$$R^p({}^\phi\mathbb{C}^\tau G) \cong \begin{cases} K^p(\mathrm{pt})^{n/2} & n \text{ even}, \\ KO^{p+4}(\mathrm{pt}) \oplus K^p(\mathrm{pt})^{(n-1)/2} & n \text{ odd}. \end{cases}$$

Representation rings of twisted group algebras form an excellent way to introduce the ten-fold way of topological insulators. Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, $\phi : G \to \mathbb{Z}_2$ the product map and $c : G \to \mathbb{Z}_2$ projection onto the second coordinate. Denote the generator of the first factor of G by T and the generator of the second factor by C. Let $H \subseteq G$ be a subgroup. Recall that

- if H is trivial, then $H^2(H, U(1)_\phi) = 0$;
- if H is the diagonal subgroup, then ϕ is trivial, so that $H^2(H, U(1)_\phi) = 0$ (see Lemma A.21 in the appendix);
- if $H = 1 \times \mathbb{Z}_2$ or $H = \mathbb{Z}_2 \times 1$, then $H^2(H, U(1)_\phi) = \mathbb{Z}_2$ depending on whether in the twisted group algebra the generator squares to ± 1;
- if $H = G$, then $H^2(H, U(1)_\phi) = \mathbb{Z}_2^2$ depending on whether $C^2 = \pm 1$ and $T^2 = \pm 1$.

The representation rings ${}^{\phi}R^{\tau,c}(H)$ are then determined as in the following table, which is easily derived using the theory of Appendix B. The notation of the *CT-type* is as follows: the two symbols denote the first and second generator respectively. The symbol 0 implies that the corresponding element is not in H. The symbols $\pm$ imply that the corresponding element is in H and its square in the twisted group algebra is ± 1.

CT-type of (H, ϕ, τ, c)	00	diag	$+0$	$+-$	$0-$	$--$	-0	$-+$	$0+$	$++$
${}^{\phi}R^{\tau,c}(H)$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$

The reader acquainted with K-theory will note the resemblance with the K-theory of a point. In fact, the table above can be generalized to multiple settings. First of all, we can compute the twisted representation ring of $H \times G$, which we make into an extended quantum symmetry group by pulling back the structure on H. Then the table will contain the (either real or complex) representation ring of G shifted with the appropriate degree. See Appendix B and Paragraph 8 of Freed & Moore for more information. Secondly, Freed & Moore formulate a generalization of such a statement to twisted equivariant K-theory as Corollary 10.25.

2. Condensed Matter

Under the right physical conditions, large classes of materials occurring in nature can be condensed into highly symmetric configurations called crystals. Condensed matter theory mostly concerns systems of electrons moving inside such a fixed lattice of atom cores. Our interest will therefore be restricted to quantum systems that are periodic under a perfect lattice, possibly with more crystal symmetries. The intuitive concepts of lattices and crystals will be developed in a formal setting and the usual notion of the Brillouin zone will naturally appear. Restricting to the example of wave functions, we recover band theory as the study of the Hilbert bundle of Bloch waves over the Brillouin zone. The induced action of a classical symmetry group on wave functions will give this Bloch bundle the structure of a so-called twisted equivariant vector bundle. The reader should see this chapter as a journey to examine the exact mathematical structure on the Bloch bundle that is conserved under continuous deformations of the Hamiltonian.

2.1. Lattices

To define symmetries of perfect configurations of atoms, the mathematical definitions related to lattices will be discussed.

Definition 2.1. Let V be a d-dimensional real vector space. An *n-dimensional lattice* is a subgroup $L \subseteq V$ isomorphic to $\mathbb{Z}^n$ such that the real linear span of L is a n-dimensional vector space. The lattice is called *full* if $d = n$.

Note that a group homomorphism $k : L \to 2\pi\mathbb{Z}$ from a full lattice, defines a unique linear functional on V, which will also be written k. Therefore the set of homomorphisms $\mathrm{Hom}(L, 2\pi\mathbb{Z})$ can be identified with the elements of V^* that map L into $2\pi\mathbb{Z}$ in that case.[1]

Definition 2.2. The *reciprocal space* or *momentum space* of the space V is the dual space V^*. Given a full lattice $L \subseteq V$ in space, the *reciprocal lattice* inside reciprocal space is the subgroup $L^\vee := \mathrm{Hom}(L, 2\pi\mathbb{Z}) \subseteq V^*$. The *Brillouin zone* is the quotient $T^\vee := V^*/L^\vee$.

Note that if L is full, then $T^\vee$ is topologically a torus. It is in some sense dual to the torus $T := V/L$. Mathematically one could think about the Brillouin zone as follows:

Proposition 2.3. *The Brillouin zone of a full lattice L is isomorphic to the so-called Pontrjagin dual of L, which is given by $\hat{L} := \mathrm{Hom}(L, U(1))$.*

Proof. Let $\phi : V^* \to \hat{L}$ be the map $\phi_k(v) = e^{ik(v)}$ for $k \in V^*$ and $v \in L$. We see that the map $\phi_k : L \to U(1)$ is identically equal to one if and only if $k(L) \subseteq 2\pi\mathbb{Z}$. Hence $L^\vee = \ker \phi$. By picking a $\mathbb{Z}$-basis of L, it is easy to see that ϕ is surjective. □

[1] The 2π appear in these formulae because of the physicist's convention to denote momentum space, which will be motivated by the proposition below.

2.2. Bloch Theory

An important quantum system associated to a d-dimensional Euclidean space V is the space of wave functions; L^2-functions $\mathcal{H} = L^2(V, W)$ with values in a fixed finite-dimensional vector space W of internal degrees of freedom. Suppose that we consider time-evolution in a system of electrons inside a lattice L consisting of atom cores with a periodic potential. The translation-invariant Hamiltonian gives an effective quantum theory on the Brillouin zone $T^\vee$ by Bloch's theorem, hence reducing computations to a compact space. Bloch waves on the Brillouin zone will then naturally appear as Fourier-type decompositions of wave functions. All of this well-known theory will now be developed in a mathematical context.

Definition 2.4. A *Bloch wave with Bloch momentum* $k \in T^\vee$ is a function $\psi_k : V \to \mathbb{C}$ of the form

$$\psi_k(v) = e^{ik(v)}u(v),$$

where $u : V \to \mathbb{C}$ is a periodic (i.e. L-invariant) function inducing a L^2-function on T.

More generally, Bloch waves could take values in a complex vector space W. The reader should be warned that nontrivial Bloch waves are themselves not L^2.

Lemma 2.5. *A function $\psi : V \to \mathbb{C}$ (which is L^2 on a fundamental domain) is a Bloch wave with Bloch momentum k if and only if it satisfies the* Bloch wave condition*:*

$$\psi(x + v) = e^{ik(v)}\psi(x)$$

for every $x \in V$ and $v \in L$.

Proof. One sees immediately that a Bloch wave with Bloch momentum k satisfies the Bloch wave condition. Let ψ satisfy the Bloch wave condition with momentum k. Define $u(x) := e^{ik(-x)}\psi(x)$. Then clearly $\psi(x) = e^{ik(x)}u(x)$ and

$$u(x + v) = e^{ik(-x-v)}e^{ik(v)}\psi(x) = u(x).$$

□

The importance of Bloch waves is in Bloch sums, which are necessary to state Bloch's theorem.

Definition 2.6. The *Bloch sum* of an L^2-function $f : V \to W$ is the map $\hat{f} : T^\vee \times V \to W$ given by

$$\hat{f}_k(x) = \sum_{v \in L} e^{ik(v)} f(x - v).$$

Note that f_k satisfies the Bloch wave condition.

The Bloch sum of a L^2-function can be regarded as a function of space and momentum, so it is natural to also consider Bloch waves as a function of momentum. Because the Bloch wave condition is a twisted periodicity condition, one could describe these as L^2-sections of a bundle.

Definition 2.7. The *Poincaré line bundle* $\mathcal{L}$ is the line bundle over $T^\vee \times T$ given by the quotient of the trivial complex line bundle over $T^\vee \times V$ by the L-action

$$v \cdot (k, x, z) \to (k, x + v, e^{ik(v)} z).$$

By construction, a section of the Poincaré bundle is the same as a function $f : T^\vee \times V \to \mathbb{C}$ such that $f_k(x+v) = e^{ik(v)} f_k(x)$ for all $k \in T^\vee, x \in V$ and $v \in L$. In particular, sections of the restriction $\mathcal{L}_k := \mathcal{L}|_{\{k\}\times T}$ are exactly the Bloch waves with momentum k.

It should not be surprising that Bloch sums induce a Fourier decomposition into Bloch waves. This is best summarized using the Bloch bundle, which can be constructed using the Poincaré line bundle. For this, we consider the family of Hilbert spaces $\mathcal{E}$ over the Brillouin zone, which to every point assigns the Bloch waves with momentum k. This results in the following two propositions, of which a sketch can be found in Freed & Moore:

Proposition 2.8. *Bloch waves $\mathcal{E}_k := \Gamma_{L^2}(T, \mathcal{L}_k)$ with fixed momentum $k \in T^\vee$ form a Hilbert bundle over $T^\vee$ called the* Bloch bundle. *Moreover, the Bloch sum defines an isomorphism of Hilbert spaces $F : L^2(V, W) \cong \Gamma_{L^2}(T^\vee; \mathcal{E} \otimes W)$, where $\Gamma_{L^2}(\mathcal{E} \otimes W)$ denotes L^2-sections of the Hilbert bundle $\mathcal{E}$ with values in W. The inverse of the Bloch sum maps $\psi \in \Gamma_{L^2}(T^\vee; \mathcal{E} \otimes W)$ to*

$$x \mapsto \int_{k \in T^\vee} \psi_k(x) dk.$$

Proposition 2.9 (Bloch's Theorem)**.** *Let $H : D(H) \to L^2(V, W)$ be a self-adjoint operator acting on the space of wave functions that commutes with the action of L on $L^2(V, W)$ induced by the translation action on V. Then under the Bloch sum isomorphism F of last proposition, H gives a continuous family $\{H_k\}_{k\in T^\vee}$ of self-adjoint operators (called Bloch Hamiltonians) on the fibers of the Bloch bundle. So $H_k : D(H_k) \to \mathcal{E}_k$ are such that*

$$(H\psi)(k) = H_k \psi(k) \quad \psi \in \Gamma_{L^2}(T^\vee; \mathcal{E} \otimes W).$$

2.3. Mathematical Crystallography

In general, lattices have more symmetries than just translations and these symmetries can be of physical relevance. The Hamiltonian of a quantum system on a lattice can for example be invariant under a rotational symmetry of the lattice, thereby realizing states as representations of a discrete rotation group, such as a dihedral group. Mathematically speaking, lattices are naturally free abelian groups that should be considered as translation symmetries of a crystal, but we want crystal symmetries to also include elements of the orthogonal group $O(d)$. Therefore crystals should be defined as objects in affine Euclidean space $\mathbb{E}^d$ with their symmetries inside the Euclidean symmetry group $\mathbb{E}(d)$ of d-dimensional space, also to distinguish them mathematically from lattices.[2]

[2]We write $\mathbb{E}$ for Euclidean instead of $\mathbb{R}$ to remind ourselves that there is no canonical origin chosen.

Definition 2.10. Let
$$1 \to V \to \mathbb{E}(d) \to O(d) \to 1$$
be the (non-canonically) split exact sequence, where $V \subseteq \mathbb{E}(d)$ is the subgroup of translations. A *crystal* is a subset $C \subseteq \mathbb{E}^d$ with the property that its translational symmetries

$$L(C) := \{v \in V : C + v = C\} \subseteq V \subseteq \mathbb{E}(d)$$

form a lattice. The *space group* $S(C)$ of a crystal is the subgroup of $\mathbb{E}(n)$ preserving the lattice:
$$S(C) := \{T \in \mathbb{E}(d) : T(C) = C\}.$$
The *point group* $P(C)$ is the quotient of $S(C)$ by $L(C)$. In other words, there is a group extension, which is a subextension of the usual Euclidean group extension as follows.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L(C) & \longrightarrow & S(C) & \longrightarrow & P(C) & \longrightarrow & 1 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 1 & \longrightarrow & V & \longrightarrow & \mathbb{E}(d) & \longrightarrow & O(d) & \longrightarrow & 1 \end{array}$$

The crystal is called *symmorphic* if the first of the two exact sequences above splits.

Suppose we have chosen an origin $0 \in \mathbb{E}^d$, so the Euclidean group extension splits into a semi-direct product. Although the point group is naturally a subgroup $P(C) \subseteq O(d)$, the origin gives this a realization of elements as rotations and reflections with respect to the origin 0. In condensed matter a computational notation - called the *Seitz notation* - is often used in this context. To make connections with the literature, this notation will be explained in terms of the abstract definitions given here. The element of $O(d) \ltimes V$ which projects down to the elements $R \in O(n)$ and $v \in V$ is written $\{R|v\}$. Then the multiplication rule of the semidirect product is

$$\{R_1|v_1\}\{R_2|v_2\} = \{R_1R_2|v_1 + R_1v_2\}.$$

In particular, the inverse of $\{R|v\}$ is just $\{R^{-1}| - R^{-1}v\}$.

In order to briefly discuss the classification of crystal symmetries, it is convenient to convert this problem to the setting of abstract groups. First of all, it should be noted that in this document the only information of a crystal that is actually used, is the symmetry data that has to be specified in order to be able to talk about symmetry-protected topological phases. Therefore, two crystals are said to be *isomorphic* if their space groups are isomorphic. The next question that arises is when an abstract group can be realized as a space group of a crystal. We start by finding constraints on the groups $P(C)$ and $S(C)$. The first step is given by Zassenhaus' theorem.

Theorem 2.11 (Zassenhaus [71])**.** *A discrete group G can be realized as a subgroup of $\mathbb{E}(d)$ such that $V \cap G$ spans V if and only if G contains a free abelian subgroup $L \subseteq G$ of rank d that is maximally abelian such that G/L is finite.*

So this translates the problem of classifying space groups into classifying extensions of certain groups by lattices. These abstract point groups must certainly be finite subgroups of $O(d)$, which can be classified at least for small d. As our interest lies in $d \leq 3$, we state the following proposition for reference:

Proposition 2.12. *The finite subgroups of $O(2)$ are all isomorphic to cyclic groups and dihedral groups. Finite subgroups of $SO(3)$ are isomorphic to exactly one of the following:*

- *a cyclic group C_n of order n with $n \geq 1$;*
- *a dihedral group D_n of order $2n$ with $n \geq 2$;*
- *the symmetry group of a platonic solid, of which there are three nonisomorphic examples:*
  - *The rotational symmetry group A_4 of the tetrahedron;*
  - *The rotational symmetry group S_4 of the cube or the octahedron;*
  - *The rotational symmetry group A_5 of the dodecahedron or the icosahedron.*

Proof. See for example Chapter 19 of Armstrong's book [2]. □

To account more generally for finite subgroups of $O(3)$ we have to include group extensions of the above with $\mathbb{Z}_2$. This makes the classification slightly more messy and an exposition is therefore excluded.

A second constraint on the finite group G/L is that it must act on the lattice $L \cong \mathbb{Z}^d$ by conjugation. If $G/L \cong P(C)$ comes from the point group of a lattice, this action must be faithful, since an element $g \in P(C)$ is uniquely determined by its action on the lattice L. Hence $P(C)$ must be a finite subgroup of $\operatorname{Aut} \mathbb{Z}^d$ as well. This gives the so-called crystallographic restriction, thereby reducing the number of possible space groups to a finite number.[3] Although there is a result for general dimensions, we only consider $d \leq 3$.

Theorem 2.13 (Crystallographic Restriction)**.** *For $d \leq 3$, finite subgroups of $\operatorname{Aut} \mathbb{Z}^d$ can only contain elements of order $1, 2, 3, 4$ and 6.*

Proof. The result for general d can be found in [47], Theorem 2.7. □

Hence we are left with the problem of classifying group extensions of finite subgroups P of $O(3)$ with elements of order $1, 2, 3, 4$ and 6. The problem of classifying group extensions is solved using group cohomology, see Appendix A.1 for details. Hence we get the following:

Corollary 2.14. *Given an abstract d-dimensional point group P, the abstract d-dimensional space groups with point group P are classified as follows: fix a d-dimensional integral representation ω of P, i.e. a group homomorphism $\omega : P \to \operatorname{Aut}(\mathbb{Z}^d)$. Then there is a one-to-one correspondence between elements of $H^2(P, \mathbb{Z}^d_\omega)$ and isomorphism classes (in the sense of extensions) of space group with point group G such that the corresponding action is ω.*

[3] This is actually the solution to the 18th problem of Hilbert. [34]

The question when two actions ω, ω' are considered equivalent has different possible answers and will not be considered here, see Hiller [34] for a nice exposition. We will also not elaborate on the extremely complicated zoo of notations for space groups and point groups, but merely mention that in this document the standard IUC notation for two-dimensional space groups and the standard international short symbol for three-dimensional space groups is used.

To make contact with the condensed matter notation, we write everything concretely in Seitz notation. Suppose we have a d-dimensional crystal with crystallographic group extension

$$1 \to \mathbb{Z}^d \to S \to P \to 1.$$

A choice of embedding $P \subseteq O(d)$ gives a splitting $R \mapsto \{R|0\} \in P$. A short computation yields that the integral representation corresponding to this group extension equals

$$\omega_R(\{1|v\}) = \{R|0\}\{1|v\}\{R|0\}^{-1} = \{1|Rv\}.$$

2.4. The Twisted Equivariant Bloch Bundle

This section sketches how twisted equivariant bundles appear naturally in the setting described in this section, motivating the development of twisted equivariant K-theory in the coming chapters. Most of the results in this section have been formalized in the paper of Freed & Moore. Since the final aim of this document is to compute the relevant K-theory groups in some basic examples, the main goal of this section is motivational. Precise definitions of relevant concepts will be given in later chapters.

Consider a d-dimensional crystal $C \subseteq \mathbb{R}^d$ in d-dimensional space with the following crystal extension.

$$\begin{array}{ccccccccc} 1 & \longrightarrow & L & \longrightarrow & S & \longrightarrow & P & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & V & \longrightarrow & \mathbb{E}(d) & \longrightarrow & O(d) & \longrightarrow & 1 \end{array}$$

The space $\mathcal{H} = L^2(\mathbb{R}^d, W)$ of d-dimensional wave functions carries a natural action of $\mathbb{E}(d)$, which restricts to an action of S. Recall that by Proposition 2.8, we have an isomorphism $\mathcal{H} \cong \Gamma_{L^2}(X, \mathcal{E} \otimes W)$, where $X = T^\vee$ is the Brillouin zone and $\mathcal{E}$ is the Bloch bundle. Following the action through the isomorphism gives an action of $\{R|v\} \in S$ on Bloch waves, but the momentum is changed. This happens though the action of P on representations of L given by conjugation using the crystal extension. In other words: there is a nontrivial action of S on the Brillouin zone and the Bloch bundle is naturally an S-equivariant bundle. Indeed, if $f \in \Gamma_{L^2}(X, \mathcal{E})$ and R maps the lattice onto itself, then using Proposition 2.8, a Euclidean transformation $\{R|v\}$ maps f to

$$\begin{aligned}(\{R|v\}\cdot f)_k(x) &= \sum_{v'\in L} e^{ik(v')}\int_T f_{k'}(\{R|v\}^{-1}\cdot(x-v'))dk' \\ &= \sum_{v'\in L} e^{ik(v')}\int_T f_{k'}(R^{-1}x-R^{-1}v'-R^{-1}v)dk' \\ &= \sum_{v'\in L} e^{ik(Rv')}\int_T f_{k'}(R^{-1}x-v'-R^{-1}v)dk' \\ &= f_{k\circ R}(R^{-1}x-R^{-1}v).\end{aligned}$$

The action of the lattice L on the Brillouin zone is trivial and therefore the Brillouin zone is also a P-space. However, the lattice does not act nontrivially on the Bloch bundle, due to the quasiperiodicity condition. Note that this implies that L acts by multiplication by phases on Bloch waves, but the phase does depend on the fiber:

$$(\{I|v\}\cdot f)_k(x) = f_k(x-v) = e^{-ik(v)}f_k(x).$$

The Bloch bundle hence becomes a twisted P-equivariant bundle; a global analogue of a twised representation of P to be defined in the next chapter. The corresponding cocycle (which happens to live in $Z^2(P, C(X, U(1)))$) is given by $\tau_k(g_1, g_2) = e^{ik(\nu(g_1,g_2))}$, where $\nu \in Z^2(P, L)$ is the cocycle corresponding to the crystal extension, $k \in X$ and $g_1, g_2 \in P$. Note that ϕ is trivial. This τ will be called the *crystal twist*.

More generally, one can consider an extended quantum symmetry group (G', ϕ, τ, c) that contains a d-dimensional lattice L on which all twisting data restricts trivially. If (ρ, H) makes wave functions $\mathcal{H}$ into a gapped extended quantum system, the Bloch bundle has a structure of an action of $G := G'/L$ twisted by the cocycle ν corresponding to the extension of G'/L by L, as well as the anomaly τ. The main idea is then the following: if G is compact, then this construction gives a correspondence between topological phases protected by (G, ϕ, τ, c) and $(\phi, \nu\cdot\tau)$-twisted G'-equivariant bundles $\mathcal{E}$ over the Brillouin zone together with a decomposition $\mathcal{E} = \mathcal{E}^+ \oplus \mathcal{E}^-$ into a conduction band and a valence band. However, this correspondence is not always perfect. We have to quotient out by certain 'trivial' phases and certain 'trivial' bundles on the other side. This gives an isomorphism between the group of reduced topological phases protected by (G', ϕ, τ, c) and a certain $(\phi, \nu\cdot\tau)$-twisted c-graded G-equivariant K-theory, to be defined in the next section. See paragraph 10 of Freed & Moore for mathematical details in this argument.

3. Twisted Equivariant K-theory

In the second chapter, it was argued that all relevant topological information of a condensed matter system protected by a quantum symmetry group (G, τ, c) is contained in the Bloch states. Since the deformations are required to preserve the symmetry, this information also includes how the Bloch states form (ϕ, τ)-twisted representations of G (Definition 1.9) and how they are split into particles living in the valence band and in the conduction band. Moreover, if the crystal is nonsymmorphic, the representations are twisted by the crystal twist. In other words, the relevant information is a combination of representation-theoretic and bundle topology data, which is expected to form a version of equivariant K-theory. In this chapter this K-theory will be defined in general and its cohomological properties will be developed on a suitable category of twisted G-spaces. Representatives of classes in this K-theory are bundles over the Brillouin zone together with the data of symmetries, quantum anomalies and nonsymmorphicity. Spaces of Bloch eigenstates for some concrete Hamiltonian form a model for such classes. The theory here will be developed for finite groups and a finite number of bands, but is expected to generalize to compact groups and Hilbert space fibers, see for example [27] and [31]. Therefore thoughout this section G will be a finite group and $\phi : G \to \mathbb{Z}_2$ a homomorphism determining whether elements of G act unitarily or anti-unitarily on the Hilbert space of quantum states. Also X will denote a compact Hausdorff G-space throughout this section. Since our final aim is to explicitly compute these K-theory groups and making the computations as combinatorial as possible, special attention is paid to the case in which X is a finite G-CW complex (see Appendix C.2 for the basic definitions of equivariant algebraic topology). Of course the most important example for X will be the Brillouin zone torus.

3.1. Abstract Twists and Twisted Spaces

Looking at the mathematical structures arising from the physics studied in the first two chapters, spaces with symmetries should come equipped with a twist under which the states form projective representations. These twists abstract the twists found on the Brillouin zone torus, which appear because of quantum anomalies and nonsymmorphic crystals. Consistency conditions will show that such a twist should be a cocycle, see Remark 3.12.

Definition 3.1. Let G be a finite group acting continuously on a compact Hausdorff space X and let $\phi : G \to \mathbb{Z}_2$ be a group homomorphism. A *ϕ-twist* on X is a unital group cocycle

$$\tau \in Z^2(G, C(X, U(1))_\phi),$$

where $C(X, U(1))_\phi$ is the G-module given by $(g \cdot f)(x) = f(g^{-1}x)^{\phi(g)}$. Here unital means that $\tau_x(g, 1) = \tau_x(1, g) = 1$ for all $g \in G$ and $x \in X$. We make ϕ-twists into a category as a special case of Definition A.3, i.e. by setting the morphisms between two twists τ_1, τ_2 to be the set of unital 1-cochains λ such that $\tau_1 = \tau_2 d\lambda$ (here unital means $\lambda_x(1) = 1$ for all $x \in X$).[1] An isomorphism class of ϕ-twists is a group cohomology class [2]

$$\tau \in H^2(G, C(X, U(1))_\phi).$$

Write ${}^\phi\operatorname{Twist}_G^X = Z^2(G, C(X, U(1))_\phi$ for the group of twists.

Example 3.2 (Anomaly Twist)*.* Consider an anomaly cocycle $\tau \in Z^2(G, U(1)_\phi)$ as in Definition 1.7. This is clearly equivalent to giving a ϕ-twist $\tau \in {}^\phi\operatorname{Twist}_G^X$ that is constant in X. Such constant cocycles will therefore also be called anomaly twists in the future. Using relations between group cohomology and extensions, this is equivalent to an extension G^τ of G by $U(1)$ such that elements $g \in G$ with $\phi(g) = -1$ anticommute with $U(1)$ and commute otherwise. See Appendix A.1 for more information on the relation between extensions and group cohomology and see Section 1.3 for the relation with quantum anomalies in quantum mechanics.

Example 3.3 (Crystal Cocycle)*.* Suppose

$$0 \to \mathbb{Z}^d \to S \to P \to 0$$

is a crystal extension, $X := \operatorname{Hom}(\mathbb{Z}^d, U(1))$ the Brillouin zone and we are given a homomorphism $\gamma : G \to P$. From the relation between extensions and group cohomology (see Appendix A.1) we know that a choice of a section of $S \to P$ gives a group cocycle $\nu \in Z^2(P, \mathbb{Z}^d_\omega)$. Here $\mathbb{Z}^d_\omega$ is the P-module coming from the conjugation action of P on $\mathbb{Z}^d$ via the short exact sequence. Via γ we get an action of G on $\mathbb{Z}^d$, which is denoted $\alpha : G \to \operatorname{Aut}\mathbb{Z}^d$. The pull-back under γ gives a crystal cocycle $\gamma^*\nu \in H^2(G, \mathbb{Z}^d_\alpha)$. Now α gives an action of G on X by precomposition, so that X becomes a (compact) G-space. Finally we get a ϕ-twist on X by

$$\tau_\lambda(g_1, g_2) := \lambda(\gamma^*\nu(g_1, g_2)) = \lambda(\nu(\gamma(g_1), \gamma(g_2))).$$

Note that in positive dimension, τ can only be an anomaly if $\gamma^*\nu$ is trivial.

For constructing an equivariant K-theory for which the twists depend on the space, the category on which the K-theory is a cohomology theory should be some category of twisted G-spaces. As a warm-up the reader can consult Appendix C.2 for a review of the well-known theory of cohomology theories on the category of (nontwisted) G-spaces and their use in relation to ordinary complex equivariant K-theory of Segal [57].

Definition 3.4. Let $\mathbf{Top}_G^t$ denote the category in which

- objects are *twisted G-spaces*, which are pairs (X, τ), where X is a (compact) G-space and $\tau \in {}^\phi\operatorname{Twist}_G^X$ is a ϕ-twist on X;

[1] Note that we consider unital 1-cocycles λ because $d\lambda$ is unital if and only if λ is (and twists are required to be unital).

[2] Every cohomology class contains a unital representative, see Lemma A.16.

- morphisms $(X_1,\tau_1) \to (X_2,\tau_2)$ are pairs (f,λ), where f is a continuous G-map $f: X_1 \to X_2$ and $\lambda \in C^1(G, C(X, U(1)_\phi))$ is a 1-cochain such that $f^*(\tau_2) = \tau_1 \cdot d\lambda$. Composition of (f_1,λ_1) and (f_2,λ_2) is defined by $(f_2 \circ f_1, \lambda_1 \cdot f_1^*\lambda_2)$.

Remark 3.5. The pullback map $f^* : {}^\phi\,\mathrm{Twist}_G^{X_2} \to {}^\phi\,\mathrm{Twist}_G^{X_2}$ is defined by

$$f^*\tau_x(g_1,g_2) := \tau_{f(x)}(g_1,g_2).$$

It is easy to check - using the fact that f is a G-map - that it is well-defined in the sense that $f^*\tau$ is again a cocycle. Note that the composition is well-defined, because

$$f_1^*\tau_2 = \tau_1 d\lambda_1, \quad f_2^*\tau_3 = \tau_2 d\lambda_2 \implies (f_2f_1)^*(\tau_3) = f_1^*(\tau_2 d\lambda_2) = \tau_1 d\lambda_1 f_1^* d\lambda_2 = d(\lambda_1 f_1^*\lambda_2).$$

Because the pull-back operation is functorial, $\mathbf{Top}_G^t$ is indeed a category.

Example 3.6. Let $f: X_1 \to X_2$ a map of G-spaces and $\tau \in \mathrm{Twist}_G^{X_2}$ a twist on X_2. Then $f: (X_1, f^*\tau) \to (X_2,\tau)$ is a map of G-spaces with trivial 1-cochain λ. An important subexample: if $A \subseteq X$ is a G-subspace then $(A, \tau|_A)$ is a twisted G-space and the inclusion map is a morphism of twisted G-spaces.

Example 3.7. If $\tau \in Z^2(G, U(1)_\phi)$ is an anomaly, then the projection mapping $\pi: (X,\tau) \to (\mathrm{pt},\tau)$ is a map of twisted G-spaces with trivial 1-cochain. Note that if τ were not isomorphic to a twist constant in X, then there is no map of twisted G-spaces of the form $\pi: (X,\tau) \to (\mathrm{pt},\nu)$.

Example 3.8. Let X be a G-space and $\tau_1,\tau_2 \in \mathrm{Twist}_G^X$. Then the identity map $id: (X,\tau_1) \to (X,\tau_2)$ can be made into a morphism of twisted G-spaces if and only if τ_1 and τ_2 are isomorphic.

Example 3.9. Let (X,τ) be a twisted G-space. Given $x \in X$, a restricted class on the point $\tau_x \in Z^2(G, U(1)_\phi)$ exists if and only if x is a fixed point of the action. In that case, the inclusion $i: (\mathrm{pt},\tau_x) \to (X,\tau)$ is a map of twisted spaces, since i is equivariant. More generally, let $G_x \subseteq G$ be the stabilizer group at x. Then we get a map $i: (\mathrm{pt}, (\tau|_{G_x})_x) \to (X, \tau|_{G_x})$, where $(X,\tau|_{G_x})$ is the twisted G_x-space obtained by restriction. Therefore a twist in particular gives a ϕ-twisted extension of G_x at every point x of X.

3.2. Comparison of Different Definitions of Twists

There are various definitions of twists in the literature which are related in some way or another. In this paragraph, these definitions and their relation will be discussed. To improve readability, not all terms will be defined, but instead the reader will be referred to the literature.

The most common twists can be summarized as follows. Note that most references include the grading morphism c in the twist data, but we decided to exclude it in the comparison.

- The classical (nonequivariant) twists as in Atiyah and Segal's work [60], defined using projective bundles over the space, are classified by degree three ordinary cohomology $H^3(X,\mathbb{Z})$;

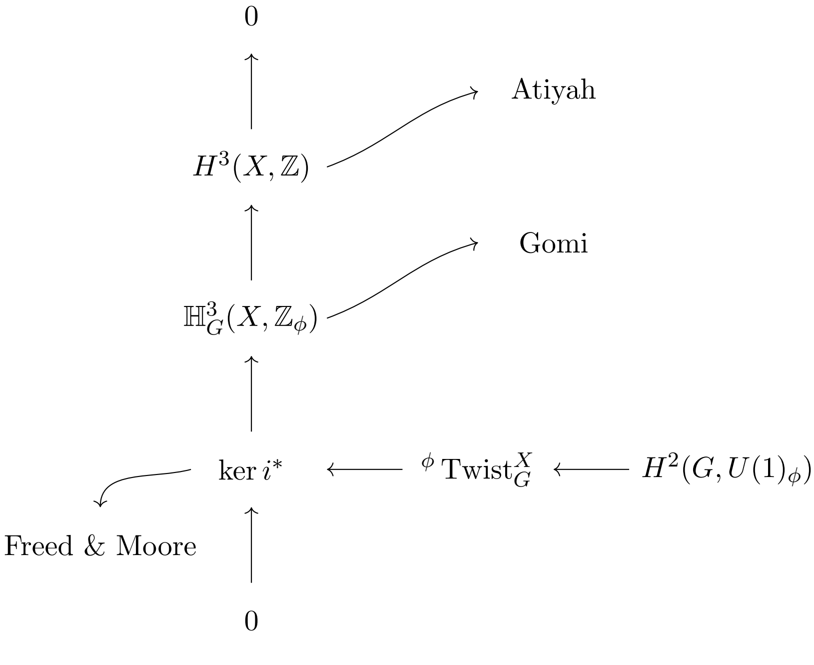
- The quantum anomaly twists in $H^2(G, U(1)_\phi)$ - often called the discrete torsion - i.e. the twists constant in X;

- The twists of Freed & Moore are the the ϕ-twisted central extensions of the groupoid $X/\!\!/G$. They can also be formulated as being the subset of the twists of Gomi [31] that restrict to the trivial Atiyah-twist;

- The most general twists, defined by Gomi [31], consist of an equivalence of the groupoid $X/\!\!/G$ with a local quotient groupoid $\mathcal{X}$ and a ϕ-twisted central extension of $\mathcal{X}$. The twists form a category for which isomorphism classes of objects are classified by the third degree Borel equivariant cohomology with twisted coefficients $\mathbb{H}^3_G(X, \mathbb{Z}_\phi) := H^3(X \times_G EG, \mathbb{Z}_\phi)$.[3] In the complex case (ϕ trivial), this theory has been developed in detail for proper Lie groupoids in Tu et al. [66];

- Finally the twists ${}^\phi\,\mathrm{Twist}^X_G$ considered in this document, classified by $H^2(G, C(X, U(1))_\phi)$. These correspond exactly to the twists of Freed & Moore that are trivial as line bundles (the cocycle in $Z^2(G, C(X, U(1))_\phi$ can be constructed in a similar way to Remark 7.12 of Freed & Moore while taking proper care of the dependence on X, also see the remark under Definition 2.2 of Gomi [31]).

Hence, the definition of ϕ-twists as given in this document fits in this picture as a strict subset of the Freed & Moore twists and has empty intersection with the twists of Atiyah and Segal [60]. An overview of the next proposition is given in the picture below.



[3]Here $X \times_G EG$ is $X \times EG$ quotiented by the diagonal action of G. The local system $\mathbb{Z}_\phi$ on $X \times_G EG$ is the pull-back of the local system on BG coming from the map $\pi_1(BG) \cong G \xrightarrow{\phi} \mathbb{Z}_2$. Borel equivariant cohomology should neither be confused with Borel measurable cohomology, nor with Bredon equivariant cohomology H_G, developed in Appendix A.1 and C.3 respecitvely.

Proposition 3.10. *Suppose X is path-connected and has at least one fixed point. The fibration $X \xrightarrow{i} EG \times_G X \to BG$ induces maps on cohomology*

$$H^2(G, U(1)_\phi) \cong H^3(BG, \mathbb{Z}_\phi) \to H^3(X \times_G EG, \mathbb{Z}_\phi) = \mathbb{H}^3_G(X, \mathbb{Z}_\phi) \to H^3(X, \mathbb{Z}).$$

The associated Serre spectral sequence gives a filtration on $\mathbb{H}^3_G(X, \mathbb{Z}_\phi)$ denoted

$$\mathbb{H}^3_G(X, \mathbb{Z}_\phi) \supseteq F^1\mathbb{H}^3_G(X, \mathbb{Z}_\phi) \supseteq F^2\mathbb{H}^3_G(X, \mathbb{Z}_\phi) \supseteq F^3\mathbb{H}^3_G(X, \mathbb{Z}_\phi) \supseteq 0.$$

Then

- $F^1\mathbb{H}^3_G(X, \mathbb{Z}_\phi)$ *consists of isomorphism classes of the Freed & Moore twists, i.e. ϕ-twisted central extensions of $X/\!\!/G$;*
- $F^2\mathbb{H}^3_G(X, \mathbb{Z}_\phi)$ *are the isomorphism classes of ϕ-twists as defined in this document;*
- $F^3\mathbb{H}^3_G(X, \mathbb{Z}_\phi)$ *is the part constant in X, hence isomorphic to $H^2(G, U(1)_\phi)$, i.e. it consists of the quantum anomaly ϕ-twists.*

Proof. See Gomi's work on twists on tori [30], Proposition 5.1. □

3.3. Twisted Equivariant Vector Bundles

In order to define the right notion of twisted equivariant K-theory of bundles, i.e. so that they correspond with Bloch bundles over the Brillouin zone, we develop a theory of twisted equivariant bundles over twisted G-spaces, which are the natural global analogues of (ϕ, τ)-twisted representations. For the rest of this section, let (X, τ) be a twisted G-space. In order to keep the amount of data manageable, we start by developing twisted equivariant K-theory of degree zero without chiral symmetries, i.e. with trivial grading homomorphism $c : G \to \mathbb{Z}_2$.

Definition 3.11. A *(ϕ, τ)-twisted equivariant vector bundle* over (X, τ), written (E, ρ) is a complex vector bundle $E \to X$ equipped with a family of real bundle maps $\rho(g) : E \to E$ for all $g \in G$ such that:

1. ρ covers the action of G on X, so $\rho(g)(E_x) \subseteq E_{gx}$ for all $g \in G$ and $x \in X$;
2. ϕ determines whether $\rho(g)$ is complex linear or complex anti-linear, i.e. $i\rho(g) = \phi(g)\rho(g)i$ for all $g \in G$;
3. ρ is a projective representation with cocycle τ at every point;

   $$\rho(1) = 1 \quad \text{and} \quad \rho(g)\rho(h) = \tau(g, h)\rho(gh).$$

   More precisely, for every $x \in X$ and $g, h \in G$,

   $$\rho_x(1) = id_{E_x} \quad \text{and} \quad \rho(g)_{hx}\rho(h)_x = \tau(g, h)_{ghx}\rho(gh)_x$$

   as linear maps $E_x \to E_{ghx}$.

Remark 3.12. In order for (ϕ, τ)-twisted equivariant bundles to exist, i.e. for them to be compatible with the associativity of the group, τ has to be a cocycle with the group action as in Definition 3.1. To see this, let $g_1, g_2, g_3 \in G$ and $x \in X$. Then on the one hand

$$\begin{aligned}\rho(g_1)_{g_2g_3x}\rho(g_2)_{g_3x}\rho(g_3)_x &= \rho(g_1)_{g_2g_3x}\tau(g_2,g_3)_{g_2g_3x}\rho(g_2g_3)_x \\ &= \tau(g_2,g_3)^{\phi(g_1)}_{g_2g_3x}\rho(g_1)_{g_2g_3x}\rho(g_2g_3)_x \\ &= \tau(g_2,g_3)^{\phi(g_1)}_{g_2g_3x}\tau(g_1,g_2g_3)_{g_1g_2g_3x}\rho(g_1g_2g_3)_x,\end{aligned}$$

while on the other hand

$$\begin{aligned}\rho(g_1)_{g_2g_3x}\rho(g_2)_{g_3x}\rho(g_3)_x &= \tau_{g_1g_2g_3x}(g_1,g_2)\rho(g_1g_2)_{g_3x}\rho(g_3)_x \\ &= \tau_{g_1g_2g_3x}(g_1,g_2)\tau_{g_1g_2g_3x}(g_1g_2,g_2)\rho(g_1g_2g_3)_x.\end{aligned}$$

By picking $y = g_3^{-1}g_2^{-1}g_1^{-1}x$, we see that the necessary and sufficient condition for τ to be consistent with the associativity of G is exactly

$$\tau_{g_1^{-1}y}(g_2,g_3)^{\phi(g_1)}\tau_y(g_1g_2,g_3)^{-1}\tau_y(g_1,g_2g_3)\tau_y(g_1,g_2)^{-1} = 1$$

for all $g_1, g_2, g_3 \in G$ and $y \in X$. In other words, we see that τ is a group cocycle with the desired G-module structure on $C(X, U(1))$.

Remark 3.13. Note that twisted equivariant vector bundles would not exist for nonunital twists. Indeed, for a bundle (E, ρ) to exist we necessarily have

$$\rho(x) = \rho(1 \cdot x) = \tau(1,x)\rho(1)\rho(x) = \tau(1,x)\rho(x).$$

This motivates the choice of defining all twists to be unital.

Example 3.14. If X is a point, then a (ϕ, τ)-twisted equivariant bundle is a (ϕ, τ)-twisted representation of G in the sense of Definition 1.9 into a finite dimensional vector space.

Example 3.15. If ϕ and τ are trivial, then we recover the notion of a G-vector bundle as defined by Segal [57] (not to be confused with (e.g. principal) G-bundles which are fiber bundles with a fiberwise G-action).

Example 3.16. Take $G = \mathbb{Z}_2$, let ϕ be the isomorphism and let $\tau \in Z^2(G, U(1)_\phi)$ be an anomaly. If τ is the trivial cocycle, then a (ϕ, τ)-twisted equivariant bundle is a complex vector bundle with an anti-unitary involution j covering the action of the nontrivial element of G on X. This gives a so-called Real bundle over the Real space X as in Atiyah's KR-theory [3]. Note that if the action of G on X is trivial, then G gives a real structure on the vector bundle. Hence in that case a (ϕ, τ)-twisted equivariant bundle is the same as a real vector bundle.

Example 3.17. Now we consider the above example, but with an anomaly τ which is not a coboundary. In Section A.3 it will be shown that $H^2(\mathbb{Z}_2, U(1)_\phi) \cong \mathbb{Z}_2$ and the nontrivial class can be represented by

$$\tau((-1)^{m_1}, (-1)^{m_2}) = (-1)^{m_1 m_2}.$$

Therefore a (ϕ, τ)-twisted equivariant bundle is a complex bundle with an anti-unitary map covering the action of the nontrivial element of $\mathbb{Z}_2$ on X squaring to -1. In analogy with the real case, this could be called a quaternionic bundle; if the action is trivial, then $\mathbb{Z}_2$ makes the fibers of the bundle into quaternionic vector spaces.

Definition 3.18. Let $f : (X_1, f^*\tau) \to (X_2, \tau)$ be a map of twisted G-spaces with trivial 1-cochain. Suppose (E_1, ρ_1) and (E_2, ρ_2) are twisted equivariant vector bundles over $(X_1, f^*\tau)$ and (X_2, τ) respectively. A *morphism* $\psi : E_1 \to E_2$ *of twisted equivariant bundles covering* f is a complex vector bundle map covering f that intertwines the action of G.

Remark 3.19. It is possible to define morphisms of twisted equivariant bundles for arbitrary morphisms f, but we will restrict to this case for now. Morphisms of vector bundles coming from a morphism of twisted spaces that is the identity on the space (so just multiplication of τ by some $d\lambda$) will appear for graded bundles in the next section.

Example 3.20. A map $(E_1, \rho_1) \to (E_2, \rho_2)$ of (nontwisted) $\mathbb{Z}_2$-equivariant bundles with ϕ nontrivial is a complex vector bundle map intertwining the Real structures of E_1 and E_2. in case the action of $\mathbb{Z}_2$ on X is trivial, this is equivalent to a real-linear map between the underlying real vector bundles.

Now that the main ingredients are introduced, the first version of K-theory can be introduced as a functor. Note that the direct sum of two (ϕ, τ)-twisted equivariant bundles is naturally again a (ϕ, τ)-twisted equivariant bundle.

Definition 3.21. Isomorphism classes of (ϕ, τ)-twisted G-equivariant bundles form a commutative monoid under direct sum. The Grothendieck completion of this monoid, written

$$^{\phi}K_G^{\tau}(X)$$

is called the (ϕ, τ)*-twisted equivariant K-theory group of* X.

Remark 3.22. Note that unlike ordinary equivariant K-theory, $^{\phi}K_G^{\tau}(X)$ is not naturally a ring; the tensor product of a τ_1-twisted bundle and a τ_2-twisted bundle is naturally a $\tau_1 \cdot \tau_2$-twisted bundle. It therefore is a module over $^{\phi}K_G(X)$, the twisted K-theory with $\tau = 0$.[4] However, since we do not see a natural physical interpretation of this algebraic structure, we only develop the K-theory as an abelian group. [5]

Example 3.23. Suppose τ and ϕ are trivial. Then (ϕ, τ)-twisted equivariant K-theory is just the usual complex equivariant K-theory developed in Segal's work [57].

Example 3.24. Take $G = \mathbb{Z}_2$, let ϕ be the isomorphism and let $\tau \in Z^2(G, U(1)_\phi)$ an anomaly. Recall that if τ is the trivial cocycle, then (ϕ, τ)-twisted equivariant bundles are exactly Real bundles over the Real space X. The (ϕ, τ)-twisted equivariant K-theory

[4] Alternatively, we could consider the direct sum of the twisted equivariant K-theories over all isomorphism classes of twists and consider that as a ring.

[5] In our point of view, direct sums of bundles correspond to stacking independent physical systems under the isomorphism of Fock spaces $\bigwedge(E_1 \oplus E_1) \cong \bigwedge E_1 \otimes \bigwedge E_2$. Therefore it can be argued that this abelian group structure is physically relevant.

${}^{\phi}K_G^{\tau}(X)$ is exactly Atiyah's KR-theory $KR^0(X)$. In particular, if the action on X is trivial, the involution is a real structure on the bundle. Therefore we get the K-theory of real vector bundles $KO(X)$.

Now we consider a constant cocycle τ which is not a coboundary, which gives us quaterionic bundles. It is a well-known fact that this is quaternionic K-theory $KQ(X) = KR^4(X)$ (see for example [21]). Note that if the action on X is trivial, the anti-unitary map is a quaternionic structure on the bundle.

3.4. Clifford Algebras and Graded Bundles

We now introduce gradings of our vector bundles, allowing K-theory groups to be defined in more generality than in the last section. The first motivation to do this, is the natural grading of the Bloch bundle which appears as the splitting into the valence and conduction band. Namely, whenever particle-hole symmetries are present, the valence band and conduction band can no longer be split as separate topological entities, but have to be considered simultaneously as a graded twisted equivariant bundle. The second motivation has a mathematical origin; it is the natural extension of K-theory to a cohomology theory by including Clifford actions. The cohomological grading of K-theory has two points of view: the more general topological point of view (for example taken by Atiyah and Segal in [57]), which introduces the grading via suspensions, and the more algebraic point of view using Clifford algebras (classically used by Atiyah, Bott and Shapiro in [4] and advocated by Karoubi, for example in [41]). These points of view are two sides of the same coin, but in general it takes some work to connect them. For example, Donovan and Karoubi state such a connection in a version of twisted K-theory in which twists are bundles of real superalgebras (Theorem 15 in [19]). Because for us the local algebraic theory and Bott periodicity is of higher interest than suspensions, we adopt the algebraic point of view.

We now also implement chiral symmetries using a homomorphism $c : G \to \mathbb{Z}_2$. We have seen that the Bloch bundle comes equipped with symmetries that flip the spectrum whenever they satisfy $c(g) = -1$. In order to incorporate these types of symmetries in the general setting, the extra grading on the twisted equivariant bundles that should be introduced is the following:

Definition 3.25. Let (E, ρ) be a twisted equivariant bundle over (X, τ). A *c-grading* of (E, ρ) is a decomposition of vector bundles $E = E_0 \oplus E_1$ such that $\rho(g)$ is even if $c(g) = 1$ and odd if $c(g) = -1$. For $p, q \geq 0$ integers, a (p, q)-Clifford action on E is a complex linear continuous fiberwise action of the real superalgebra $Cl_{p,q}$ compatible with the grading. In other words, it consists of a family $\gamma_1, \dots, \gamma_{p+q} : E \to E$ of complex vector bundle maps of odd degree covering the identity map on X such that

- $\gamma_j \gamma_k = -\gamma_k \gamma_j$ if $j \neq k$;
- $$\gamma_j^2 = \begin{cases} -1 & \text{if } j = 1, \dots p, \\ 1 & \text{if } j = p+1, \dots, p+q; \end{cases}$$

- $\gamma_j \rho(g) = c(g)\rho(g)\gamma_j$.

A morphism $f : E \to E'$ of c-graded (ϕ, τ)-twisted equivariant bundles of degree $k \in \mathbb{Z}_2$ is a map of graded complex vector bundles of degree k such that

$$f \circ \rho(g) = c(g)^k \rho'(g) \circ f$$

for all $g \in G$. We also call a map of degree zero even and a map of degree one odd. Such a morphism is said to intertwine the Clifford action if

$$f \circ \gamma_j = (-1)^k \gamma'_j \circ f \quad \forall j = 1, \ldots, p+q.$$

To ease the notation slightly, we will call a c-graded (ϕ, τ)-twisted G-equivariant vector bundle with (p, q)-Clifford action a (G, ϕ, τ, c, p, q)-bundle and a morphism of c-graded (ϕ, τ)-twisted G-equivariant vector bundles that intertwines the Clifford action a morphism of (G, ϕ, τ, c, p, q)-bundles. For the convenience of the reader, we summarize the definition of a (G, ϕ, τ, c, p, q)-bundle (E, ρ) over a twisted space (X, τ), which consists of:

- a complex vector bundle E;
- a family $\{\rho(g)\}_{g \in G}$ of (continuous) morphisms of real vector bundles $E \to E$;
- a direct sum decomposition $E = E_0 \oplus E_1$;
- a continuous, fiber-preserving complex linear action of the Clifford algebra $Cl_{p,q}$ on E;

such that the following conditions hold:

- $\rho(1) = id_E$;
- $\rho(g)_{hx}\rho(h)_x = \tau(g, h)_{ghx}\rho(gh)_x$;
- $\rho(g)$ is complex linear if $\phi(g) = 1$ and complex anti-linear if $\phi(g) = -1$;
- $\rho(g)$ is an even vector bundle map with respect to the direct sum decomposition if $c(g) = 1$ and an odd map if $c(g) = -1$;
- under the natural grading of $Cl_{p,q}$, odd elements $\gamma \in Cl_{p,q}$ act by odd maps and even elements by even maps;
- elements of $Cl_{p,q}$ and G graded-commute in the sense that $\gamma\rho(g) = c(g)\rho(g)\gamma$ if $\gamma \in Cl_{p,q}$ is odd and $\gamma\rho(g) = \rho(g)\gamma$ if $\gamma \in Cl_{p,q}$ is even.

Remark 3.26. Readers familiar with the twisted K-theory of Donavan & Karoubi as in [19], may be tempted to see the bundles in this document as modules over the trivial superalgebra bundle with fiber ${}^{\phi}\mathbb{C}^{\tau, c+(p,q)}G$. However, if the base X has nontrivial G-action, there is no notion of a module structure over the twisted group algebra. [6] Hence on first sight the theory developed by Donovan & Karoubi can only be used if the equivariant structure on the level of spaces is trivial.

[6] What would be the meaning of an expression such as $(x_{g_1} + x_{g_2})v$ if $v \in E_x$ and $g_1 x \neq g_2 x$?

Example 3.27. A c-graded equivariant vector bundle over a twisted point (pt, τ) is a (ϕ, τ)-twisted c-graded representation of G in the sense of Definition 1.17. More generally, a (G, ϕ, τ, c, p, q)-bundle over such a point is a supermodule over the twisted group algebra ${}^{\phi}\mathbb{C}^{\tau,c+(p,q)}G = {}^{\phi}\mathbb{C}^{\tau,c}G\hat{\otimes}_{\mathbb{R}}Cl_{p,q}$. An even morphism of (G, ϕ, τ, c, p, q)-bundles $M_1 \to M_2$ is a map of supermodules, while an odd morphism is a map $M_1 \to \widehat{M_2}$, where $\widehat{M_2}$ is the module M_2 with the opposite grading.

More generally, if (X, τ) is a twisted G-space with anomaly twist τ and M is a supermodule over the twisted group algebra ${}^{\phi}\mathbb{C}^{\tau,c+(p,q)}G$, then we can form the trivial (G, ϕ, τ, c, p, q)-bundle $\theta_M(X) = \theta_M$ with fiber M as follows. As a vector bundle, define $\theta_M := X \times M$. The action is then just given by $g \cdot (x, m) = (gx, x_g m)$ and $\gamma \cdot (x, m) = (x, \gamma m)$, where x_g is the element of the twisted group algebra corresponding to $g \in G$ and γ is a Clifford algebra element.

Definition 3.28. Let E be a (G, ϕ, τ, c, p, q)-bundle over a twisted G-space (X, τ), where τ is an anomaly. Let M be a supermodule over ${}^{\phi}\mathbb{C}^{\tau,c+(p,q)}G$. A *local trivialization of E with fiber M* is an open G-subspace $U \subseteq X$ together with an even isomorphism $E|_U \cong M \times U = \theta_M(U)$ of (G, ϕ, τ, c, p, q)-bundles. E is called *trivializable* if it is isomorphic to a bundle of the form θ_M.[7]

Example 3.29. If c is trivial, a c-graded (ϕ, τ)-twisted equivariant bundle is just a (ϕ, τ)-twisted equivariant bundle with a $\mathbb{Z}_2$-grading such that all maps $\rho(g)$ are of degree zero. Equivalently, it can be seen as a pair (E_1, E_2) of two (ϕ, τ)-twisted equivariant bundles. From that point of view, a morphism of c-graded (ϕ, τ)-twisted equivariant bundles $(E_1, E_2) \to (E_1', E_2')$ of even degree is just a pair of morphisms of (ϕ, τ)-twisted equivariant bundles $E_1 \to E_1'$ and $E_2 \to E_2'$. An odd morphism instead consists of maps $E_1 \to E_2'$ and $E_2 \to E_1'$.

As is motivated in Section 1.7 for the zero-dimensional case, we quotient out topological phases that admit an odd automorphism squaring to 1 in order to ensure that we can fully reproduce the Atland-Zirnbauer classification, which makes the role of Clifford algebras clearer as well. Recall that in this way we depart from Freed & Moore in the sense that they do not demand the odd automorphism to square to 1. Moreover, in an attempt to make this twisted K-theory into a full-blown cohomology theory, we make use of Clifford algebra actions. The definition is motivated by the exposition in [4], in the most suggestive way. So finally the most general version of twisted equivariant K-theory that will be studied in this document can be defined.

Definition 3.30. Let ${}^{\phi}\,\mathrm{Vect}_G^{\tau,c+(p,q)}(X)$ denote isomorphism classes of (G, ϕ, τ, c, p, q)-bundles (under morphisms of degree zero). They form a monoid under direct sum. Let ${}^{\phi}\,\mathrm{Triv}_G^{\tau,c}(X) \subseteq {}^{\phi}\,\mathrm{Vect}_G^{\tau,c+(p,q)}(X)$ denote the submonoid of supertrivial bundles; the bundles admitting an odd automorphism squaring to one. The *(ϕ, τ)-twisted $c + (p, q)$-graded*

[7] Note that we take trivializations to have globally constant fiber. For example, a bundle over a discrete space X with trivial action is trivializable if and only if the even isomorphism class of its fiber as a supermodule is locally constant in X.

G-equivariant K-theory group of X now equals

$$ {}^{\phi}K_G^{\tau,c+(p,q)}(X) := \frac{{}^{\phi}\operatorname{Vect}_G^{\tau,c+(p,q)}(X)}{{}^{\phi}\operatorname{Triv}_G^{\tau,c+(p,q)}(X)}. $$

Remark 3.31. We must immediately disappoint the reader and share the strong suspicion that for p, q nonzero, the seemingly reasonable definition of K-theory as above does not seem to be correct. For an argument for this suspicion and suggestions to resolve it, consider Section 3.8.

Remark 3.32. Suppose C is a crystal with lattice group L, space group S and magnetic point group P. Let $c : P \to \mathbb{Z}_2$ be a homomorphism and let $\phi := t \cdot c$, where $t : P \to \mathbb{Z}_2$ is the time-reversal homomorphism. Write X for the Brillouin zone. Let τ_1 be the crystal twist corresponding to the crystal extension, let τ_2 be an anomaly twist on P and take τ to be the product. Analogously to Section 10 of Freed & Moore, we suspect the (ϕ, τ)-twisted $c + (0, 0)$-graded P-equivariant K-theory of X to be isomorphic to the abelian group of reduced topological phases with extended quantum symmetry group (S, ϕ, τ_2, c), where ϕ, τ_2 and c are extended to S by pull-back (see Definition 1.16).

Remark 3.33. K-theory indeed forms an abelian group; the inverse of a (G, ϕ, τ, c, p, q)-bundle (E, ρ) will be the bundle $(\hat{E}, \hat{\rho})$ in which the grading is reversed; $(\hat{E})_0 = E_1$ and $(\hat{E})_1 = E_0$ with the obvious action. It is easy to check that this will again be a (G, ϕ, τ, c, p, q)-bundle and $E \oplus \hat{E}$ has an odd isomorphism squaring to one induced by the identity map on E. It now follows that K-theory forms a group by basic monoid theory, see Lemma B.18 in Appendix B.3.

By the following lemma the definition above generalizes the earlier K-theory group:

Lemma 3.34. *Suppose c is the trivial homomorphism and $p = q = 0$. Then the (ϕ, τ)-twisted c-graded equivariant K-theory is isomorphic to the (nongraded) τ-equivariant K-theory. The map ${}^{\phi}K_G^{\tau,c+(0,0)}(X) \to {}^{\phi}K_G^{\tau}(X)$ is given by*

$$ [E_0 \oplus E_1] \mapsto [E_0] - [E_1]. $$

Proof. Let $E_0 \oplus E_1$ be a (ϕ, τ)-twisted c-graded equivariant bundle. Since $\rho(g)$ must always be even, $(E_0, \rho|_{E_0})$ and $(E_1, \rho|_{E_1})$ are (ϕ, τ)-twisted G-equivariant bundles. An odd isomorphism of $E_0 \oplus E_1$ gives an isomorphism $E_0 \cong E_1$, so that $[E_0] = [E_1]$. Therefore the map is well-defined by Lemma B.18. It is clearly a map of monoids.

Given E_0, E_1, two (ϕ, τ)-twisted equivariant bundles, we can always construct a $\mathbb{Z}_2$-graded (ϕ, τ)-twisted equivariant bundle by taking the direct sum $E_0 \oplus E_1$, proving that the map is surjective. Now suppose that $[E_0] - [E_1] = 0$. Then there exists a (ϕ, τ)-twisted equivariant bundle F such that $E_0 \oplus F \cong E_1 \oplus F$. But then if $\bar{F} := F \oplus F$ is the $\mathbb{Z}_2$-graded bundle with F as both components, we have

$$ [E_0 \oplus E_1] + [\bar{F}] = [(E_0 \oplus F) \oplus (E_1 \oplus F)] = 0, $$

since $(E_0 \oplus F) \oplus (E_1 \oplus F)$ admits an odd isomorphism squaring to one by assumption. As $\bar{F}$ admits an odd automorphism squaring to one, we therefore also have $[E_0 \oplus E_1] = 0$. □

Example 3.35. The (ϕ, τ)-twisted $c + (p,q)$-graded equivariant K-theory group of a point forms the (ϕ, τ)-twisted $c+(p,q)$-graded representation ring ${}^{\phi}R^{\tau,c+(p,q)}(G)$ as in Section 1.7. Note that unlike the ungraded version, this representation ring does not agree with definition 7.1(iii) of Freed & Moore, since our odd automorphism is demanded to square to 1. For an explicit example in which our definitions differ and, see Section 1.7 on the algebraic theory of these types of representation rings. In Section 3.9 the connection with the local structure of twisted equivariant K-theory will be developed with care.

Example 3.36. Suppose $G = \mathbb{Z}_2$, ϕ is trivial, τ is trivial and c is nontrivial. Then we get the K-theory of Atiyah and Hopkins developed in [5].

Lemma 3.37. *Let $\tau \in {}^{\phi}\operatorname{Twist}_G^X$ be a ϕ-twist and let $c : G \to \mathbb{Z}_2$ be a grading homomorphism. Let $E = E_0 \oplus E_1$ be a (G, ϕ, τ, c, p, q)-bundle. Then there exists a Hermitian metric on E making*

- *$\rho(g)$ unitary for all $g \in G$ with $\phi(g) = 1$;*
- *$\rho(g)$ anti-unitary for all $g \in G$ with $\phi(g) = -1$;*
- *the decomposition $E = E_0 \oplus E_1$ orthogonal.*

Proof. Since X is in particular paracompact, we can construct a Hermitian metric on E_0 and E_1. Let $\langle ., . \rangle'$ denote the Hermitian metric on $E_0 \oplus E_1$ such that E_0 and E_1 are orthogonal. Now for $x \in X$ and $v_1, v_2 \in E_x$, define

$$\langle v_1, v_2 \rangle_x := \sum_{g \in G} {}^{\phi(g)}\overline{\langle \rho(g)_x v_1, \rho(g)_x v_2 \rangle'_{gx}}.$$

Note that the summand is continuous in x. Linearity follows since

$$\langle z v_1, v_2 \rangle_x = \sum_{g \in G} {}^{\phi(g)}\overline{\langle {}^{\phi(g)}\bar{z} \rho(g)_x v_1, \rho(g)_x v_2 \rangle'_{gx}} = z \langle v_1, v_2 \rangle_x.$$

By the following computation, $\rho(g)$ becomes unitary if $\phi(g) = 1$ and anti-unitary if $\phi(g) = -1$:

$$\begin{aligned}
\langle \rho(g)_x v_1, \rho(g)_x v_2 \rangle_{gx} &= \sum_{g' \in G} {}^{\phi(g')}\overline{\langle \rho(g')_{gx} \rho(g)_x v_1, \rho(g')_{gx} \rho(g)_x v_2 \rangle'_{g'gx}} \\
&= \sum_{g' \in G} {}^{\phi(g')}\overline{\langle \tau(g', g)_{g'gx} \rho(g'g)_x v_1, \tau(g', g)_{g'gx} \rho(g'g)_x v_2 \rangle'_{g'gx}} \\
&= \sum_{g' \in G} {}^{\phi(g')}\overline{\langle \rho(g'g)_x v_1, \rho(g'g)_x v_2 \rangle'_{g'gx}} \\
&= \sum_{g' \in G} {}^{\phi(g'g^{-1})}\overline{\langle \rho(g')_x v_1, \rho(g')_x v_2 \rangle'_{g'x}} \\
&= {}^{\phi(g)}\overline{\sum_{g' \in G} {}^{\phi(g')}\overline{\langle \rho(g')_x v_1, \rho(g')_x v_2 \rangle'_{g'x}}} \\
&= {}^{\phi(g)}\overline{\langle v_1, v_2 \rangle_x}.
\end{aligned}$$

The other properties follow easily. ∎

3.5. Functorial Properties of Twisted Equivariant K-theory

In order to define K-theory as a functor on the category of twisted G-spaces, we first develop the theory of pull-back bundles, which is somewhat subtle due to the twists.

Lemma 3.38. *Let $f : (X_1, f^*\tau) \to (X_2, \tau)$ be a map of twisted G-spaces with trivial 1-cochain. Let (E, ρ) be a (ϕ, τ)-twisted c-graded equivariant vector bundle over X_2. Then the pull-back vector bundle f^*E has a unique structure of a c-graded $f^*\tau$-twisted equivariant vector bundle $(f^*E, f^*\rho)$ over $(X_1, f^*\tau)$ such that the map of complex vector bundles $\bar{f} : f^*E \to E$ becomes a morphism of twisted equivariant bundles.*

Proof. Let $g \in G$. Then $(f^*\rho)(g)$ is uniquely determined by demanding that the following diagram commutes:

$$\begin{array}{ccc} E & \xleftarrow[\bar{f}]{} & f^*E \\ \big\downarrow{\scriptstyle \rho(g)} & & \big\downarrow{\scriptstyle (f^*\rho)(g)} \\ E & \xleftarrow[\bar{f}]{} & f^*E. \end{array}$$

More explicitly, if $x \in X_1$ and $v \in (f^*E)_x = E_{f(x)}$, then $(f^*\rho)_x(g)v = \rho_{f(x)}(g)v$. It is easy to show using a trivializing neighbourhood that $f^*\rho$ is continuous. Note that $f^*\rho$ covers the action of G:

$$(f^*\rho)((f^*E)_x) = \rho_{f(x)}(g)(E_{f(x)}) \subseteq E_{gf(x)} = E_{f(gx)} = f^*(E_{gx})$$

by equivariance of f. Obviously, $f^*\rho(g)$ is complex linear if $\phi(g) = 1$ and complex anti-linear if $\phi(g) = -1$. Now let $g_1, g_2 \in G$. Then a short computation verifies the projective representation property:

$$\begin{aligned} (f^*\rho)_{g_2x}(g_1)(f^*\rho)_x(g_2) &= \rho(g_1)_{f(g_2x)}\rho(g_2)_{f(x)} = \rho(g_1)_{g_2f(x)}\rho(g_2)_{f(x)} \\ &= \tau_{g_1g_2f(x)}(g_1, g_2)\rho_{f(x)}(g_1g_2) \\ &= \tau_{f(g_1g_2x)}(g_1, g_2)\rho_{f(x)}(g_1g_2) \\ &= (f^*\tau)_{g_1g_2x}(g_1, g_2)(f^*\rho)_x(g_1g_2). \end{aligned}$$

Using analogous ideas, we can define the pull-back of the Clifford actions on E. Under the grading $f^*(E) = f^*E_0 \oplus f^*E_1$, it is immediate that odd elements act oddly and even elements evenly. □

Not for all maps of twisted spaces the 1-cocycle is trivial. This is solved using the following construction:

Lemma 3.39. *Let (X, τ) be a twisted G-space and (E, ρ) a (G, ϕ, τ, c, p, q)-bundle over X. Let $\lambda \in C^1(G, C(X, U(1))_\phi)$ be a unital 1-cochain. Then $(E, \lambda \cdot \rho)$ is a $(G, \phi, d\lambda \cdot \tau, c, p, q)$-bundle over X, where*

$$(\lambda \cdot \rho)_x(g) := \lambda_{gx}(g)\rho_x(g).$$

Proof. Because λ is unital, we have $(\lambda\rho)_x(1) = 1$ for all $x \in X$. Now let $g, h \in G$ and $x \in X$. Then $\lambda\rho$ satisfies the projective representation condition:

$$\begin{aligned}(\lambda \cdot \rho)_{hx}(g)(\lambda \cdot \rho)_x(h) &= \lambda_{ghx}(g)\rho_{hx}(g)\lambda_{hx}(h)\rho_x(h)\\ &= \lambda_{ghx}(g)\lambda_{hx}(h)^{\phi(g)}\tau_{ghx}(g,h)\rho_x(gh)\\ &= d\lambda_{ghx}(g,h)\tau_{ghx}(g,h)\lambda_{ghx}(gh)\rho_x(gh)\\ &= (d\lambda \cdot \tau)_{ghx}(g,h)(\lambda \cdot \rho)_x(gh),\end{aligned}$$

where the reader is reminded that

$$d\lambda_x(g,h) = \lambda_{g^{-1}x}(h)^{\phi(g)}\lambda_x(gh)^{-1}\lambda_x(h).$$

The other conditions are immediate. □

Theorem 3.40. *The assignment* ${}^{\phi}K_G^{\bullet,c+(p,q)} : (X,\tau) \mapsto {}^{\phi}K_G^{\tau,c+(p,q)}(X)$ *defines a contravariant functor* $\mathbf{Top}_G^t \to \mathbf{Ab}$.

Proof. Let $(f,\lambda) : (X_1,\tau_1) \to (X_2,\tau_2)$ be a map of twisted G-spaces, such that λ establishes an isomorphism $\tau_1 \cong f^*\tau_2$. Define a map $(f,\lambda)^* : {}^{\phi}K_G^{\tau_2,c+(p,q)}(X_2) \to {}^{\phi}K_G^{\tau_1,c+(p,q)}(X_1)$ by $(f,\lambda)^*([(E,\rho)]) = [(f^*E, \lambda \cdot f^*\rho)]$. This map is well-defined; if $(E,\rho) \cong (E',\rho')$ by an isomorphism of degree $k \in \mathbb{Z}_2$, then there is a unique isomorphism $\psi : (f^*E, f^*\rho) \cong (f^*E', f^*\rho')$ of degree k such that the following diagram commutes:

$$\begin{array}{ccc} E & \xleftarrow[\bar{f}]{} & f^*E \\ \big\downarrow{\cong} & & \big\downarrow{\cong} \\ E' & \xleftarrow[\bar{f}]{} & f^*E'. \end{array}$$

However, ψ is complex linear and hence also defines an isomorphism $(f^*E, \lambda \cdot f^*\rho) \cong (f^*E', \lambda \cdot f^*\rho')$ of degree k. So indeed f^* is a map ${}^{\phi}K_G^{\tau_2,c+(p,q)}(X_2) \to {}^{\phi}K_G^{\tau_1,c+(p,q)}(X_1)$ by Lemma B.18.

For functoriality, let $(f_1,\lambda_1) : (X_1,\tau_1) \to (X_2,\tau_2), (f_2,\lambda_2) : (X_2,\tau_2) \to (X_3,\tau_3)$ be morphisms of twisted G-spaces. Then we see that

$$\begin{aligned}((f_2 \circ f_2)^*E, \lambda_1 f_1^*\lambda_2(f_2 \circ f_1)^*\rho) &\cong (f_1^*f_2^*E, \lambda_1 f_1^*\lambda_2 f_1^*f_2^*\rho)\\ &\cong (f_1^*f_2^*E, \lambda_1 f_1^*(\lambda_2 f_2^*\rho))\end{aligned}$$

by even isomorphisms. Hence K-theory becomes a functor. □

Remark 3.41. One interesting consequence of last theorem is that ${}^{\phi}K_G^{\tau,c+(p,q)}(X) \cong {}^{\phi}K_G^{\tau\cdot d\lambda,c+(p,q)}(X)$ if $\lambda \in C^1(G, C(X, U(1)_\phi)$. Indeed, such an isomorphism is induced by the morphism of twisted spaces (id_X, λ).

Up until now, we have considered our K-theory under a fixed group, but we will also consider maps between equivariant K-theory groups, of which the most notable is restriction of the action. To implement this into our framework in the most general setting, we let $\alpha : H \to G$ be a homomorphism of finite groups.

Proposition 3.42. *There is a covariant functor $F_\alpha : \mathbf{Top}^t_G \to \mathbf{Top}^t_H$ induced by α. Moreover, it induces a map on K-theory*

$$\alpha^* : {}^\phi K_G^{\tau,c+(p,q)}(X) \to^{\phi\circ\alpha} K_H^{\alpha^*\tau,c\circ\alpha+(p,q)}(X).$$

Proof. Let $(X_1,\tau_1) \overset{(f,\lambda)}{\to} (X_2,\tau_2)$ be a map of twisted G-spaces. We let $F_\alpha(X_1,\tau_1) = (X_1,\alpha^*\tau_1)$, where X_1 is the H-space $h\cdot x = \alpha(h)x$ and $\alpha^*\tau_1$ is the ordinary pull-back of cocycles

$$(\alpha^*\tau_1)_x(h_1,h_2) = (\tau_1)_x(\alpha(h_1),\alpha(h_2)),$$

which is easily verified to be well-defined in the sense that $\alpha^*\tau_1 \in^{\phi\circ\alpha} \mathrm{Twist}_H^X$. On morphisms, the functor is defined as $F_\alpha(f,\lambda) = (f,\alpha^*\lambda) : (X_1,\alpha^*\tau_1) \to (X_2,\alpha^*\tau_2)$. Clearly f becomes an H-equivariant map. Hence the only thing left to show for $F_\alpha(f)$ to be a map of twisted G-spaces is

$$f^*\alpha^*\tau_2 = \alpha^* f^*\tau_2 = \alpha^*(d\lambda\tau_1) = \alpha^*(d\lambda)\alpha^*\tau = d(\alpha^*\lambda)\alpha^*\tau.$$

Functoriality of F_α is obvious.

We now define a map on the level of K-theory. Let (E,ρ) be a (G,ϕ,τ,c,p,q)-bundle over X. Define $\alpha^*\rho(h) := \rho(\alpha(h))$. It is easy to check that (E,ρ) is an $(H,\phi\circ\alpha,\alpha^*\tau,c\circ\alpha,p,q)$-bundle over X. If $f : (E,\rho) \to (E',\rho')$ is an isomorphism of degree $k \in \mathbb{Z}_2$, it is easy to construct an isomorphism $\alpha^* f; (E,\alpha^*\rho) \to (E',\alpha^*\rho')$. □

An important example of this proposition is the restriction map $Res^G_H := j^* \colon {}^\phi K_G^{\tau,c+(p,q)}(X) \to {}^{\phi|_H} K_H^{\tau|_H,c|_H+(p,q)}(X)$ associated to a subgroup $j : H \hookrightarrow G$. More explicitly, it is given by seeing a twisted G-equivariant bundle as a twisted H-equivariant bundle by restricting the action. In particular, if $x \in X$, we can get a map ${}^\phi K_G^{\tau,c+(p,q)}(X) \to {}^\phi R^{\tau,c+(p,q)}(G_x)$ given by

$${}^\phi K_G^{\tau,c+(p,q)}(X) \overset{Res^G_{G_x}}{\to} {}^{\phi|_{G_x}} K_{G_x}^{\tau|_{G_x},c|_{G_x}+(p,q)}(X) \overset{i^*}{\to} {}^\phi R^{\tau|_{x\times G_x},c|_{G_x}+(p,q)}(G_x),$$

where i is the inclusion map $(\mathrm{pt},\tau|_{x\times G_x}) \to (X,\tau|_{G_x})$ of twisted G_x-spaces.

3.6. Homotopy Invariance

We will now show a suitable version of homotopy invariance, following Gomi [31]. The claim is that the proper notion of homotopy is the following:

Definition 3.43. Let $(f_0,\eta_0),(f_1,\eta_1) : (X_1,\tau_1) \to (X_2,\tau_2)$ be two maps of twisted G-spaces. Consider the pull-back twist $p^*\tau_1$ on $X_1\times[0,1]$ under the projection map $p : X_1\times[0,1] \to X_1$. A *twisted G-homotopy from f_0 to f_1* is a morphism of twisted G-spaces $(f,\eta) : (X_1\times[0,1],p^*\tau_1) \to (X_2,\tau_2)$ such that $f|_{X_1\times 0} = f_0$ and $f|_{X_1\times 1} = f_1$. If there exists such a homotopy from f_0 to f_1, we say that f_0 and f_1 are *twisted G-homotopic*.

We start with a couple of lemmas as in Segal's exposition of equivariant K-theory [57].

Lemma 3.44. *Let $(E_1, \rho_1), (E_2, \rho_2)$ be two (G, ϕ, τ, c, p, q)-bundles over X. Then there exists a $(G, \phi, 0, 0, 0, 0)$-bundle $\mathrm{Hom}^{p,q}(E_1, E_2)$ over X such that its G-equivariant sections correspond one-to-one with morphisms of (G, ϕ, τ, c, p, q)-bundles $E_1 \to E_2$.*

Proof. Note first that since $\mathrm{Hom}^{p,q}(E_1, E_2)$ is supposed to have a trivial twist $\tau' = 0$, the bundle $\mathrm{Hom}^{p,q}(E_1, E_2)$ is itself a G-space. Hence the notion of G-equivariant sections is defined.

Now let $\mathrm{Lin}(E_1, E_2)$ be the complex vector bundle over X for which its sections correspond to ordinary complex vector bundle maps $E_1 \to E_2$ (covering the identity as always). Note that this bundle is naturally $\mathbb{Z}_2$-graded by whether the bundle maps are odd or even with respect to the gradings of E_1 and E_2. Let $\mathrm{Hom}^{p,q}(E_1, E_2)$ be the subbundle of $\mathrm{Lin}(E_1, E_2)$ defined fiberwise as the complex linear maps $(E_1)_x \to (E_2)_x$ that are maps of supermodules over $Cl_{p,q}$ under the given Clifford algebra action. Note indeed that in an open $U \subseteq X$ that trivializes both E_1 and E_2, the bundle $\mathrm{Lin}(E_1, E_2)$ trivializes as $U \times \mathrm{Lin}(V_1, V_2)$, where V_1, V_2 are the fibers of E_1 and E_2 at some fixed reference basepoint $x \in U$. Since $\mathrm{Hom}^{p,q}(V_1, V_2) \subseteq \mathrm{Lin}(V_1, V_2)$ is a complex linear subspace (the Clifford action is required to be complex linear) and the subspaces vary continuously under the trivialization, we see that $\mathrm{Hom}^{p,q}(E_1, E_2) \subseteq \mathrm{Lin}(E_1, E_2)$ is a subbundle.

Now consider the G-action on $\mathrm{Hom}^{p,q}(E_1, E_2)$ given by

$$(g \cdot f)(e_{gx}) = c(g)\rho_2(g) f(\rho_1(g)^{-1} e_{gx}) \quad x \in X, g \in G, e_{gx} \in (E_1)_{gx}.$$

First note that this action maps elements of $\mathrm{Hom}^{p,q}(E_1, E_2)$ to elements of $\mathrm{Hom}^{p,q}(E_1, E_2)$, since the Clifford algebra action graded commutes with the action of G. Also g acts complex linearly if $\phi(g) = 1$ and complex anti-linearly if $\phi(g) = -1$:

$$\begin{aligned}(g \cdot zf)(e_{gx}) &= c(g)\rho_2(g) z f(\rho_1(g)^{-1} e_{gx}) = \bar{z}^{\phi(g)} c(g)\rho_2(g) f(\rho_1(g)^{-1} e_{gx}) \\ &= \bar{z}^{\phi(g)} (g \cdot f)(e_{gx}).\end{aligned}$$

Moreover, the action sends even maps to even maps and odd maps to odd maps, so the action of G is always even (i.e. c' is trivial). We now check that we have a genuine G-action:

$$\begin{aligned}(g_2 \cdot (g_1 \cdot f))(e_{g_1 g_2 x}) &= c(g_2)\rho_2(g_2)(g_1 \cdot f)(\rho_1(g_2)^{-1} e_{g_1 g_2 x}) \\ &= c(g_2)\rho_2(g_2) c(g_1)\rho_2(g_1) f(\rho_1(g_1)^{-1}\rho_1(g_2)^{-1} e_{g_1 g_2 x}) \\ &= c(g_2)c(g_1)\tau_{g_1 g_2 x}(g_2, g_1)\rho_2(g_2 g_1) \cdot \\ &\quad\; f((\tau_{g_1 g_2 x}(g_2, g_1)\rho_1(g_2 g_1))^{-1} e_{g_1 g_2 x}) \\ &= c(g_2 g_1)\tau_{g_1 g_2 x}(g_2, g_1)\rho_2(g_2 g_1) \cdot \\ &\quad\; f((\rho_1(g_2 g_1)\tau_{g_1 g_2 x}(g_2, g_1)^{\phi(g_1 g_2)})^{-1} e_{g_1 g_2 x}) \\ &= c(g_2 g_1)\tau_{g_1 g_2 x}(g_2, g_1)\rho_2(g_2 g_1)\tau_{g_1 g_2 x}(g_2, g_1)^{-\phi(g_1 g_2)} \cdot \\ &\quad\; f(\rho_1(g_2 g_1)^{-1} e_{g_1 g_2 x}) \\ &= c(g_2 g_1)\rho_2(g_2 g_1) f(\rho_1(g_2 g_1)^{-1} e_{g_1 g_2 x}) \\ &= ((g_2 \cdot g_1) \cdot f))(e_{g_1 g_2 x}).\end{aligned}$$

Now by construction the equivariant sections of $\mathrm{Hom}(E_1, E_2)$ are exactly the maps of (G, ϕ, τ, c, p, q)-bundles $E_1 \to E_2$. $\square$

Lemma 3.45. *Let $A \subseteq X$ be a closed G-subspace, E a $(G, \phi, 0, 0, 0, 0)$-bundle over X and $s \in \Gamma_G(E|_A)$ an equivariant section. Then s has an equivariant extension to X. If s maps into the even part of $E|_A$, then we can choose the equivariant extension to map into the even part of E.*

Proof. This result is well-known for ordinary sections of vector bundles; one just applies Tietze's extension theorem locally (X is compact Hausdorff) and a partition of unity argument. So let $\tilde{t} \in \Gamma(E)$ be any section extending s. Average $\tilde{t}$ over G by setting

$$t(x) := \frac{1}{|G|} \sum_{g \in G} \rho(g) \tilde{t}(g^{-1}x).$$

Then clearly $t(a) = \tilde{t}(a) = s(a)$ for all $a \in A$. Moreover, for all $h \in G$ we have

$$\begin{aligned} t(h^{-1}x) &= \frac{1}{|G|} \sum_{g \in G} \rho(g) \tilde{t}(g^{-1}h^{-1}x) \\ &= \frac{1}{|G|} \sum_{g' \in G} \rho(h^{-1}g') \tilde{t}(g'^{-1}x) \\ &= \rho(h^{-1}) \frac{1}{|G|} \sum_{g' \in G} \rho(g') \tilde{t}(g'^{-1}x) \\ &= (h^{-1} \cdot t)(x). \end{aligned}$$

It should be noted that we used $\tau = 0$. Also, since all group elements are assumed to act by even maps, t maps into the even part of E whenever s does. $\square$

Lemma 3.46. *Let $(E_1, \rho_1), (E_2, \rho_2)$ be two (G, ϕ, τ, c, p, q)-bundles over X and $A \subseteq X$ be a closed G-subspace. Suppose $f : E_1|_A \to E_2|_A$ is an isomorphism of (G, ϕ, τ, c, p, q)-bundles. Then there exists an open G-neighbourhood U of A and an extension $\tilde{f} : E_1|_U \to E_2|_U$ that is still an isomorphism of (G, ϕ, τ, c, p, q)-bundles. If f is even, then $\tilde{f}$ can be chosen to be even.*

Proof. We apply Lemma 3.45 to the bundle of Lemma 3.44. Then we get a G-equivariant section t of the bundle $\mathrm{Hom}^{p,q}(E_1, E_2)$ such that $t|_A = f$. Define

$$U := \{x \in X : t_x : (E_1)_x \to (E_2)_x \text{ is an isomorphism of complex vector spaces}\}.$$

Clearly U contains A. It is also a G-subspace, because t is a G-invariant section:

$$t_{gx}(e_{gx}) = \rho_2(g) t_x(\rho_1(g)^{-1} e_x).$$

Also note that $t|_U$ is actually an isomorphism of (G, ϕ, τ, c, p, q)-bundles; by basic theory of vector bundles we know that $t|_U$ has an inverse $(t|_U)^{-1}$ in the category of ordinary complex vector bundles. But $t|_U$ is also a morphism of (G, ϕ, τ, c, p, q)-bundles and the condition of being a (G,ϕ,τ,c,p,q)-bundle is local, so that $(t|_U)^{-1}$ is also a map of (G,ϕ,τ,c,p,q)-bundles. Finally we show that U is open. Let $x \in U$ and let $U' \subseteq X$ be a trivializing neighbourhood of E_1 and E_2 around x. Then we have seen that locally $\mathrm{Hom}^{p,q}(E_1,E_2)|_{U'} \cong \mathrm{Hom}(V_1,V_2)\times U'$, where V_1 and V_2 are respectively the fibers of E_1 and E_2 at x and $\mathrm{Hom}^{p,q}(V_1,V_2)$ denotes the linear maps commuting with the Clifford action. Under these correspondences, we conclude that t can be seen as a map $t : U' \to \mathrm{Hom}^{p,q}(V_1,V_2)$ and

$$U' \cap U = \{x \in U' : t \text{ maps } x \text{ into invertibles}\}.$$

However, the invertible maps in $\mathrm{Hom}^{p,q}(V_1,V_2)$ form an open set and t is continuous. Hence U is open.

To show that $\tilde{f}$ can be chosen to be even if f is even, pick the section t to map into the even part of $\mathrm{Hom}^{p,q}(E_1,E_2)$ using Lemma 3.45. Then t_x is an even isomorphism of complex vector spaces for $x \in U$. The rest of the proof is analogous. □

Proposition 3.47. *Suppose that $(f,\eta) : (X_1 \times [0,1], p^*\tau_1) \to (X_2,\tau_2)$ is a twisted G-homotopy from (f_0,η_0) to (f_1,η_1) and E is a (G,ϕ,c,τ_2,p,q)-bundle over X_2. Then there exists a $\lambda \in C^1(G,C(X,U(1)_\phi))$ such that $f_0^*\tau_2 \cdot d\lambda = f_1^*\tau_2$ and*

$$(f_1^*E, f_1^*\rho) \cong (f_0^*E, \lambda \cdot f_0^*\rho)$$

by an even isomorphism.

Proof. Consider the following diagram of twisted G-spaces for some fixed $t_0 \in [0,1]$:

$$\begin{array}{ccc} (X_1\times[0,1], f^*\tau_2) & \xrightarrow{\quad f \quad} & (X_2,\tau_2) \\ \Updownarrow & & \uparrow{\scriptstyle f_{t_0}} \\ (X_1\times[0,1], \pi_{t_0}^* f_{t_0}^*\tau_2) & \xrightarrow{\pi_{t_0}} & (X_1\times\{t_0\}, f_{t_0}^*\tau_2). \end{array}$$

We want to relate $(X_1\times[0,1], f^*\tau_2)$ to $(X_1\times[0,1], \pi_{t_0}^* f_{t_0}^*\tau_2)$ and eventually f^*E to $\pi_{t_0}^* f_{t_0}^*E$. Note that more explicitly, the cocycle η satisfies

$$(\tau_2)_{f_t(x)} = (f^*\tau_2)_{(x,t)} = (\tau_1)_x \cdot d\eta_{(x,t)},$$

for $x \in X_1$ and $t \in [0,1]$. This also gives us

$$\begin{aligned} (f^*\tau_2)_{(x,t)} &= (\tau_1)_x \cdot d\eta_{(x,t_0)} d\eta^{-1}_{(x,t_0)} d\eta_{(x,t)} = (f^*\tau_2)_{(x,t_0)} d\eta^{-1}_{(x,t_0)} d\eta_{(x,t)} \\ &= (\pi_{t_0}^* f_{t_0}^*)_{(x,t)} (p^* i_{t_0}^* d\eta)^{-1}_{(x,t)} d\eta_{(x,t)}, \end{aligned}$$

where $i_{t_0} : X \to X\times[0,1]$ is the inclusion of X as the subspace $X \times \{t_0\}$. So we see that $f^*\tau_2$ and $\pi_{t_0}^* f_{t_0}^*\tau_2$ are isomorphic via $\lambda_{t_0} := d(\eta \cdot p^* i_{t_0}^*(\eta)^{-1})$.

We will now show that $(f_0^*E, f_0^*\rho) \cong (f_1^*E, f_1^*\rho \cdot \lambda_1)$. For this purpose, consider the set

$$I := \{t \in [0,1] : (f_0^*E, f_0^*\rho) \cong (f_t^*E, f_t^*\rho \cdot \lambda_t) \text{ evenly}\}.$$

It will now be shown that this set is clopen, thereby finishing the proof. For fixed $t_0 \in [0,1]$, consider the G-subspace $A_{t_0} := X_1 \times \{t_0\} \subseteq X_1 \times [0,1]$. Then the pull-back bundles $(f^*E, f^*\rho)$ and $(f_{t_0}^*\pi_{t_0}^*E, \lambda_{t_0} \cdot f_{t_0}^*\pi_{t_0}^*)$ restricted to A_{t_0} are isomorphic, because the maps and twists are equal. Therefore, by the last lemma, there is an open neighbourhood $U \subseteq X_1 \times [0,1]$ such that restricted to U the bundles are still isomorphic. Now since the collection

$$\{X_1 \times ([0, t_0 - \epsilon) \cup (t_0 + \epsilon, 1]) : \epsilon > 0\}$$

covers the compact set $(X_1 \times [0,1]) \setminus U$, there exists an $\epsilon > 0$ such that $[t_0 - \epsilon, t_0 + \epsilon] \subseteq U$. In other words, given any $t \in (t_0 - \epsilon, t_0 + \epsilon)$, we have

$$f_t^*E \cong f^*E|_{X_1 \times \{t\}} \cong f_{t_0}^*\pi_{t_0}^*E|_{X_1 \times \{t\}} \cong f_{t_0}^*E.$$

So isomorphism classes are locally constant in t. Hence if we take $t_0 \in I$, we see that $(t_0 - \epsilon, t_0 + \epsilon) \subseteq I$, so I is open. Conversely, if we take $t_0 \in [0,1] \setminus I$, then given $t \in (t_0 - \epsilon, t_0 + \epsilon)$ we have $(f_t^*E, f_t^*\rho \cdot \lambda_t) \cong (f_{t_0}^*E, f_{t_0}^*\rho \cdot \lambda_{t_0})$, so that $(f_t^*E, f_t^*\rho \cdot \lambda_t) \ncong (f_0^*E, f_0^*\rho \cdot \lambda_0)$. Hence $(t_0 - \epsilon, t_0 + \epsilon) \subseteq [0,1] \setminus I$ and we get that I is closed. □

Homotopy invariance as in this proposition immediately gives the following important fact:

Corollary 3.48. *If f_0 and f_1 are homotopic maps, they induce the same map on K-theory.*

The corollary can be stated in a fancier way by saying that the K-theory functor factors through the associated (naive) homotopy category. In this category we replace morphisms between twisted G-spaces by equivalence classes of these under twisted G-homotopy. This abstraction also applies to maps on the levels of groups; if $\alpha : H \to G$ is a homomorphism, then F_α factors through the homotopy category. Indeed, if $f : (X_1 \times [0,1], p^*\tau_1) \to (X_2, \tau_2)$ is a twisted G-homotopy, then $F_\alpha(f) : (X_1, p^*\alpha^*\tau_1) \to (X_2, \alpha^*\tau_2)$ is a twisted H-homotopy from $F_\alpha(f_0)$ to $F_\alpha(f_1)$.

3.7. Reduced K-theory, Stable Equivalence and Pairs of Twisted Spaces

To construct the reduced theory, a pointed category is needed.

Definition 3.49. A *pointed twisted G-space* is a triple (X, x_0, τ), where (X, x_0) is a pointed G-space[8] and $\tau \in {}^\phi\,\mathrm{Twist}_G^X$ a ϕ-twist. A map of pointed twisted G-spaces is a map of pointed G-spaces that is also a map of twisted spaces: $f : (X, x_0, \tau_1) \to (Y, f(x_0), \tau_2)$ such that $f^*\tau_2 \cong \tau_1$. In other words, it is a G-map that preserves the twists under pull-back. The *reduced twisted equivariant K-theory* of a pointed twisted G-space (X, x_0, τ) is

$${}^\phi\widetilde{K}_G^{\tau,c+(p,q)}(X) := \ker\left(i_{x_0}^* : {}^\phi K_G^{\tau,c+(p,q)}(X) \to {}^\phi K_G^{\tau_{x_0},c+(p,q)}(\mathrm{pt}, \tau_x)\right),$$

where i_{x_0} is inclusion of the point x_0.

[8]Recall in particular that this implies that x_0 is a fixed point under G. See Appendix C.2 for basic definitions involving G-spaces.

Remark 3.50. Since x_0 is a fixed point of the action, i_{x_0} is a map of twisted G-spaces, so the reduced theory is well-defined.

Proposition 3.51. *A map of pointed twisted G-spaces $f : (X_1, x_0, \tau_1) \to (X_2, f(x_0), \tau_2)$ induces a map f^* on reduced twisted equivariant K-theory. This makes reduced K-theory into a functor on pointed twisted spaces. Moreover, if f_0 is twisted G-homotopic to f_1 relative to x_0, then $(f_0)^* = (f_1)^*$, so that the functor factors through the relevant homotopy category.*

Proof. The map f^* on unreduced twisted equivariant K-theory induces a map on reduced equivariant K-theory because if $[E] \in {}^{\phi}\widetilde{K}_G^{\tau,c+(p,q)}(X)$ is an isomorphism class of a c-graded (ϕ, τ)-twisted bundle E, then

$$i_{x_0}^* \circ f^*([E]) = (f \circ i_{x_0})^*([E]) = i_{f(x_0)}^*([E]) = 0$$

so that indeed $f^*([E]) \in \ker i_{x_0}^*$. For the homotopy statement, note that since $f_t(x_0)$ is independent of t, all maps $(f_t)^*$ come from the same reduced K-theory group. Hence the statement $(f_0)^* = (f_1)^*$ makes sense. Now the proposition follows immediately from the unreduced case. □

In twisted K-theory, the K-theory of a point depends on the twist and not all twists are constant in X. This fact is relevant; given a twisted space (X, τ), the projection map $\pi : X \to \mathrm{pt}$ can be made into a map of twisted G-spaces if and only if τ is isomorphic to an anomaly. In that case it induces a map of abelian groups $\pi^* : {}^{\phi}R^{\tau,c}(G) \to {}^{\phi}K_G^{\tau,c}(X)$, which moreover is a section of any inclusion map $\mathrm{pt} \hookrightarrow X$ by functoriality. Therefore in the case of an anomaly twist ${}^{\phi}K_G^{\tau,c}(X)$ becomes a module over the c-graded (ϕ, τ)-twisted representation ring of G. An insightful way to summarize these algebraic facts is as follows:

Lemma 3.52. *Let (X, x, τ) be a pointed twisted G-space. There is an exact sequence*

$$0 \to {}^{\phi}\widetilde{K}_G^{\tau,c+(p,q)}(X) \hookrightarrow {}^{\phi}K_G^{\tau,c+(p,q)}(X) \to {}^{\phi}R^{\tau_x,c+(p,q)}(G) \to 0.$$

If τ is an anomaly twist, the sequence is split by the inclusion of the trivial bundle $\pi^* : {}^{\phi}R^{\tau,c+(p,q)}(G) \to {}^{\phi}K_G^{\tau,c+(p,q)}(X)$.

Note that this lemma implies that unlike for nontwisted K-theory, the reduced theory is a priori not canonically defined for a connected twisted G-space without basepoint. This should not be too surprising, since reduced theories are naturally constructed on pointed spaces. However, in the case of an anomaly twist $\tau \in H^2(G, U(1)_\phi)$, we can realize the reduced theory using stable equivalence classes of bundles as in [57]. This approach will now be developed.

Proposition 3.53. *Let E be a (G, ϕ, τ, c, p, q)-bundle over (X, τ), where τ is an anomaly. Then E is contained in a trivial bundle θ_M, where M is a supermodule over ${}^{\phi}\mathbb{C}^{\tau,c+(p,q)}G$.*

Proof. We follow Segal's proof for the ordinary equivariant case (Proposition 2.4 of [57]). The set $\Gamma(E)$ of all continuous sections of E forms a (possibly infinite-dimensional) supermodule over ${}^{\phi}\mathbb{C}^{\tau,c+(p,q)}G$ by

$$(x_g \cdot s)(x) = \rho(g)s(x).$$

Given any $x \in X$, there is a surjective complex-linear map $\psi_x : \Gamma(E) \to E_x$ given by evaluation at x. Let $\sigma_x \subseteq \Gamma(E)$ be a finite subset so that $\psi_x(\sigma_x)$ complex-spans E_x. Let U_x be an open neighbourhood of x such that for all $y \in U_x$, $\psi_y(\sigma_x)$ spans E_y, which exists because spanning is an open condition. Now consider the open covering

$$\mathcal{V} = \{U_x : x \in X\}$$

of X. Let $U_{x_1}, \ldots, U_{x_n}$ be a finite subcover and let $M \subseteq \Gamma(E)$ be the submodule generated by $\sigma_{x_1}, \ldots, \sigma_{x_n}$. Since ${}^{\phi}\mathbb{C}^{\tau,c+(p,q)}G$ is finite-dimensional, so is M. Now consider the map

$$\theta_M \to E, \quad (x, s) \mapsto s(x).$$

The map is fiber-preserving and it is an even morphism of (G, ϕ, τ, c, p, q)-bundles by definition of the module structure of $\Gamma_a(E)$. Moreover, the map is surjective by construction: if $y \in X$ and $v \in E_y$, take i such that $y \in U_{x_i}$. Then since $\psi_y(\sigma_{x_i})$ spans E_y, there exist $s_1, \ldots, s_k \in M$ and $a_1, \ldots, a_k \in \mathbb{C}$ such that

$$v = \sum_i a_i \psi_y(s_i) = \sum_i s_i(y) \implies \sum_i a_i s_i \in M \text{ is mapped to } v \in E.$$

□

Corollary 3.54. *Let E be a (G, ϕ, τ, c, p, q)-bundle over (X, τ), where τ is an anomaly. Then there exists a (G, ϕ, τ, c, p, q)-bundle F over (X, τ) such that $E \oplus F$ is a trivial bundle.*

Proof. Using the proposition, take a supermodule M over ${}^{\phi}\mathbb{C}^{\tau,c+(p,q)}$ and an embedding $E \hookrightarrow \theta_M$. Let $\langle ., . \rangle$ be a hermitian product on M as in (the local case of) Lemma 3.37. Then the usual complex vector bundle $E^{\perp} \subseteq \theta_M$ can be made into a (G, ϕ, τ, c, p, q)-bundle by restricting the action and grading on θ_M. Clearly $E \oplus E^{\perp} \cong \theta_M$. □

Proposition 3.55. *Suppose $c = 1$, $p = q = 0$ and τ is an anomaly. Then reduced K-theory can be realized by stable equivalence classes of (ϕ, τ)-twisted equivariant bundles. More precisely, let ${}^{\phi}\widetilde{K}_G^{\tau}(X)$ be the monoid consisting of isomorphism classes of (ϕ, τ)-twisted G-equivariant bundles over X quotiented by the submonoid consisting of trivializable bundles. Then the map*

$$\psi : {}^{\phi}\widetilde{K}_G^{\tau}(X) \to {}^{\phi}\widetilde{K}_G^{\tau,c+(0,0)}(X), \quad [E] \mapsto [E] - [\theta_{E_{x_0}}],$$

where $x_0 \in X$ is the basepoint, is an isomorphism.

Proof. By ψ is well-defined by Lemma B.18, because if M is a module over ${}^\phi\mathbb{C}^\tau G$, then $[\theta_M]$ is mapped to $[\theta_M] - [\theta_M] = 0$.

To show that ψ is surjective, let $[E_1] - [E_2] \in {}^\phi\widetilde{K}_G^{\tau,c+(p,q)}(X)$, where E_1 and E_2 are (ϕ,τ)-twisted equivariant bundles. Using Lemma 3.54, we get a (ϕ,τ)-twisted equivariant bundle F such that $E_2 \oplus F \cong \theta_M$ for some module M over ${}^\phi\mathbb{C}^\tau G$. Hence

$$[E_1] - [E_2] = [E_1] + [F] - [E_2] - [F] = [E_1 \oplus F] - [\theta_M].$$

So every element of ${}^\phi K_G^\tau(X)$ can be written as $[E] - [\theta_M]$ for some (ϕ,τ)-twisted equivariant bundle E and ${}^\phi\mathbb{C}^\tau G$-module M. Now note that

$$i^*([E] - [\theta_M]) = [E_{x_0}] - [M] = 0 \iff E_{x_0} \cong M$$

as ${}^\phi\mathbb{C}^\tau G$-modules. So the fact that $[E] - [\theta_M] \in \ker i^*$ gives us that $[E] \in {}^\phi\widetilde{K}_G^\tau(X)$ is a pre-image of $[E] - [\theta_M]$.

For injectivity, suppose that $[E] - [\theta_{E_{x_0}}] = 0$. Then there is a (ϕ,τ)-twisted equivariant bundle F such that $E \oplus F \cong \theta_{E_{x_0}} \oplus F$. By embedding F into a trivial bundle, we can assume without loss of generality that $F = \theta_{M'}$ for some module M' over ${}^\phi\mathbb{C}^\tau G$. But then we have in ${}^\phi\widetilde{K}_G^\tau(X)$ that

$$[E] + [\theta_{M'}] = [\theta_{E_{x_0} \oplus M'}] \implies [E] = 0.$$

□

The setting of reduced K-theory can be generalized to pairs, but one has to be slightly careful defining relative K-theory; just as in ordinary K-theory it is no longer the kernel of the restriction map since the subspace is not necessarily contractible. In case the twist on the subspace is isomorphic to an anomaly cocycle, K-theory of pairs of spaces can be defined by the reduced K-theory of the quotient by excision. However, otherwise it seems that the relative K-theory is not naturally defined in this Freed & Moore formulation and one really has to consider another type of K-theory formulation, see Section 3.8.

Definition 3.56. A *pair of twisted G-spaces* (X, A) consists of a twisted G-space (X, τ) and a closed G-subspace $A \subseteq X$. Morphisms of pairs $(X, A) \to (Y, B)$ are morphisms of twisted G-spaces that map A into B.

Lemma 3.57. *Let (X, A) be a pair of G-spaces with twist $\tau \in \mathrm{Twist}_G^X$ and let $\pi : (X, A) \to (X/A, \mathrm{pt})$ denote the quotient map, which is a morphism in the category of pairs of G-spaces. Then $(X/A, \mathrm{pt})$ can be made into a twisted G-space such that π is a morphism of pairs of twisted G-spaces if and only if τ is isomorphic to a twist $\tau' \in \mathrm{Twist}_G^X$ such that $\tau'|_A$ is constant in A.*

Proof. Suppose first that $\tilde{\tau} \in \mathrm{Twist}_G^{X/A}$ is such that $\pi : (X, \tau) \to (X/A, \tilde{\tau})$ is a map of twisted G-spaces. Define $\tau' := \pi^*(\tilde{\tau})$. Then, because π is a map of twisted G-spaces, $\tau' \cong \tau$. Moreover, given some $a \in A$ we have $(\tau')_a = (\pi^*\tilde{\tau})_a = \tilde{\tau}_{\pi(a)}$, which is constant in a.

Conversely, suppose $\tau' \in \mathrm{Twist}_G^X$ is a twist satisfying the given assumption. Define $(\tilde{\tau})_{\pi(x)} := \tau'_x$. This is well-defined because τ' is constant on A. Moreover, the map $X/A \to U(1)$ given by $\pi(x) \mapsto (\tilde{\tau})_{\pi(x)}(g_1, g_2)$ is continuous for all $g_1, g_2 \in G$. The fact that $\tilde{\tau}$ is a 2-cocycle follows immediately from the fact that τ' is a 2-cocycle. By construction we have $\pi^*(\tilde{\tau}) = \tau'$ and therefore π is a map of twisted G-spaces with respect to $\tilde{\tau}$. □

Remark 3.58. Even though a twist τ on X isomorphic to a twist constant in A gives a twist on X/A, the choice of the twist on X/A depends on the chosen isomorphism of τ with the twist constant in A. Hence for X/A to be canonically a twisted G-space, we need to assume that $\tau|_A$ is an anomaly twist itself.

Proposition 3.59. *The assignment $(X, A) \mapsto {}^{\phi}\widetilde{K}_G^{\tau, c+(p,q)}(X/A)$ is a functor on the full subcategory $\mathbf{Pairs}_G^t$ of pairs of twisted G-spaces consisting of spaces for which the twist on X restricted to A is an anomaly twist.*

Proof. Let $f : (X, A, \tau_X) \to (Y, B, \tau_Y)$ be a morphism in $\mathbf{Pairs}_G^t$. Consider the following commutative diagram of G-maps:

$$\begin{array}{ccccc} \mathrm{pt} & \hookrightarrow & X/A & \xleftarrow{\pi_1} & X \\ \| & & \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ \mathrm{pt} & \hookrightarrow & Y/B & \xleftarrow{\pi_2} & Y \end{array}$$

where $\tilde{f}$ is the unique map (which is continuous and G-equivariant) such that the diagram commutes. Then $\tilde{f}$ is a map of pointed twisted G-spaces under the canonical twists $\tau_{X/A}$ of X/A and $\tau_{Y/A}$ of Y/A:

$$\pi_1^* \tilde{f}^* \tau_{Y/B} = f^* \pi_2^* \tau_{Y/B} = f^* \tau_Y \cong \tau_X \implies \tilde{f}^* \tau_{Y/B} \cong \tau_{X/A}.$$

So $\tilde{f}$ induces a map $\tilde{f}^* : {}^{\phi}\widetilde{K}_G^{\tau, c+(p,q)}(Y/B) \to {}^{\phi}\widetilde{K}_G^{\tau, c+(p,q)}(X/A)$. It follows immediately by functoriality of K-theory that this assignment is a functor. □

Lemma 3.60. *Let (X, A) be a pair of twisted G-spaces such that $\tau|_A$ is an anomaly and let $\tilde{\tau}$ be the canonical lift of τ to X/A. A (G, ϕ, τ, c, p, q)-bundle (E, ρ) over X comes from a $(G, \phi, \tilde{\tau}, c, p, q)$-bundle $\tilde{E}$ over X/A if and only if E is trivializable over A.*

Proof. First suppose we are given $\tilde{E}$ over X/A such that $\pi^* \tilde{E} = E$, where $\pi : (X, A) \to (X/A, \mathrm{pt})$ denotes the quotient map. Then the obvious map $\pi^* : E|_A \to A \times \tilde{E}_{\mathrm{pt}}$, where $\mathrm{pt} \in X/A$ is the basepoint, is clearly an isomorphism of (G, ϕ, τ, c, p, q)-bundles.

Conversely, let $\alpha : E|_A \to A \times M$ be a trivialization for some choice of fiber supermodule M over ${}^{\phi}\mathbb{C}^{\tau, c+(p,q)}G$. Denote the projection maps by $p : E \to X$ and $pr_2 : A \times M \to M$ Define an equivalence relation on E by $e_1 \sim e_2$ if and only if $e_1 = e_2$ or $p(e_1), p(e_2) \in A$ and $pr_2(\alpha(e_1)) = pr_2(\alpha(e_2))$. Let $\tilde{E}$ be the topological space E modulo this relation. Define $\tilde{p} : \tilde{E} \to X/A$ by $\tilde{p}([e]) = \pi p(e)$. This is well-defined and hence continuous, because if $e_1 \sim e_2$ and $e_1 \neq e_2$, then $p(e_1), p(e_2) \in A$, so that $\pi p(e_1) = \pi p(e_2)$. We can define a twisted action of G on $\tilde{E}$ by $g \cdot [e] = [\rho(g)e]$ and similarly for the Clifford action. This action is well-defined since α is a map of (G, ϕ, τ, c, p, q)-bundles. Moreover, it is easy to see that this action is $\tilde{\tau}$-twisted.

It will now be shown that $\tilde{E}$ is a $(G, \phi, \tilde{\tau}, c, p, q)$-bundle over X/A. This finishes the proof, since by construction, $\pi^*\tilde{E} \cong E$. Moreover, it is sufficient to show that X/A admits an open cover of trivializations of $(G, \phi, \tilde{\tau}, c, p, q)$-bundles of $\tilde{E}$. For any point in $X \setminus A$, there is a sufficiently small G-neighbourhood U such that E is trivializable over U and $U \cap A = \emptyset$. Using this fact, it is easy to see that $\tilde{E}$ is locally trivial around any point in X/A unequal to the basepoint. To show that $\tilde{E}$ is locally trivial around the basepoint, we use Lemma 3.46 to get an extension of α to a trivialization of E over an open G-neighbourhood U of A, which will also be written α. Define a map $\psi : \tilde{E}|_{\pi(U)} \to \pi(U) \times M$ by $\psi([e]) = (\pi p(e), pr_2\alpha(e))$. This is well-defined and moreover $\pi^{-1}\pi(U) = U$, so that U is open. It is easy to show that ψ is an isomorphism of $(G, \phi, \tilde{\tau}, c, p, q)$-bundles. □

3.8. Alternative Ways to Define K-theory and Eilenberg-Steenrod Axioms

In Section 1.7, it was argued that when X is a point the definition of K-theory of Freed & Moore has to be altered slightly in order to reproduce the ten-fold way and agree with the work of Atiyah & Segal [7]. The claim of this chapter is that for X not equal to a point, Definition 3.30 is in general still not correct for p, q nonzero. We first illustrate this claim by an example.

Example 3.61. Consider the ordinary complex K-theory of $X = S^1$ in degree 1 in this formulation. So in the formulation of Definition 3.30, G, ϕ, τ and c are trivial, while $p = 1$ and $q = 0$. Using suspensions and reduced K-theory, we can compute that

$$K^1(S^1) \cong K^1(\mathrm{pt}) \oplus \widetilde{K}^1(S^1) = \widetilde{K}^1(S^1) \cong \widetilde{K}^0(S^0) \cong K^0(\mathrm{pt}) \cong \mathbb{Z}.$$

However, Definition 3.30 does not seem to reproduce this fact. In fact, the claim is that Definition 3.30 will result in this group to be zero.

Indeed, let $E = E_0 \oplus E_1$ be a complex supervector bundle with Clifford action of $\mathbb{C}l_1$. The action can be represented by a pair of vector bundle maps $\gamma_0 : E_0 \to E_1$ and $\gamma_1 : E_1 \to E_0$. The Clifford algebra relation implies that $\gamma_1 = \gamma_0^{-1}$, so the total odd bundle map $E \to E$ can be written $(\gamma_0, \gamma_0^{-1})$. Applying the well-known fact that every complex vector bundle over S^1 is trivializable, we can let $\psi : E_0 \to S^1 \times \mathbb{C}^n$ be a global trivialization. Using the two isomorphisms above, we can construct an even isomorphism of E with two copies of the trivial bundle $S^1 \times \mathbb{C}^n$. If we equip these two copies with the flipping action of the Clifford algebra, this isomorphism intertwines the Clifford algebra action. We have to show that for all n this bundle admits another Clifford algebra action, anticommuting with the flip. To that extent, let $\gamma' : S^1 \times \mathbb{C}^n \to S^1 \times \mathbb{C}^n$ be an isomorphism. Let us assume γ' does not depend on S^1. Just as for γ, the odd action of γ' on $S^1 \times \mathbb{C}^n \oplus \mathbb{C}^n$ can be written $(\gamma', (\gamma')^{-1})$, so

$$\gamma(t, z_1, z_2) = (t, z_2, z_1), \quad \gamma'(t, z_1, z_2) = (t, (\gamma')^{-1}(z_1), \gamma'(z_2)) \quad t \in S^1, z_1, z_2 \in \mathbb{C}^n.$$

Then γ and γ' anticommute if the following two expressions are equal for all $t \in S^1$ and $z_1, z_2 \in \mathbb{C}^n$:

$$\gamma\gamma'(t, z_1, z_2) = (t, \gamma'(z_1), (\gamma')^{-1}(z_2))$$
$$-\gamma'\gamma(t, z_1, z_2) = (t, -(\gamma')^{-1}(z_1), -\gamma'(z_2)).$$

This is equivalent to $(\gamma')^2 = -1$. Taking γ' to be multiplication by i, this is clearly satisfied. We conclude that E is supertrivial and hence vanishes in K-theory.

It is not a priori clear how Definition 3.30 can be repaired. The author shares here his wild speculation that replacing the category of vector bundles by a weaker notion that behaves better categorically, might give Definition 3.30 meaning for higher degrees. An example of a weaker notion would be allowing fibers to have varying dimension, in analogy with the notion of coherent sheaves in algebraic geometry. Anyhow, a definite correct formulation in higher degree of the K-theory defined by Freed & Moore should probably be analogous to the work of Gomi [31]. First we sketch the two primary possible formulations introduced by Gomi, the first being a Fredholm-type formulation (see Atiyah and Singer [8] and Freed, Hopkins and Teleman [26]), the second being a Karoubi type formulation (see Karoubi's work [41] and [19]). Next, in order to make sure the computational techniques applied in Chapter 4 are at least mathematically correct, we will redefine our K-theory. We end this section with a short argument using the existing the literature that shows that this redefined K-theory is correct and the exposition in this document is consistent.

The idea of the Fredholm-type formulation is finding a representing space of Fredholm operators for twisted equivariant K-theory, or more precisely: a spectrum. The choice of Fredholm operators is motivated by the Atiyah-Jänich theorem and the generalization to higher degree K-theory of Atiyah and Singer [8]. In Freed, Hopkins and Teleman [26] twisted groupoid K-theory was defined using this method, which encompasses the K-theory studies in this document for trivial ϕ. It has been generalized by Gomi to study twisted groupoid K-theory for nontrivial ϕ in Paragraph 3 of [31], see Definition 3.4. Due to the twist, classes in K-theory will no longer be homotopy classes of maps, but instead of sections of a certain bundle of Fredholm operators. For this definition of K-theory, Gomi proves Bott periodicity (Lemma 3.5) and a suspension theorem (Theorem 3.8). In Appendix E of Freed & Moore, it is shown that for p, q and c trivial, Definition 3.30 agrees with the Fredholm formulation of Gomi sketched above. However, as Gomi already points out as a remark under Proposition 3.14 [31], this argument does not work if c is nontrivial. Even worse, in the remark below Proposition 3.15 he gives a counterexample for c nontrivial. It is clear that the physical arguments that lead to the definition of Freed & Moore's K-theory should be reviewed for c nontrivial. The hope is that a 'right' definition of reduced topological phases should correspond with the 'right' definition of K-theory with nontrivial c.

Another possibile formulation, that is quite hands-on and close to the formulation of Freed & Moore, is the Karoubi triple formulation. In this formulation, classes are triples consisting of a (ϕ, τ)-twisted equivariant bundle E equipped with a (nongraded) action of $Cl_{p,q}$, together with two gradings ϵ_1, ϵ_2 of E making it into a (G, ϕ, τ, c, p, q)-bundle.

Such a triple will be called *elementary* (and is to be quotiented out) if ϵ_1 is homotopic to ϵ_2 within the topological space of admissable gradings.[9] Of course, this Karoubi triple formulation can also be done in an infinite-dimensional setting and the question whether this is equivalent is answered affirmatively by Gomi, paragraph 4.3 [31]. The Karoubi formulation also provides another choice for a spectrum of K-theory; Fredholm operators can be replaced by gradations. Although the Karoubi formulation is very close to the formulation of Definition 3.30 on first sight, they do not seem to match completely.

Let us now turn to a redefinition of K-theory and an argument as to why this is a correct approach. As the above discussion suggests more complications if c is nontrivial, we reduce the definition to the case where c is trivial. We start with the assumption that Definition 3.21 (which agrees with the definition of Freed & Moore by Lemma 3.34 and with the Fredholm formulation of Gomi by the discussion in Appendix E of Freed & Moore or alternatively Corollary 4.22 in Gomi [31]) is the correct degree zero K-theory group. Using suspensions, we can then try to artificially make twisted equivariant theory into a cohomology theory analogously to how Segal approached the problem for complex equivariant K-theory (Definition 2.7 of [57]). However, for general twist such an approach would not be immediately applicable, since the reduced suspension of a pointed twisted G-space is not canonically a pointed twisted G-space. Therefore we assume that τ is constant in space.

Definition 3.62. Let (G, ϕ, τ) be a quantum symmetry group and (X, x_0) a pointed G-space, seen as a ϕ-twisted G-space using τ. The *topologically defined reduced twisted-equivariant K-theory group of degree $-n$* is

$$
{}^{\phi}\widetilde{K}_G^{\tau-n}(X) := {}^{\phi}\widetilde{K}_G^{\tau}(\Sigma^p X),
$$

Using appropriate references and the discussion above, we aim to prove that this redefinition results in an additive equivariant cohomology theory in the sense of Appendix C.3. To arrive there, we first construct long exact sequences using Segal's development of equivariant K-theory [57]. The following lemma is essential:

Lemma 3.63. *Suppose τ is an anomaly and (X, A) is a pair of G-spaces twisted by τ. Then we have half-exactness of the pair, i.e. the sequence*

$$
{}^{\phi}\widetilde{K}_G^{\tau}(X/A) \to {}^{\phi}\widetilde{K}_G^{\tau}(X) \to {}^{\phi}\widetilde{K}_G^{\tau}(A)
$$

is exact.

Proof. We start by showing that

$$
{}^{\phi}\widetilde{K}_G^{\tau}(X/A) \to {}^{\phi}K_G^{\tau}(X) \to {}^{\phi}K_G^{\tau}(A)
$$

is exact. To show that the composition is zero, note that there is a commutative square of pairs of twisted G-spaces given by

[9] It is topologized as a subset of $\operatorname{End} E$, which is equipped with the compact-open topology.

$$\begin{array}{ccc} (A,\emptyset) & \longrightarrow & (X,\emptyset) \\ \downarrow & & \downarrow \\ (A,A) & \longrightarrow & (X,A). \end{array}$$

By applying the twisted equivariant K-theory functor of a pair, we see that the composition factors through ${}^{\phi}\widetilde{K}^{\tau}_{G}(A/A) = 0$.[10]

For the converse, suppose $[E_1] - [E_2] \in \ker i^* : {}^{\phi}K^{\tau}_{G}(X) \to {}^{\phi}K^{\tau}_{G}(A)$. Using constant τ, we can find a module M over ${}^{\phi}\mathbb{C}^{\tau}G$ such that E_2 can be embedded in $M \times X$. Hence

$$[E_1] - [E_2] = [E] - [\theta_M]$$

for $E = E_1 \oplus E_2^{\perp}$. Now we get that

$$i^*([E] - [\theta_M]) = 0$$

implies that there is a (G, ϕ, τ)-bundle F such that $E \oplus F|_A \cong \theta_M \oplus F|_A$. Using an embedding into a trivial bundle $\theta_{M'}$ again, we can assume that F is a trivial bundle, so $E \oplus \theta_{M'}$ is trivial over A. By Lemma 3.60, there exists a bundle $\tilde{E}$ over X/A such that $\pi^*\tilde{E} \cong E \oplus \theta_{M'}$. Hence we see that $j^* : {}^{\phi}\widetilde{K}^{\tau}_{G}(X/A) \to {}^{\phi}\widetilde{K}^{\tau}_{G}(X/\emptyset)$ maps $[\tilde{E}] - [\theta_{M_1 \oplus M_2}(X/A)]$ to

$$\begin{aligned} j^*([\tilde{E}] - [\theta_{M'}(X/A)] + [\theta_M(X/A)]) &= [E \oplus \theta_{M'}(X)] - [\theta_{M'}(X)] + [\theta_M(X)] \\ &= [E] - [\theta_M(X)] \end{aligned}$$

as desired.

To finish the proof, consider the following commutative diagram:

$$\begin{array}{ccccc} & & {}^{\phi}\widetilde{K}^{\tau}_{G}(X) & \xrightarrow{\tilde{i}^*} & {}^{\phi}\widetilde{K}^{\tau}_{G}(A) \\ & \overset{\tilde{j}^*}{\nearrow} & \big\downarrow \iota_X & & \big\downarrow \iota_A \\ {}^{\phi}\widetilde{K}^{\tau}_{G}(X/A) & \xrightarrow{j^*} & {}^{\phi}K^{\tau}_{G}(X) & \xrightarrow{i^*} & {}^{\phi}K^{\tau}_{G}(A) \\ & & \big\downarrow i_A^* & & \big\downarrow i_A^* \\ & & {}^{\phi}K^{\tau}_{G}(\mathrm{pt}) & = & {}^{\phi}K^{\tau}_{G}(\mathrm{pt}) \end{array}$$

By Lemma 3.52, the two columns are fully exact and by the first part of this proof, the middle row is half-exact. A simple diagram chasing argument using these facts - which will now be given for completion - shows that our desired sequence is half-exact. Let $\alpha \in {}^{\phi}\widetilde{K}^{\tau}_{G}(X/A)$. Then

$$\iota_A \tilde{i}^* \tilde{j}^*(\alpha) = i^* j^*(\alpha) = 0.$$

So by injectivity of ι_A we get $\tilde{i}^*\tilde{j}^*(\alpha) = 0$, hence indeed $\tilde{i}^*\tilde{j}^* = 0$.

[10] Note that if we define $X/\emptyset := X \sqcup \mathrm{pt}$, the K-theory of a pair is also well-defined when the subspace is empty.

If $\alpha \in \ker \tilde{i}^*$ instead, then $\iota_X \alpha \in \ker i^*$ since

$$i^* \iota_X(\alpha) = \iota_A \tilde{i}^*(\alpha) = \iota_A(0) = 0.$$

Hence we get a $\beta \in {}^{\phi}\widetilde{K}_G^{\tau}(X/A)$ such that $j^*\beta = \iota_X \alpha$. But then

$$\iota_X(\tilde{j}\beta - \alpha) = \iota_X \tilde{j}^* \beta - \iota_X \alpha = j^*\beta - \iota_X \alpha = 0,$$

so that by injectivity of ι_X we get $\alpha = \tilde{j}^*\beta$ as desired. □

This lemma shows that we can artificially make twisted equivariant theory into a cohomology theory. Indeed, a standard procedure extends the reduced half-exact sequence infinitely to the left for the topologically defined reduced higher twisted-equivariant K-theory of Definition 3.62 given by ${}^{\phi}\widetilde{K}_G^{\tau-p}(X) = {}^{\phi}\widetilde{K}_G^{\tau}(\Sigma^p X)$. Then a proof of Bott periodicity (of order 8) in this context would result in the sequence closing into a 24-periodic exact sequence, just as in Real K-theory. We do not provide such a proof, but we provide the relevant references. First we briefly review the construction of the long exact sequence from the above lemma and for future reference we explicitly give the boundary map. More details can be found in Segal's work on equivariant K-theory [57]. For simplicity and applications, we reduce to the case where X is a finite G-CW complex.

To this extent, let (X, A) be a pair of finite G-CW complexes with anomaly twist τ. We want to give a map $\delta : {}^{\phi}\widetilde{K}_G^{\tau-p}(A) \to {}^{\phi}\widetilde{K}_G^{\tau-(p-1)}(X/A)$. First of all, since ${}^{\phi}\widetilde{K}_G^{\tau-p}(A) \cong {}^{\phi}\widetilde{K}_G^{\tau-(p-1)}(\Sigma A)$, it is sufficient to give a map of pointed twisted G-spaces $X/A \to \Sigma A$. For this, consider the twisted G-space $X \cup_A KA$, where we attached the reduced cone KA to X along A. By quotienting out the cone we get X/A, while by quotienting out X we get ΣA:

$$\Sigma A \cong \frac{X \cup_A KA}{X} \longleftarrow X \cup_A KA \xrightarrow{\ q\ } \frac{X \cup_A KA}{KA} \cong X/A.$$

The claim is that the quotient map $q : X \cup_A KA \longrightarrow X/A$ is a twisted G-homotopy equivalence relative to the basepoints. If this is the case, we can select a homotopy inverse of this map, compose it with the map $X \cup_A KA \to \Sigma A$ and get the desired map $X/A \to \Sigma A$. Moreover, two choices f, f' of a homotopy inverse of q would be homotopic since

$$f = f \circ id_{X/A} \sim f \circ q \circ f' \sim id_{X \cup_A KA} \circ f' = f',$$

so they induce the same map on K-theory. To show that the quotient map is a homotopy equivalence, we need some technical lemmas in order to use the proper analogue of homotopy extension properties in the context of twisted G-spaces.

Definition 3.64. Let (X, A) be a pair of twisted G-spaces. Then (X, A) is said to satisfy the *homotopy extension property* if for any twisted G-space Y, for any morphism of twisted G-spaces $F_0 : X \to Y$ and for any twisted G-homotopy $f : A \times [0,1] \to Y$ with $f_0 = F_0|_A$, there exists an extension of f to a twisted G-homotopy $F : X \times [0,1] \to Y$ that agrees with the given F_0.

Note that for an anomaly twist, a twisted G-homotopy is basically the same as a normal G-homotopy. It is well-known that G-CW pairs satisfy the homotopy extension property, see the book of May [29], Theorem 3.1.

Lemma 3.65. *Let (X, A, a_0) be a triple of twisted G-spaces with anomaly twist τ, where (X, A) satisfies the homotopy extension property. Suppose that the identity map id_A is twisted G-homotopic relative a_0 to the map $A \to A$ constantly equal to a_0. Then the quotient map $q : X \to X/A$ is a twisted G-homotopy equivalence relative basepoints.*

Proof. Let $H : A \times [0,1] \to X$ be a map of twisted G-spaces such that $H_t(a_0) = a_0, H_0(a) = a$ and $H_1(a) = a_0$. Apply the homotopy extension property to H and the map $F = id_X$. Then we get a homotopy $F : X \times [0,1] \to X$ with the following properties:

$$F_0(x) = x, \quad F_t(a_0) = a_0, \quad F_t(A) \subseteq A, \quad F_1(A) = \{a_0\}.$$

So F is a kind of deformation retraction relative to a_0, with corresponding retraction F_1.

We expect $f : X/A \to X$ given by $f([x]) = F_1(x)$ to be the homotopy inverse of the quotient map. The map is well-defined, for if $a, a' \in A$, then $F_1(a) = a_0 = F_0(a)$. In particular f maps basepoint to basepoint. Clearly f is a map of twisted G-spaces as well.

It will now be shown that f is a homotopy inverse of q. First of all

$$fq(x) = F_1(x),$$

which is homotopic to id_X via F. On the other hand we have $qf(x) = [F_1(x)]$. So we define $\bar{F} : X/A \times [0,1] \to X/A$ by $\bar{F}_t([x]) := [F_t(x)]$. This is well-defined for if $a, a' \in A$ then $[F_t(a)] = [F_t(a')]$ is the basepoint of X/A. Hence $\bar{F}$ is a well-defined homotopy between $id_{X/A}$ and qf. □

Corollary 3.66. *Let (X, A, a_0) be a G-CW triple. The quotient map $X \cup_A KA \to X/A$ is a twisted G-homotopy equivalence.*

Proof. We show that the triple $(X \cup_A KA, KA, a_0)$ satisfies the conditions of Lemma 3.65. Since $(X \cup_A KA, KA)$ can be made into a G-CW pair, it satisfies the homotopy extension property. Let $H : KA \times [0,1] \to KA$ be the map $H_t[a, s] = [a, ts]$, where $a \in A$ and $t, s \in [0,1]$. This is a well-defined map of pointed G-spaces, because $(a, 0)$ is mapped to itself and any representative (a_0, s) of the basepoint is mapped to a representative of the basepoint (a_0, st). Moreover, $H_1 = id_A$ and H_0 is identically equal to the basepoint. □

Proposition 3.67. *Let (X, A, a) be a G-CW triple and τ an anomaly. Then there exists a long exact sequence*

$$\ldots \xrightarrow{\delta} {}^{\phi}\widetilde{K}_G^{\tau-p}(X,A) \xrightarrow{k} {}^{\phi}\widetilde{K}_G^{\tau-p}(X) \xrightarrow{i} {}^{\phi}\widetilde{K}_G^{\tau-p}(A) \xrightarrow{\delta} {}^{\phi}\widetilde{K}_G^{\tau-p+1}(X,A) \xrightarrow{k} \ldots \xrightarrow{i} {}^{\phi}\widetilde{K}_G^{\tau}(A)$$

that is natural with respect to maps $(X, A, a) \to (Y, B, b)$ of triples.

The proof of this proposition is analogous to the proof for equivariant K-theory of Segal, see the discussion under Proposition 2.6 in [57].

We now prove additivity of K-theory. However, let us first show how to take wedge sums of general twisted spaces:

Lemma 3.68. *Let $\{(X_i, x_i, \tau_i) : i \in I\}$ be a collection of pointed twisted G-spaces such that $(\tau_i)_{x_i} = (\tau_j)_{x_j}$ for every $i, j \in I$. Then:*

1. *there exists a unique pointed twisted G-space*

$$\left(\bigvee_{i\in I} X_i, \bigvee_{i\in I} \tau_i, x\right)$$

such that the inclusion maps $\iota_i : X_i \hookrightarrow \bigvee_j X_j$ are morphisms of twisted spaces;

2. *given a collection of $(G, \phi, \tau_i, c, p, q)$-bundles $\{E_i : i \in I\}$ over (X_i, τ_i) and identifications of the fibers $\psi_i : (E_i)_{x_i} \xrightarrow{\sim} M$ with a fixed supermodule M, there is a unique (G, ϕ, τ, c, p, q)-bundle $\bigvee_i E_i$ over $(\bigvee_i X_i, \bigvee_i \tau_i, x)$ such that $\bigvee_i E_i|_{E_j} = E_j$ for all $j \in I$.*

Proof. 1. We consider the pair of twisted G-spaces (X', A, τ) given by

$$\left(\bigsqcup_{i\in I} X_i, \{x_i : i \in I\}, \bigsqcup_{i\in I} \tau_i\right),$$

where $\bigsqcup_i \tau_i$ is defined in an obvious way. Then there is an obvious (G, ϕ, τ, c, p, q)-bundle E' over X' such that $E'|_{X_i} = E_i$. As we are given an equality $(\tau_i)_{x_i} = (\tau_j)_{x_j}$, the twist $\tau|_A$ is an anomaly. Hence Lemma 3.57 gives a canonical twist over $X := X'/A = \bigvee_i X_i$ such that $\iota_i : X_i \hookrightarrow X$ are morphisms of twisted G-spaces.

2. Since the given identifications ψ_i of the fibers form exactly a trivialization of E' over A, Lemma 3.60 gives a bundle E over X such that the pull-back under the quotient map equals E'. Clearly $\bigvee_i E_i|_{E_j} = E_j$ for all $j \in I$.

□

Corollary 3.69. *Let τ be an anomaly and let $(X_1, x_1), (X_2, x_2)$ be pointed G-spaces. Then there is an isomorphism of abelian groups*

$${}^{\phi}\widetilde{K}_G^{\tau_1}(X_1) \oplus {}^{\phi}\widetilde{K}_G^{\tau_2}(X_2) \cong {}^{\phi}\widetilde{K}_G^{\tau}(X_1 \vee X_2).$$

Proof. Let $([E_1], [E_2]) \in {}^{\phi}\widetilde{K}_G^{\tau_1}(X_1) \oplus {}^{\phi}\widetilde{K}_G^{\tau_2}(X_2)$ be a pair of stable equivalence classes. By adding well-chosen copies of the trivial bundle, we can assume that E_1 has the same fiber at x_1 as E_2 has at x_2. Now consider the map ${}^{\phi}\widetilde{K}_G^{\tau}(X_1) \oplus {}^{\phi}\widetilde{K}_G^{\tau}(X_2) \to {}^{\phi}\widetilde{K}_G^{\tau}(X_1 \vee X_2)$ given by mapping (E_1, E_2) to the bundle $E_1 \vee E_2$ over $X_1 \vee X_2$ given in part 2 of Lemma 3.68. It is easy to see that this construction yields a stable equivalence class $[E_1 \vee E_2]$ independent of the stable isomorphism class of E_1 and E_2. By the uniqueness condition in Lemma 3.68, the constructed map is an isomorphism. □

Theorem 3.70. *The topologically defined reduced twisted-equivariant K-theory group is periodic of degree 8 in n, so that the theory extends to positive degrees. These groups naturally form a reduced equivariant cohomology theory and there is a corresponding nonreduced equivariant cohomology theory.*

Proof. By Proposition 3.14 of Gomi [31], ${}^{\phi}K_G^{\tau}(X)$ agrees with his Fredholm formulation of K-theory in degree zero. In Theorem 3.8 he proves that his formulation satisfies the suspension axiom. Hence it agrees with our Definition 3.62 of topologically defined K-theory in all degrees. Finally, Lemma 3.5 proves that his definition of K-theory satisfies Bott periodicity of degree 8. We can immediately conclude that topologically defined K-theory is periodic of degree 8 and we can define topologically defined K-theory in positive degrees using this periodicity. A reduced equivariant cohomology theory can be artificially made into a nonreduced equivariant cohomology theory by defining ${}^{\phi}K_G^{\tau-n}(X,A) := {}^{\phi}\widetilde{K}_G^{\tau-n}(X/A)$. It is now immediate from the results of this section that this is a finitely additive equivariant cohomology theory in the sense of Definition C.7. □

In the rest of this thesis we will assume the following conjecture. The conjecture is explicitely shown to hold in real and complex K-theory in the work of Atiyah, Bott and Shapiro [4]. It can also be shown to hold for nontwisted real and complex equivariant K-theory using the results of Section 8 of Atiyah and Segal's work on the completion theorem [7].

Conjecture 3.71. *The topologically defined nonreduced higher twisted-equivariant K-theory of a point equals the higher representation ring of the twisted group algebra in the sense of Definition 1.26. In other words, it agrees with Definition 3.30 for a point.*

The best approach to a proof of this conjecture is probably to use a Karoubi-type classifying space as in Gomi [31] Section 4, as gradations seem to be closer to the formulation of Definition 3.30. However, the possibility of an important role that Fredholm operators could play is not excluded. The Karoubi formulation is defined as homotopy classes of sections of a bundle of certain gradations on a universal Hilbert bundle. Let us sketch the definition for the K-theory of a point in this formulation. Let $A := {}^{\phi}\mathbb{C}^{\tau}G\hat{\otimes}_{\mathbb{R}}Cl_{p,q}$ be the twisted group algebra together with the appropriate number of Clifford actions. In case the space is a point, the universal Hilbert bundle is a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$ that is an infinite-dimensional supermodule over A which contains all isomorphism classes of finite-dimensional supermodules over A. The set of gradations $\mathrm{Gr}(\mathcal{H})$ then consists of direct sum decompositions of $\mathcal{H}$ as a graded A-module that only differ by a compact operator from the direct sum decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$[11]. This compact difference is expected to give a designated finite-dimensional supermodule M over A. To get a map to the representation ring, it then has to be shown that two such gradations are homotopic if and only if M is supertrivial. The proof of the above conjecture is postponed to later work.

3.9. Local Algebraic Structure of K-theory

We now determine the value of our original twisted equivariant K-theory (Definition 3.30) on the twisted orbit category $\mathcal{O}_G^t \subseteq \mathbf{Top}_G^t$, which is the subcategory of twisted spaces

[11] In the sense that the difference between the grading operators that assign $+1$ to the first summand and -1 to the second summand are compact.

for which the underlying space is a homogeneous space G/H of cosets with the obvious action. The restriction to this category is essential for computations involving the spectral sequence, since the coefficients of the Bredon cohomology appearing on the second page are equal to this restriction. It is conjectured that for a point, Definition 3.30 agrees with Definition 3.62. For starters: just as in equivariant K-theory of Segal, the K-theory of homogeneous spaces can be reduced to the K-theory of a point equivariant over the stabilizer group. We restrict to anomaly twist for simplicity.

Lemma 3.72. *Let G be a finite group with homomorphisms $\phi, c : G \to \mathbb{Z}_2$ and $H \subseteq G$ a subgroup. Let $\tau \in Z^2(G, U(1)_\phi)$ be an anomaly, making G/H into a twisted G-space. The map ${}^{\phi}K_G^{\tau,c+(p,q)}(G/H) \to {}^{\phi|_H}K_H^{\tau|_H,c|_H+(p,q)}(\mathrm{pt})$ given by taking the fiber at H of a twisted G-equivariant bundle over G/H and seeing it as a twisted representation of H is an isomorphism.*

Proof. The map is well-defined as it can be defined as the composition

$$ {}^{\phi}K_G^{\tau,c+(p,q)}(G/H) \xrightarrow{j^*} {}^{\phi|_H}K_H^{\tau|_H,c|_H+(p,q)}(G/H) \xrightarrow{i^*} {}^{\phi|_H}K_H^{\tau|_H,c|_H+(p,q)}(\mathrm{pt}), $$

where $j : H \hookrightarrow G$ is the inclusion homomorphism and $i : (\mathrm{pt}, \tau|_H) \to (G/H, \tau|_H)$ the inclusion of a twisted point that is a map of twisted H-spaces as H acts trivially on G/H. We show that the map is injective and surjective.

For injectivity, suppose that (E, ρ) is a (ϕ, τ)-twisted c-graded G-equivariant bundle over G/H with $Cl_{p,0}$-action such that the fiber E_H is supertrivial as a $\tau|_H$-twisted $c|_H$-graded representation of H. To show is that (E, ρ) is supertrivial as a (ϕ, τ)-twisted c-graded equivariant bundle with $Cl_{p,0}$-action. Let $\varphi : E_H \to E_H$ be an odd automorphism of $\tau|_H$-twisted $c|_H$-graded representations squaring to one. Define a family of maps $\psi_g : E_{gH} \to E_{gH}$ for every $g \in G$ by

$$ \psi_g = c(g)\rho_H(g)\varphi\rho_H(g)^{-1}, $$

where $\rho_H(g) : E_H \to E_{gH}$ is given by the twisted G-action on E. Then the ψ_g are odd complex linear maps that anti-commute with the Clifford algebra action. Obviously ψ_g squares to one. Also, ψ_g only depends on the coset gH, for if $h \in H$, then

$$ \begin{aligned} \psi_{gh} &= c(gh)\rho_H(gh)\varphi\rho_H(gh)^{-1} \\ &= c(g)c(h)\tau_{gH}(g,h)^{-1}\rho_H(g)\rho_H(h)\varphi\rho_H(h)^{-1}\rho_H(g)^{-1}\tau_{gH}(g,h) \\ &= c(g)c(h)\tau_{gH}(g,h)^{-1}\rho_H(g)c(h)\varphi\rho_H(g)^{-1}\tau_{gH}(g,h) \\ &= c(g)\rho_H(g)\varphi\rho_H(g)^{-1} = \psi_g. \end{aligned} $$

Moreover if $g_1, g_2 \in G$, then the following computation:

$$\begin{aligned}
\psi_{g_1 g_2}\rho_{g_2 H}(g_1) &= c(g_1g_2)\rho_H(g_1g_2)\varphi\rho_H(g_1g_2)^{-1}\rho_{g_2H}(g_1) \\
&= c(g_1g_2)\rho_H(g_1g_2)\varphi\left(\tau_{g_1g_2H}(g_1,g_2)^{-1}\rho_{g_2H}(g_1)\rho_H(g_2)\right)^{-1}\rho_{g_2H}(g_1) \\
&= c(g_1g_2)\rho_H(g_1g_2)\varphi\rho_H(g_2)^{-1}\tau_{g_1g_2H}(g_1,g_2)^{\phi(g_1)} \\
&= c(g_1)c(g_2)\tau_{g_1g_2H}(g_1,g_2)^{-1}\rho_{g_2H}(g_1)\rho_H(g_2)\varphi\rho_H(g_2)^{-1}\tau_{g_1g_2H}(g_1,g_2)^{\phi(g_1)} \\
&= c(g_1)\rho_{g_2H}(g_1)c(g_2)\rho_H(g_2)\varphi\rho_H(g_2)^{-1} \\
&= c(g_1)\rho_{g_2H}(g_1)\psi_{g_2}
\end{aligned}$$

shows that ψ_g graded commutes with the projective G-action. Hence the collection $\{\psi_g\}$ for g ranging over a full set of representatives G/H, forms an odd twisted equivariant bundle automorphism $\psi : E \to E$ squaring to one.

For surjectivity, let M be a $c|_H$-graded $\tau|_H$-twisted representation of H with $Cl_{p,q}$-action, which we can equivalently see as a supermodule over the real superalgebra ${}^{\phi|_H}\mathbb{C}^{\tau|_H, c|_H+(p,q)}(H)$, the twisted group algebra. We want to construct a c-graded (ϕ,τ)-twisted G-equivariant vector bundle with $Cl_{p,q}$-action over G/H such that the fiber over H - seen as a twisted representation of H - gives us M back. Define the induced supermodule over ${}^{\phi}\mathbb{C}^{\tau,c+(p,q)}(G)$ by

$$M' := {}^{\phi}\mathbb{C}^{\tau,c+(p,q)}(G)\hat{\otimes}_{{}^{\phi|_H}\mathbb{C}^{\tau|_H,c|_H+(p,q)}(H)}M,$$

where we see ${}^{\phi}\mathbb{C}^{\tau,c+(p,q)}(G)$ as a $({}^{\phi}\mathbb{C}^{\tau,c+(p,q)}(G), {}^{\phi|_H}\mathbb{C}^{\tau|_H,c|_H+(p,q)}(H))$-superbimodule via the inclusion superalgebra morphism ${}^{\phi|_H}\mathbb{C}^{\tau|_H,c|_H+(p,q)}(H) \hookrightarrow {}^{\phi}\mathbb{C}^{\tau,c+(p,q)}(G)$ (note the assumption of constant τ). Then M' becomes a ${}^{\phi}\mathbb{C}^{\tau,c+(p,q)}(G)$-module. We construct a twisted G-equivariant bundle E over G/H from M' by declaring the fibers to be

$$E_{gH} := \mathbb{C}x_g\hat{\otimes}M \subseteq M',$$

where we omitted the subscript of the tensor product for readability. This is a well-defined complex vector bundle over G/H since

$$\mathbb{C}x_{gh}\hat{\otimes}M = \mathbb{C}\tau(g,h)^{-1}x_gx_h\hat{\otimes}M = \mathbb{C}x_g\hat{\otimes}x_hM = \mathbb{C}x_g\hat{\otimes}M.$$

Moreover, E comes equipped with a natural projective action of g given by multiplication with x_g. Note that the action of g_1 maps E_{g_2H} into $E_{g_1g_2H}$:

$$x_{g_1}\cdot\mathbb{C}x_{g_2}\hat{\otimes}M = \mathbb{C}x_{g_1g_2}\hat{\otimes}M.$$

It is easy to see that g acts anti-linearly if $\phi(g) = -1$ and linearily if $\phi(g) = 1$. By definition of the grading of the graded tensor product, g acts by an even map if $c(g) = 1$ and by an odd map if $c(g) = -1$. Also clearly the action of G on the bundle is projective with cocycle τ by construction. Finally, it is easy to show that the ${}^{\phi|_H}\mathbb{C}^{\tau|_H,c|_H+(p,q)}(H)$-supermodule ${}^{\phi|_H}\mathbb{C}^{\tau|_H,c|_H+(p,q)}(H)\hat{\otimes}_{{}^{\phi|_H}\mathbb{C}^{\tau|_H,c|_H+(p,q)}(H)}M$ is isomorphic to M.

□

Proposition 3.73. *Under applying the isomorphism of Lemma 3.72, the value of the twisted equivariant K-theory functor on the twisted orbit category*

$$^{\phi}\mathcal{R}^{\tau,c+(p,q)} := {}^{\phi}K_G^{\tau,c+(p,q)}|_{\mathcal{O}_G^t} : \mathcal{O}_G^t \to \mathbf{Ab}$$

is given as follows:[12]

$$\begin{aligned} {}^{\phi}\mathcal{R}^{\tau,c+(p,q)}(G/H) &= {}^{\phi}R^{\tau,c+(p,q)}(H) \\ {}^{\phi}\mathcal{R}^{\tau,c+(p,q)}(G/H \overset{\pi}{\to} G/K) &= {}^{\phi}R^{\tau,c+(p,q)}(K) \overset{rest}{\to} {}^{\phi}R^{\tau,c+(p,q)}(H) \\ {}^{\phi}\mathcal{R}^{\tau,c+(p,q)}(G/(gHg^{-1}) \overset{\hat{g}}{\to} G/H) &= {}^{\phi}R^{\tau,c+(p,q)}(H) \overset{x_g}{\to} {}^{\phi}R^{\tau,c+(p,q)}(gHg^{-1}), \end{aligned}$$

where $H \subseteq K \subseteq G$ are subgroups and the restriction of twisting data to subgroups of G is omitted for readability. The maps are given as follows:

- *$\pi\colon G/H \to G/K$ is the quotient map;*
- *$\hat{g}\colon G/(gHg^{-1}) \to G/H$ is given by right-multiplication with some fixed $g \in G$;*
- *$rest\colon {}^{\phi}R^{\tau,c+(p,q)}(K) \to {}^{\phi}R^{\tau,c+(p,q)}(H)$ is the map induced by restriction of supermodules to the supersubalgebra ${}^{\phi}\mathbb{C}^{\tau,c+(p,q)}H \subseteq {}^{\phi}\mathbb{C}^{\tau,c+(p,q)}K$;*
- *$x_g\colon {}^{\phi}R^{\tau,c+(p,q)}(H) \to {}^{\phi}R^{\tau,c+(p,q)}(gHg^{-1})$ is the map induced by the map on supermodules given by $M \mapsto x_gM$.*

Proof. The value of ${}^{\phi}\mathcal{R}^{\tau,c+(p,q)}$ on objects follows immediately by Lemma 3.72. For the value on quotient maps $G/H \to G/K$, let $[M] \in {}^{\phi}R^{\tau,c+(p,q)}(K)$. Considering the proof of Lemma 3.72, we make M into a $(G,\phi,\tau,c,p,q)-$bundle over G/K by taking the fiber over gK to be the expression $x_gM := \mathbb{C}x_g\hat{\otimes}_{{}^{\phi}\mathbb{C}^{\tau,c+(p,q)}K}M$. Pulling back this bundle under $G/H \to G/K$ and taking the fiber at H now gives the module M restricted to the subalgebra ${}^{\phi}\mathbb{C}^{\tau,c+(p,q)}H \subseteq {}^{\phi}\mathbb{C}^{\tau,c+(p,q)}K$.

Now consider the (left) G-map $\hat{g} : G/(gHg^{-1}) \to G/H$ given by $\hat{g}(agHg^{-1}) = agH$. To deduce its induced map on representation rings, let $[M] \in {}^{\phi}R^{\tau,c+(p,q)}(H)$ and consider again the induced bundle (E,ρ) over G/H with fibers $E_{gH} = x_gM$. Then the pull-back of this bundle under $\hat{g}$ gives a bundle over $G/(gHg^{-1})$ with

$$\begin{aligned} &(\hat{g}^*E)_{g_1gHg^{-1}} = E_{g_1gH} = x_{g_1g}M \\ &(\hat{g}^*\rho)_{g_1gHg^{-1}}(g_2) = \rho_{g_1gH}(g_2) : x_{g_1g}M = (\hat{g}^*E)_{g_1gHg^{-1}} \to (\hat{g}^*E)_{g_2g_1gHg^{-1}} = x_{g_2g_1g}M \end{aligned}$$

Then we take the fiber of $\hat{g}^*E$ at gHg^{-1} to get the supermodule x_gM over $\mathbb{C}^{\tau,c+(p,q)}(gHg^{-1})$ given by

$$\begin{aligned} i(x_gm) &= x_g(\phi(g)im) \\ x_gx_hx_g^{-1}(x_gm) &= x_g(x_hm) \\ \gamma(x_gm) &= x_g(c(g)\gamma m) \\ |x_gm| &= c(g)|m|. \end{aligned}$$

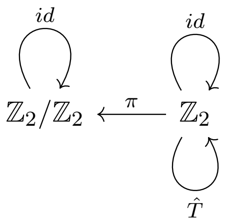
□

[12] All morphisms of the twisted orbit category are compositions of these two types of maps, see Appendix C.2.

Remark 3.74. In equivariant K-theory the maps induced by right multiplication by $g \in G$ are just conjugation of representations by g. One may therefore a priori expect that in general these maps are trivial under identification of the representation theory of conjugate subgroups. However, the action can be nontrivial in a subtle way, mainly due to ϕ and c, as can be seen in the next example.

Example 3.75. Unexpected behavior already appears in the simple example in which $G = \mathbb{Z}_2$, c is trivial and ϕ is nontrivial. We determine the functor ${}^{\phi}\mathcal{R}_G^{\tau+(p,q)} : \mathcal{O}_G \to \mathbf{Ab}$ for certain p, q. It is shown in Lemma 1.24 that $A := {}^{\phi}\mathbb{C}^{\tau}G$ is $M_2(\mathbb{R})$ if the twist is trivial and $\mathbb{H}$ if the twist is nontrivial.

There are two subgroups of G and the orbit category of G looks as follows:



The subalgebra corresponding to the trivial subgroup is $\mathbb{C}$ embedded in A in the obvious way. Corollary B.24 gives the (ϕ, τ)-twisted representation rings in different degrees. This determines the (ϕ, τ)-twisted G-equivariant K-theory on the objects of the orbit category, using Proposition 3.73:

$$ {}^{\phi}\mathcal{R}_G^{\tau+(p,q)}(G/H) = {}^{\phi|_H}R^{\tau|_H+(p,q)}(H). $$

The result is summarized in the following table, where τ_0 is the trivial twist and τ_1 is the nontrivial twist. By Morita equivalence we have the equality ${}^{\phi}\mathcal{R}_G^{\tau+(p+1,q+1)} = {}^{\phi}\mathcal{R}_G^{\tau+(p,q)}$ so that we can write ${}^{\phi}\mathcal{R}_G^{\tau+(p,q)}$ as ${}^{\phi}\mathcal{R}_G^{\tau+p-q}$ by abuse of notation, see Corollary B.22.

	${}^{\phi}\mathcal{R}^{\tau_0-p}(\mathbb{Z}_2/\mathbb{Z}_2)$	${}^{\phi}\mathcal{R}^{\tau_1-p}(\mathbb{Z}_2/\mathbb{Z}_2)$	${}^{\phi}\mathcal{R}^{\tau_0-p}(\mathbb{Z}_2)$	${}^{\phi}\mathcal{R}^{\tau_1-p}(\mathbb{Z}_2)$
$p = 0$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$p = 1$	$\mathbb{Z}_2$	0	0	0
$p = 2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$
$p = 3$	0	0	0	0
$p = 4$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$p = 5$	0	$\mathbb{Z}_2$	0	0
$p = 6$	0	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$
$p = 7$	0	0	0	0
$p = 8$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$
$\vdots$				

What does the K-theory functor do with the two nontrivial morphisms in the orbit category? For this we consult Proposition 3.73 again. First of all, the map ${}^{\phi}\mathcal{R}_G^{\tau-p}(\pi)$:

$R^pA \to R^p\mathbb{C}$ is induced by restricting modules over $A^{0,p}$ to modules over $\mathbb{C}\hat{\otimes}_{\mathbb{R}}Cl_{0,p} \cong \mathbb{C}l_p$. In particular, for $p = 0$ this maps a module over A to its underlying complex vector space. We see that if $\tau = \tau_0$ so that $A = M_2(\mathbb{R})$, the map is the identity for $p = 0$, while for $\tau = \tau_1$, so that $A = \mathbb{H}$, the map is multiplication by two. For $p = -1, -2, -3$ we see that the map is always trivial for algebraic reasons, see the table above.

So far, nothing very remarkable has occurred; just as in the case where ϕ is trivial, quotient maps in the orbit category give restrictions of representations in the representation rings. But the nontrivial automorphism of the object $G \in \mathcal{O}_G$ gives something unexpected. Firstly note that according to Proposition 3.73, this automorphism on the level of representation rings is induced by the map $M \mapsto \bar{M}$, where $\bar{M}$ is the module over $\mathbb{C}l_p$ that is only altered by the sign of multiplication by i. So for $p = 0$, we get the map from isomorphism classes of complex vector spaces to itself given by $V \mapsto \bar{V}$. This map is clearly the identity, hence giving that the induced automorphism $R^0(\mathbb{C}) \cong \mathbb{Z} \to \mathbb{Z}R^0(\mathbb{C})$ is just the identity map.

However, it will now be shown that for $p = 2$, the map $M \mapsto \bar{M}$ induces a nontrivial automorphism of $R^{-2}(\mathbb{C}) \to R^{-2}(\mathbb{C})$. For this, note first that the even isomorphism $\mathbb{C}l_2 \cong M_{1|1}(\mathbb{C})$ can be realized by

$$i \mapsto \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} \quad \gamma_1 \mapsto \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \gamma_2 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $\gamma_1, \gamma_2 \in \mathbb{C}l_2$ are Clifford generators. Indeed, γ_1 and γ_2 are mapped to odd elements, square to unity and anticommute. Now let $M = M_0 \oplus M_1$ be the standard irreducible module of $M_{1|1}(\mathbb{C})$ in the sense that $M_0 = \mathbb{C}, M_1 = \mathbb{C}$ and the action ρ is just ordinary matrix multiplication. Let $(\bar{M}, \bar{\rho})$ be the complex conjugate module. Using the decomposition

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

we see that

$$\begin{aligned} \bar{\rho}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &= \rho\left(-\frac{1}{2}\begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= \rho\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \end{aligned}$$

Completely analogously, one can see that

$$\bar{\rho}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \rho\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore $\bar{M}$ is isomorphic to the oppositely graded module $\hat{M}$, which in particular is the inverse of M in the representation ring of $M_{1|1}(\mathbb{C})$. Since M generates the representation ring, we conclude that the automorphism on $R^{-2}\mathbb{C} \cong \mathbb{Z}$ is given by -1.

3.10. Equivariant James Splitting and Thom Isomorphism

Given the specific product form of our space of interest $\mathbb{T}^d$, it would be reasonable to find a Künneth-type formula to reduce K-theory computations to computations on circles. This turns out to be at least partially true and the idea is based on the following result, that is taken from the talk [53] of Royer:

Lemma 3.76. *Let X be a based G-space. Then there is a $G^d \rtimes S_d$-equivariant stable homotopy equivalence*

$$X^d \sim_s \bigvee_{n=1}^{d} \binom{d}{n} X^{\wedge n}.$$

The following corollary is obvious and consistent with Theorem 11.8 of Freed & Moore.

Corollary 3.77. *Suppose the action of G on the Brillouin zone $\mathbb{T}^d$ can be realized as the restriction of the action of $H^d \rtimes S_d$ on $\mathbb{T}^d$ obtained by some action of H on $\mathbb{T}^1$. Then there is a G-equivariant stable homotopy equivalence*

$$\mathbb{T}^d \sim_s \begin{cases} S^1 \vee S^1 \vee S^2 & d = 2, \\ S^1 \vee S^1 \vee S^1 \vee S^2 \vee S^2 \vee S^2 \vee S^3 & d = 3. \end{cases}$$

A reasonable idea to determine the twisted equivariant K-theory of the torus is first to compute it for representation spheres S^V - i.e. the G-spaces that are one point compactifications of real representations of G - and then use the corollary above. To illustrate this, suppose that we consider a definition of twisted equivariant K-theory that is invariant under stable homotopy, i.e. a definition that satisfies the suspension axiom and homotopy invariance. Let us consider topologically defined twisted-equivariant K-theory of Definition 3.62, for which Bott periodicity holds by Theorem 3.70 and homotopy invariance holds by Corollary 3.48. Consider $d = 2$ for simplicity and assume the action of G on the Brillouin zone $\mathbb{T}^2$ can be realized as the restriction of the action of $H^2 \times \mathbb{Z}_2$ on $\mathbb{T}^2$ obtained by some action of H on $\mathbb{T}^1$. If the exact sequence of Lemma 3.52 splits (for example, if we have a quantum anomaly twists), then it follows by Corollary 3.69 that

$$\begin{aligned} {}^{\phi}K_G^{\tau-n}(\mathbb{T}^2) &\cong {}^{\phi}K_G^{\tau-n}(\mathrm{pt}) \oplus {}^{\phi}\widetilde{K}_G^{\tau-n}(\mathbb{T}^2) \\ &\cong {}^{\phi}K_G^{\tau-n}(\mathrm{pt}) \oplus {}^{\phi}\widetilde{K}_G^{\tau-n}(S^2) \oplus {}^{\phi}\widetilde{K}_G^{\tau-n}(S^1 \vee S^1). \end{aligned}$$

The last K-group however, does not split in general if G is not contained in H^2.[13] If $G \subseteq H^2$ we can determine the twisted equivariant K-theory of the torus if we know it for representation spheres, at least for simple actions:

$$\begin{aligned} {}^{\phi}K_G^{\tau-n}(\mathbb{T}^2) \cong {}^{\phi}K_G^{\tau-n}(\mathrm{pt}) &\oplus {}^{\phi}\widetilde{K}_G^{\tau-n}(S^2) \\ &\oplus {}^{\phi}\widetilde{K}_G^{\tau-n}(S^1) \oplus {}^{\phi}\widetilde{K}_G^{\tau-n}(S^1). \end{aligned}$$

[13] Consider for example $p4$ in which the action of the point group $\mathbb{Z}_4$ is given by rotation by $\pi/2$, so $(k_1, k_2) \mapsto (-k_2, k_1)$. Then the group action on the wedge of the two circles $S^1 \vee S^1$ intertwines the two copies, so that we cannot split the K-theory further.

Of course there are similar statements in the case $d = 3$.

Let us illustrate why the above reasoning asks for an analogue of the Thom isomorphism for twisted equivariant bundles, of which the following is a version in complex nontwisted equivariant K-theory:

Theorem 3.78 (Segal). *Let G finite act on a compact X. Let E be a complex G-vector bundle over X. Then $K^i_G(E) \cong K^i_G(X)$.*

Proof. See Segal's paper [57]. □

By taking X to be a point, we see that

$$\widetilde{K}^i_G(S^V) = K^i_G(V) = K^i_G(\mathrm{pt}) = \begin{cases} R(G) & \text{if } i \text{ even,} \\ 0 & \text{if } i \text{ odd.} \end{cases}$$

However, in our computations of the K-theory of the torus, V can also be a real representation, so we want a real analogue of the Thom theorem. This seemingly small generalization introduces some complications, but Karoubi gives a definite answer in [39]. See also Section 4.2 on ordinary complex equivariant K-theory computations using the equivariant James splitting.

3.11. An Atiyah-Hirzebruch Spectral Sequence

In this section, the natural filtration $X^0 \subseteq \cdots \subseteq X^d = X$ on a finite G-CW complex X will be used to construct a spectral sequence converging to twisted equivariant K-theory. We follow the unpublished work of Hatcher [33]. In analogy with [26] and [9], the second page of the spectral sequence is conjectured to be a version of twisted Bredon equivariant cohomology (see Appendix C.3 for basic information on Bredon cohomology).[14]

Consider the commutative grid consisting of the long exact sequences of pairs (X^p, X^{p-1}) together with the natural restriction maps as in Figure 3.1. The 'staircases' as in the flow of the red arrow in the grid are exact, since they are exactly the long exact sequence of a pair. Moreover, all the relative K-theory groups involved in the diagram are groups of Bredon cochains as one would expect from ordinary cohomology. Indeed first note that

$$\begin{aligned} {}^\phi K^{\tau+n}_G(X^p, X^{p-1}) &= {}^\phi \widetilde{K}^{\tau+n}_G(X^p/X^{p-1}) \\ &\cong {}^\phi \widetilde{K}^{\tau+n}_G\left(\bigvee_\sigma \bigvee_{gH_\sigma} S^p_{gH_\sigma} \right). \end{aligned}$$

[14]This method should generalize to arbitrary (sufficiently nice) topological spaces using general techniques from Segal [56], [57]. For this, let ${}^\phi\mathcal{K}^{\tau+p}_G$ be the equivariant sheaf on X associated to the presheaf $U \mapsto {}^\phi K^{\tau|_U+p}_G(U)$. The second page of the Segal spectral sequence should be something like equivariant sheaf cohomology over X with values in ${}^\phi\mathcal{K}^{\tau+p}_G$. The stalks of this sheaf are $\left({}^\phi\mathcal{K}^{\tau+p}_G\right)_x = {}^\phi K^{\tau'+p}_G(G/H_x) \cong {}^\phi K^{\tau|_{H_x}+p}_{H_x}(\mathrm{pt})$ if H_x is the stabilizer of x and τ' is some unknown twist, probably related to τ_{gx} for $g \in G$ (note that this is subtle: elements $x, y \in X$ such that $gx = y$ can have different stabilizator groups, although they must be conjugate). If τ is an anomaly twist, τ' should definitely just be τ. Otherwise it will be more complicated and it should be possible that τ' is nonconstant in space.

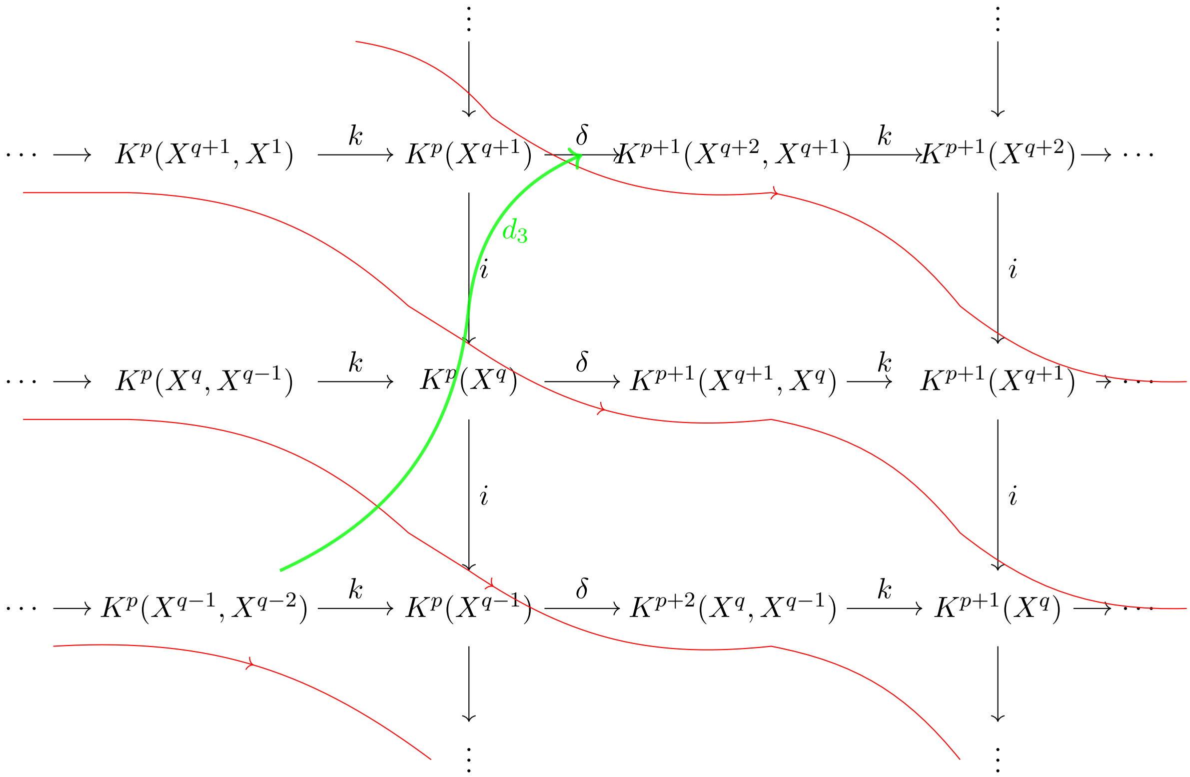


Figure 3.1.: The commutative grid of natural maps in K-theory analogous to the commutative grid in Hatcher's book on spectral sequences [33]. Following the red arrows gives the long exact sequences of pairs (X^{q+1}, X^q). The Bredon differential -which is the first differential in the spectral sequence- is $\delta \circ k$. The green arrow depicts the direction of third differential.

Here the first wedge product is over equivariant p-cells σ of X, which look like $S^p_\sigma \times G/H_\sigma$, where $H_\sigma \subseteq G$ is the stabilizer of the cell. The second wedge product is over all ordinary cells $S^p_{gH_\sigma}$ contained in the equivariant cell σ, i.e. over all cosets gH_σ of the stabilizer group. Now by additivity and the suspension axiom,

$$\begin{aligned}
{}^{\phi}K_G^{\tau+n}(X^p, X^{p-1}) &= {}^{\phi}\widetilde{K}_G^{\tau+n}\left(\bigvee_{\sigma}\bigvee_{gH_\sigma} S^p_{gH_\sigma}\right) \\
&\cong \bigoplus_{\sigma\ p\text{-cell}} {}^{\phi}\widetilde{K}_G^{\tau+n}\left(\bigvee_{gH_\sigma} S^p_{gH_\sigma}\right) \\
&= \bigoplus_{\sigma\ p\text{-cell}} {}^{\phi}\widetilde{K}_G^{\tau+n}(\Sigma^p(G/H_\sigma \sqcup *)) \\
&\cong \bigoplus_{\sigma\ p\text{-cell}} {}^{\phi}\widetilde{K}_G^{\tau+n-p}(G/H_\sigma \sqcup *) \\
&\cong \bigoplus_{\sigma\ p\text{-cell}} {}^{\phi}K_G^{\tau+n-p}(G/H_\sigma) \\
&\cong \bigoplus_{\sigma\ p\text{-cell}} {}^{\phi}K_{H_\sigma}^{\tau+n-p}(\mathrm{pt}) \\
&\cong \bigoplus_{\sigma\ p\text{-cell}} {}^{\phi}R^{\tau+n-p}(H_\sigma) \\
&\cong C^p_G(X, {}^{\phi}\mathcal{R}_G^{\tau+n-p}),
\end{aligned}$$

which is the group of Bredon cochains with coefficient functor ${}^{\phi}\mathcal{R}^{\tau,c+n-p}$. We now make an exact couple out of the grid, giving an Atiyah-Hirzebruch spectral sequence (see Appendix C.1 for basics of spectral sequences, exact couples and our grading conventions):

$$\begin{aligned}
E_1^{p,q} &= {}^{\phi}K_G^{\tau+p+q}(X^p, X^{p-1}), \\
A_1^{p,q} &= {}^{\phi}K_G^{\tau+p+q}(X^p), \\
i : A_1^{p,q} &\to A_1^{p-1,q+1} \text{ restriction}, \\
k : E_1^{p,q} &\to A_1^{p,q} \text{ extension to } X^p, \\
\delta : A_1^{p,q} &\to E_1^{p+1,q} \text{ boundary map}.
\end{aligned}$$

Recall that the differential d_1 of E_1 is defined by $d_1 = \delta k$. The derived exact couple is given as follows. Firstly, $A_2^{p,q} = \mathrm{Im}(A_1^{p+1,q-1} \to A_1^{p,q})$, i.e. the vector bundles on X^p that are restrictions of a vector bundle on X^{p+1}. Secondly, $E_2^{p,q}$ is the cohomology of $E_1^{p,q}$ with respect to d_1. We can continue taking derived exact couples in this fashion to produce the other pages of the spectral sequence, until at $E_d^{p,q}$, when the sequence collapses due to the fact that we only consider finite CW complexes. The differential of the second page is then abstractly defined as $d_2([e]) = [\delta a]$ where a is such that $ia = ke$. More concretely, the second differential of a bundle over X^p should be given as follows. Since the first differential vanishes, the bundle can be lifted to a bundle over X^{p+1}. Then the boundary map δ can be applied to get a class in relative K-theory one degree higher and one dimension higher. The degree r differential is defined similarly by first lifting a bundle over X^p to a bundle over X^{p+r-1}. See the green arrow in Figure 3.1 for a schematic representation of the third differential.

Now Proposition C.3 from Appendix C.1 will be applied in this setting to show that this spectral sequence exists. We check that the assumptions of this proposition are satisfied. First of all note that the differentials are of the right degree. Now for fixed $i \in \mathbb{Z}$ we have that for $q < i$ the group, $A^{i-q,q}$ vanishes since $X^n = \emptyset$ for $n < 0$. Also, since X is a finite CW complex, the restriction maps $X^{n+1} \to X^n$ become identities for $n > \dim X$ so that $A^{i-q,q} = A^{i-q+1,q-1} = \cdots = {}^{\phi}K_G^{\tau+p+n}(X^n)$. Therefore Proposition C.3 gives the following:

Theorem 3.79. *Let (G, ϕ, τ) be a finite quantum symmetry group and let X be a G-space, considered as a twisted G-space using the anomaly τ. Suppose X comes equipped with the structure of a finite G-CW complex*

$$\emptyset = X^{-1} \subseteq X^0 \subseteq X^1 \subseteq \cdots \subseteq X^d = X$$

where $d = \dim X$. Then there is a spectral sequence $E_r^{p,q}$ converging to ${}^{\phi}K_G^{\tau+p+q}(X)$. The corresponding filtration of the K-theory group is given by the kernels of successive restriction maps

$$\ker({}^{\phi}K_G^{\tau+p+q}(X) \to {}^{\phi}K_G^{\tau+p+q}(X^k)).$$

Higher differentials d_r are abstractly defined as follows: let $f \in C_G^p(X, {}^{\phi}\mathcal{R}_G^{\tau+q})$ be a p-cochain representing an element of $E_r^{p,q}$. Under the isomorphism with K-theory, there is a corresponding class $e \in {}^{\phi}K_G^{\tau+p+q}(X^p, X^{p-1})$. Pick a class $a \in {}^{\phi}K_G^{\tau+p+q}(X^{p+r})$ that agrees with f in the sense that $a|_{X^p}$ equals the extension of e to X^p.[15] Then $d_r f = [\delta a] \in E_r^{p+r,q+1-r}$, where

$$\delta : {}^{\phi}K_G^{\tau+p+q}(X^{p+r}) \to {}^{\phi}K_G^{\tau+p+q}(X^{p+r+1}, X^{p+r})$$

is the boundary map of the long exact sequence of the pair (X^{p+r+1}, X^{p+r}).

The following statement will be assumed to be true in the rest of this document. It is well-known in for complex twisted equivariant K-theory, see Dwyer [22].

Conjecture 3.80. *Under the isomorphism ${}^{\phi}K_G^{\tau+p+q}(X^p, X^{p-1}) \cong C^p(X, {}^{\phi}\mathcal{R}^{\tau+q})$, the first differential of the spectral sequence is mapped to the cellular Bredon differential on the corresponding G-CW structure with coefficient functor ${}^{\phi}\mathcal{R}_G^{\tau+q}$. Hence the second page is isomorphic to twisted Bredon cohomology $H_G^p(X, {}^{\phi}\mathcal{R}_G^{\tau+q})$.*

The proof of this conjecture should involve an explicit computation of the boundary map δ on the equivariant cells of a G-CW complex, in particular an explicit expression for a homotopy inverse of the quotient maps $X^n \cup_{X^{n-1}} KX^{n-1} \to X^n/X^{n-1}$ in those cases. A possible 'cheat' proof that does not give the explicit expression for the differential, would

[15] This class exists exactly because of the fact that all earlier differentials vanish on f.

be showing that d_1 induces an *ordinary* equivariant cohomology theory in the sense of Definition C.7. By a theorem of Bredon [14], this implies that the cohomology theory is naturally isomorphic to Bredon cohomology. However, we will use the conjectured explicit expression for the first differential and therefore assume the stronger statement of the conjecture above.

It would be even more interesting to find explicit expressions for higher differentials, since these seem to be unknown even in simple cases. For Real K-theory, the Atiyah-Hirzebruch spectral sequence of the form considered in the document does not seem to be well-known, let alone its higher differentials. Hence we restrict this discussion to complex K-theory, for which the second differential vanishes, since ${}^{\phi}\mathcal{R}_G^{\tau+q} = 0$ for q odd. In nonequivariant twisted complex K-theory, it is known that the third differential is a Steenrod square plus a cup product with the twisting class (see Atiyah & Segal [6]). The Steenrod square maps via an abelian group of the form $\mathbb{Z}_2^k$, so it can only measure 2-torsion. An equivalent statement for the the third differential in a more general case would be desirable. However, even for nontwisted equivariant complex K-theory, a computable expression for the third differential does not seem to be known, although Bárcenas et al. [9] do mention an abstract description in Section 4.1.

Now we will briefly consider how the spectral sequence simplifies in the case of interest for topological phases; computing the zeroth degree K-theory for X a torus of dimension at most three. Consider again the commutative grid of Figure 3.1. After taking the cohomology with respect to the first differential, we get the second page of the Atiyah-Hirzebruch spectral sequence. The relevant part of this second page is given below in the upper part of Figure 3.2. Taking the cohomology with respect to the second differential d_2, we get the the lower part of Figure 3.2. In dimension 2, this is the limit of the spectral sequence, but in $d = 3$, it is still possible that $d_3 \neq 0$ and we need to take the cohomology once more to get to the final page E_4, which is omitted in the figure.

In general this results in a filtration

$$0 = F^{4,-4} \subseteq F^{3,-3} \subseteq F^{2,-2} \subseteq F^{1,-1} \subseteq F^{0,0} = {}^{\phi}K_G^{\tau}(X),$$

that can be defined as

$$F^{p,-p} := \ker({}^{\phi}K_G^{\tau}(X) \to {}^{\phi}K_G^{\tau}(X^{p-1}))$$

The filtration has the crucial property

$$\frac{F^{p,q}}{F^{p+1,q-1}} \cong E_{\infty}^{p,q}.$$

Writing out all definitions, results in the fact that all the data of the spectral sequence (including the higher differentials) is equivalent to the following list of short exact sequences:

$$0 \to \operatorname{coker}\left(d_3^{(0,-1)} : \ker d_2^{(0,-1)} \to \operatorname{coker} d_2^{(1,-2)}\right) \to F^{2,-2} \to \operatorname{coker} d_2^{(0,-1)} \to 0$$

$$0 \to F^{2,-2} \to F^{1,-1} \to \ker d_2^{(1,-1)} \to 0$$

$$0 \to F^{1,-1} \to {}^{\phi}K_G^{\tau}(X) \to \ker\left(d_3^{(0,0)} : \ker d_2^{(0,0)} \to \operatorname{coker} d_2^{(1,-1)}\right) \to 0$$

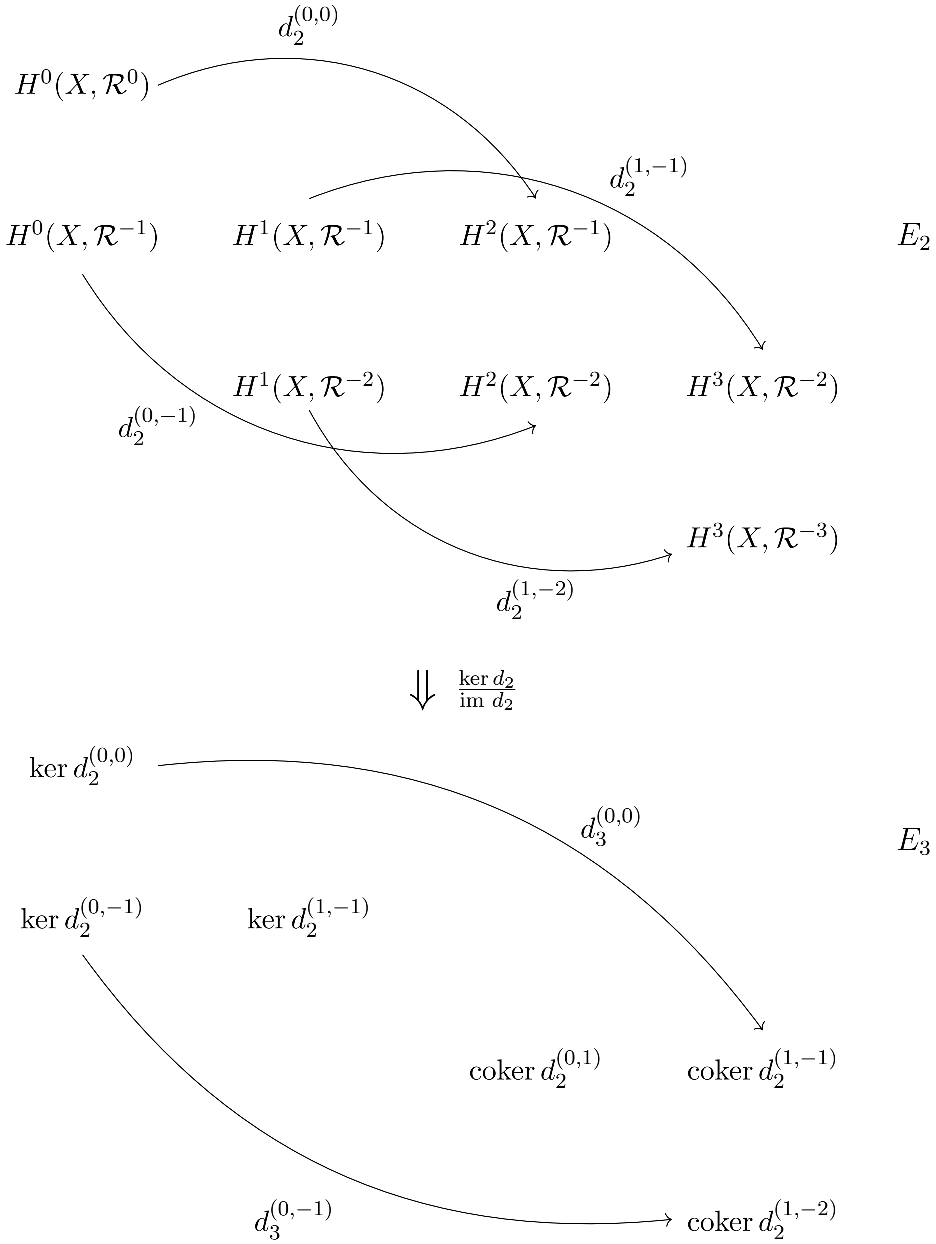


Figure 3.2.: The part of the second and third page of the Aityah-Hirzebruch spectral sequence for X of dimension at most 3 that is relevant to compute the K-theory group in degree zero. For readability, twisting labels are omitted.

Hence the K-theory can be uniquely determined by the spectral sequence (without explicitly determining the maps between K-theory groups involved) if and only if the above three sequences split. If all groups are free this splitting condition is redundant, but especially in case ϕ is nonzero, this is a very strong condition. This is because 2-torsion naturally appears on the algebraic level in the higher representation rings (e.g. $KO^{-1}(\mathrm{pt}) = \mathbb{Z}_2$). Note in particular that if all higher differentials vanish and all exact sequences split, we get a Chern-character type isomorphism

$$ {}^{\phi}K_G^{\tau}(X) \cong H_G^0(X, {}^{\phi}\mathcal{R}_G^{\tau}) \oplus H_G^1(X, {}^{\phi}\mathcal{R}_G^{\tau-1}) \oplus H_G^2(X, {}^{\phi}\mathcal{R}_G^{\tau-2}) \oplus H_G^3(X, {}^{\phi}\mathcal{R}_G^{\tau-3}). $$

In dimension two, the third differential is zero and the short exact sequences reduce to

$$ 0 \to \operatorname{coker}\left(d_2^{(0,-1)} : H_G^0(X, {}^{\phi}\mathcal{R}_G^{\tau-1}) \to H_G^2(X, {}^{\phi}\mathcal{R}_G^{\tau-2})\right) \to F^{1,-1} \to H_G^1(X, {}^{\phi}\mathcal{R}_G^{\tau-1}) \to 0 $$

$$ 0 \to F^{1,-1} \to {}^{\phi}K_G^{\tau}(X) \to \ker\left(d_2^{(0,0)} : H_G^0(X, {}^{\phi}\mathcal{R}_G^{\tau}) \to H_G^2(X, {}^{\phi}\mathcal{R}_G^{\tau-1})\right) \to 0. $$

4. Computing Topological Phases & Some Familiar K-theories

In this section, the framework from the last chapter will be applied to specific familiar settings of the condensed matter literature. On the way we reproduce the results of Kruthoff et al. [44] [43]. The theory will be mostly restricted to class A, class AI and class AII topological phases in spatial dimension $d \geq 3$, possibly protected by an extra crystal symmetry.[1] Computations are performed for symmorphic crystals in $d = 2$. A possibly interesting follow-up study would be to use the framework in this document to study topological phases of topological insulators in the other Cartan classes more carefully. We assume a perfect crystal on which we have a quantum system of free fermions without long-rang entanglement, which is invariant under lattice translations. We always take our Brillouin zone to be a torus, but the computations also apply (and simplify) for a spherical Brillouin zone. More precisely, throughout this section assume we are given some symmetry data for symmetry-protected phases in the following sense:

Definition 4.1. We define a *full set of symmetry data (for a symmetry-protected topological insulator of spatial dimension $d \leq 3$ protected extended quantum symmetry group (G, ϕ, τ_2, c))* to consist of the following data:

- a finite group G of symmetries that act on classical spacetime[2] $\mathbb{R}^{d|1}$, preserving a lattice of atoms $\mathbb{Z}^d \times \mathbb{R}_t$ fixed in time;
- a homomorphism $\phi : G \to \mathbb{Z}_2$ having the property that $\phi(g) = 1$ if the quantum operator associated to g is unitary and $\phi(g) = -1$ if it is antiunitary;
- a homomorphism $c : G \to \mathbb{Z}_2$ such that $c(g) = -1$ exactly when the symmetry of band structure induced by g is particle-hole reversing;
- a homomorphism $\beta : G \to \operatorname{Aut} \mathbb{Z}^d$, i.e. an integral representation, which represents the action of G on the crystal lattice;
- a crystal twist $\tau_1 \in H^2(G, \mathbb{Z}^d_\beta)$, which forms the data of an isomorphism class of a possibly nonsymmorphic crystal extension of (a point group part of) G by $\mathbb{Z}^d$ (which vanishes for symmorphic crystals);

[1] The reader is reminded that this means that there is no particle-hole symmetry (so no grading homomorphism c). Class A corresponds to no time reversal, class AI to time reversal squaring to 1 and class AII to time reversal squaring to -1.

[2] In our formalism, quantum symmetries are recovered from classical symmetries by a group extension using the data of an anomaly cocycle. Moreover, lattice and time translations have been quotiented out, so that G consists of point group symmetries and time reversing operations.

- finally, we have a quantum anomaly class $\tau_2 \in H^2(G, U(1)_\phi)$, containing the data of anomalous behavior of the quantum mechanical operators coming from the symmetries in G. Mathematically speaking, this data is telling us how projective G acts on the Hilbert space of quantum states through the Wigner extension.

Note that a full set of symmetry data as above is not really a quantum mechanical system in a certain phase, but instead a choice of symmetries for a perfect crystal. In other words, it specifies a Brillouin zone and for what groups Bloch states should form representations, but it does not specify anything about the topology or symmetric structure of the bands. The computation of a twisted equivariant K-theory group of the Brillouin zone answers the question of how many topologically distinct Bloch bundles can exist from purely mathematical considerations.

Since the main focus will be on topological insulators of class A,AI and AII, we state our main assumptions on the finite group G and the form of ϕ in order to suit our specific interests:

Definition 4.2. A full set of symmetry data will be called of *class A* if c and ϕ are trivial and G is a d-dimensional point group (i.e. G acts faithfully on $\mathbb{Z}^d$ via β, see Section 2.3 for more information on the form of point groups). The symmetry data will be called of *class AI* if

- c is trivial;
- $G = G_0 \times \mathbb{Z}_2$ is a d-dimensional magnetic point group[3], i.e. a d-dimensional point group G_0 together with a time reversing operation T;
- $\phi : G \to \mathbb{Z}_2$ is projection onto the second factor, so time reversal acts anti-unitarily and all other symmetries act unitarily;
- the crystal class $\tau \in H^2(G, \mathbb{Z}^d_\beta)$ has corresponding action $\beta : G_0 \times \mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z}^d)$ that is of the form $\beta(g) = \alpha(p_1(g))$, where p_1 is projection onto G_0 and α is the action of the point group G_0 on the crystal lattice;
- T squares to 1 on the quantum mechanical level in the following sense: the pull-back map $i^* : H^2(G, U(1)_\phi) \to H^2(\mathbb{Z}_2, \mathbb{Z}_\phi) \cong \mathbb{Z}_2$ maps τ_2 to the trivial element.

It is of *class AII* if all but the last of the above points hold and T squares to -1, i.e. $i^* : H^2(G, U(1)_\phi) \to H^2(\mathbb{Z}_2, \mathbb{Z}_\phi)$ maps τ_2 to the nontrivial element.

Using this data, we construct the following mathematical structures:

- The Brillouin zone $X := \mathrm{Hom}(\mathbb{Z}^d, U(1))$ is the Pontryagin dual of $\mathbb{Z}^d$, which is topologically a d-torus;
- the G-action on X given by $(g \cdot \lambda)(v) = \lambda(\alpha(g^{-1})v)^{\phi(g)} = \lambda(\beta(g)v)$. For class AI and AII data this action can be written in notation more common to the condensed matter literature as $g \cdot e^{ikr} = e^{ig(k)r}$ and $T \cdot e^{ikr} = e^{-ikr}$;

[3] See Freed & Moore Section 2 for details on magnetic point groups and a generalization to crystals that naturally include spin.

- a pointed G-CW structure on $(X, 0)^4$;
- the ϕ-twist $\tau \in H^2(G, C(X, U(1))_\phi)$ on X defined on the cocycle level by

$$\tau_\lambda(g_1, g_2) := \lambda(\tau_1(g_1, g_2))\tau_2(g_1, g_2)$$

  containing the information of the quantum anomaly twist τ_2 and the crystal twist τ_1. This twist describes how projective the representation is that should be formed by the Bloch states, which is obtained by restriction from the representation on the level of wave functions;
- The twisted equivariant K-theory group

$$^\phi K_G^{\tau,c}(X)$$

  now *classifies reduced topological phases in d dimensions protected by G with symmetry data as above.* A class in $^\phi K_G^{\tau,c}(X)$ is represented by a specific band structure over the Brillouin zone, together with the splitting into a conduction band and valence band and the structure of how Bloch waves form a (twisted) representation of G. Continuous deformations of this bundle that preserve these structures leave the class corresponding to the bundle invariant. Mathematically, for class A this K-group is a twisted version of ordinary complex equivariant K-theory and for class AI and AII it is a twisted version of KR-theory.

Example 4.3. We consider the symmorphic square crystal in $d = 2$ of type AII. It is protected by time reversal and a π-rotation symmetry such that rotation by 2π on the quantum level corresponds to a change in sign. More precisely, we set $G_0 = \mathbb{Z}_2, G = \mathbb{Z}_2 \times \mathbb{Z}_2, \phi = pr_2$. The generator of G of the first factor will be denoted r and the other one t. Define $\alpha(r)v = -v$ and let $\tau_1 \in H^2(G_0, \mathbb{Z}^2_\alpha)$ be the trivial class. It can be shown that $H^2(G, U(1)_\phi) = \mathbb{Z}_2 \times \mathbb{Z}_2$ has representative group extensions

$$1 \to U(1) \to G^\tau \to G \to 1$$

determined by whether the elements t, r of G lift to elements in G^τ satisfying $T^2 = \pm 1$ and $R^2 = \pm 1$ respectively. Moreover, we can pick the extension so that $TR = RT$. The map $i^* : \mathbb{Z}_2 \times \mathbb{Z}_2 \cong H^2(G, U(1)_\phi) \to H^2(\mathbb{Z}_2, U(1)_\phi) \cong \mathbb{Z}_2$ maps the representative group extensions above to the nontrivial class if and only if $T^2 = -1$. We take the class with $T^2 = -1$ and $R^2 = -1$, which is thus of class AII. Hence we consider the two-torus $X := ([-\pi, \pi] \times [-\pi, \pi])/\sim$ with the action of G given by $T(k) = -k$ and $R(k) = -k$. Now we want to consider complex vector bundles E over X together with two vector bundle maps $T, R : E_k \to E_{-k}$ such that R is complex linear, T is complex anti-linear, $TR = RT, T^2 = -1$ and $R^2 = -1$. In particular note that TR is a real structure on E. The Grothendieck completion of the monoid of isomorphism classes of such objects is the twisted equivariant K-theory group $^\phi K_G^\tau(X)$ that classifies topological phases. It is shown in Example 4.33 that it is either $\mathbb{Z} \oplus \mathbb{Z}_2^4$ or $\mathbb{Z} \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_4$.

The goal of the remaining sections is to compute the reduced topological phases $^\phi K_G^\tau(X)$ in a few typical cases.

[4]Because G is finite, X is a compact smooth manifold and the action is smooth, this always exists. [36]

4.1. Class A: Equivariant Chern Classes and Segal's Formula

In the coming sections, systems with broken particle-hole and time reversal symmetry will be considered. Hence throughout this section we assume $\phi = c = 1$ so that we are left with a complex twisted equivariant K-theory group. The classical example and the first ever discovered nontrivial topological phase, is the integer quantum hall effect. In our theory this can be represented by $d = 2$ without any symmetries (except of course lattice and time translations, which we always assume).[5]

Already by Thouless [65], the quantum Hall phase was described as a Chern number of the valence band over the Brillouin zone. This relates to our theory by the Chern character isomorphism in its original (nontwisted, nonequivariant) state

$$K^0(X) \otimes \mathbb{Q} \cong \bigoplus_{k \text{ even}} H^k(X, \mathbb{Q}).$$

Using the well-known fact that the cohomology ring of the torus is the alternating algebra on d variables:

$$H^k(\mathbb{T}^d, \mathbb{Z}) = \bigwedge^k \mathbb{Z}^d = \mathbb{Z}^{\binom{d}{k}},$$

this quickly gives the non-torsion part. For example, for $d = 2$:

$$K^0(\mathbb{T}^2) \otimes \mathbb{Q} \cong \mathbb{Q}^2.$$

A physicist would formulate this result by saying that the topology of the bands over a two-dimensional Brillouin zone is classified by the number of valence bands (i.e. the dimension of the even part of the Bloch bundle) and the first Chern number of it. More generally, for class A, we have Segal's formula to compute the nontorsion part, at least in the case of τ being an anomaly, i.e. a symmorphic crystal.

Lemma 4.4. *Suppose X is a finite G-CW complex and $\tau \in H^2(G, U(1))$ is an anomaly.*

- *Given $g \in G$, there is an action of $Z(g)$ on the ordinary complex K-theory group $K^0(X^g)$ given by the formula $z \cdot [E] := \alpha^*_{z^{-1}}([E])$, where $\alpha_{z^{-1}}$ is the restriction of the corresponding action map $X \to X$;*
- *the map $L^\tau_g : Z(g) \to U(1)$ given by $L^\tau_g(z) = \tau(z, g)\tau(g, z)^{-1}$ is a homomorphism, hence giving a one-dimensional representation V^τ_g of $Z(g)$.*

Proof. The proof is elementary and will be omitted. □

Theorem 4.5 (Segal's formula)**.** *With the assumptions of the lemma above,*

[5] This explanation of the quantum Hall effect is slightly oversimplified, since the quantum Hall effect can be shown to vanish unless electrons are localized. A more rigorous mathematical explanation due to Bellissard [11] realizes the Brillouin zone in slightly deformed crystals as a noncommutative torus. The K-theory of C^*-algebras and the noncommutative Chern character then provide the tools to study topological phases in nonperiodic crystals.

- *there is an isomorphism*

$$K_G^\tau(X) \otimes \mathbb{C} \cong \bigoplus_{[g]\subseteq G} (K(X^g) \otimes_{\mathbb{C}} V_g^\tau)^{Z(g)}$$

  where the direct sum is over all conjugacy classes $[g]$ *of* G *(and for every conjugacy class we have chosen a fixed representative* $g \in [g]$*) and* $(K(X^g) \otimes_{\mathbb{C}} V_g^\tau)^{Z(g)}$ *denotes the invariant elements of the representation* $K(X^g) \otimes_{\mathbb{C}} V_g^\tau$ *of* $Z(g)$*;*

- *in case the twist is trivial, the formula reduces to the more familiar formula of Segal:*

$$K_G(X) \otimes \mathbb{C} \cong \bigoplus_{[g]\subseteq G} K(X^g/Z(g)) \otimes \mathbb{C}.$$

Proof. See Theorem 7.4 in [1] for the first point. A possible reference for the second point is [35], but it is probably not the first. □

Remark 4.6. Freed, Hopkins and Teleman have proven a generalization of this formula for type A for twisted groupoids in [25].

Remark 4.7. Using the formula above and the Chern character homomorphism for ordinary complex K-theory, the computation of the non-torsion part of equivariant K-theory is reduced to computations of the representation theory of G and the cohomology of topological spaces.

Remark 4.8. To make connections with the spectral sequence approach, we also state the Bredon-cohomological version of the Segal formula, taken from Lück et al. [49], which is basically the natural setting for an equivariant Chern character isomorphism.

$$K_G(X) \otimes \mathbb{Q} \cong \bigoplus_{n \text{ even}} H_G^n(X, \mathcal{R}_G \otimes \mathbb{Q}),$$

where $\mathcal{R}_G$ is the representation ring Bredon coefficient functor, to be defined in Section 4.3.

In the following, the Segal formula is used to compute the $\mathbb{Z}$-part of a couple of class A topological phases. In other words, the nontwisted complex equivariant K-theory of $X := \mathbb{T}^d$ will be computed for a few finite groups. All computations agree with the K-theory of the reduced C^*-algebras of the corresponding space groups as in [50] and [70] and with computations using band structure combinatorics as in [44].

Example 4.9 ($d = 1$ with unitary action $k \mapsto -k$). Consider $G = \mathbb{Z}_2 = \{1, p\}$ acting on the lattice $\mathbb{Z}$ by reflection $v \mapsto -v$ around the origin. This gives the action on $X = \mathrm{Hom}(\mathbb{Z}, U(1)) \cong \mathbb{T}^1$ given by $(p \cdot \lambda)(v) = \lambda(-v) = -\lambda(v)$. Under the isomorphism of X with $\mathbb{T}^1 = [-\pi, \pi]/\sim$ this gives the action $k \mapsto -k$[6].

[6]It can be shown in a fairly general setting that the action of G on $\mathbb{T}^d$ is given by the same formula as its action on the lattice

The conjugacy classes of G are its one element subsets $\{1\}$ and $\{p\}$ and all centralizers are G. Hence we consider

$$X^p/Z(p) = \{0, \pi\}/G = \{0, \pi\},$$

which is a discrete set of two points and

$$X^1/Z(1) = \mathbb{T}^1/\mathbb{Z}_2$$

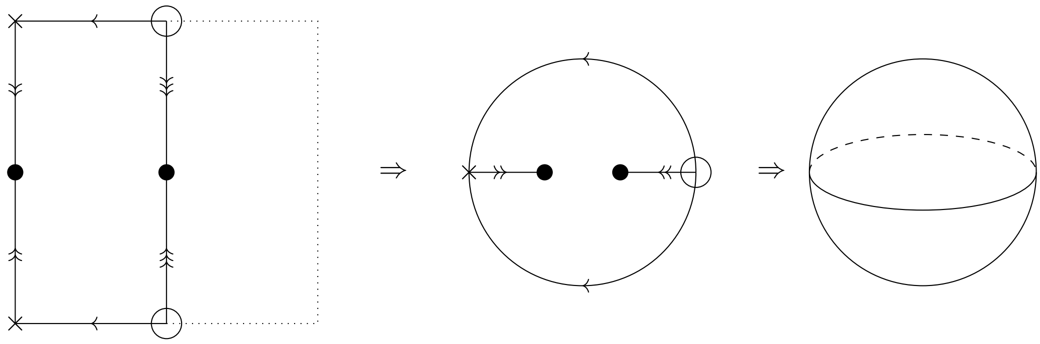
is an interval. The interval is homotopic to a point so that one can conclude by homotopy invariance that

$$K^0_{\mathbb{Z}_2}(\mathbb{T}^1) \otimes \mathbb{C} \cong H^0(\mathbb{T}^1/\mathbb{Z}_2, \mathbb{C}) \oplus H^0(\mathrm{pt} \sqcup \mathrm{pt}, \mathbb{C}) \cong \mathbb{C}^3.$$

Example 4.10 ($d = 2$ space group p_2). $G = \mathbb{Z}_2 = \{1, r\}$ acts on $\mathbb{Z}^2$ by $v \mapsto -v$, giving the action $k \mapsto -k$ on $\mathbb{T}^2 = [-\pi, \pi]^2/\sim$. In this case $rk = k$ implies $k_i = 0, \pi$ for $i = 1, 2$. Hence X^g consists of four points and

$$X^1/Z(1) = X/G \cong S^2.$$

To see how this homeomorphism is established, consider the fundamental domain $[-\pi, 0] \times [-\pi, \pi]$ of the action of G on X. Then the gluing data on the boundary indeed gives a sphere:



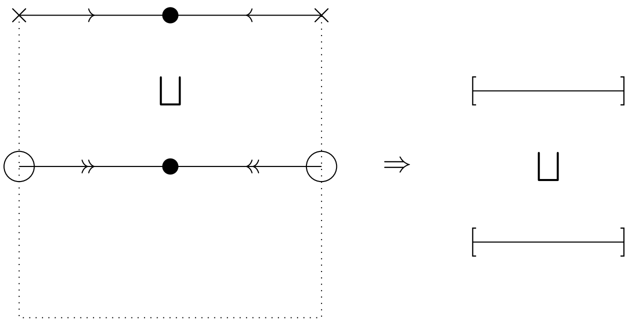
Therefore we get

$$K^0_{\mathbb{Z}_2}(\mathbb{T}^2) \otimes \mathbb{C} \cong H^0(\mathrm{pt} \sqcup \mathrm{pt} \sqcup \mathrm{pt} \sqcup \mathrm{pt}, \mathbb{C}) \oplus H^0(S^2, \mathbb{C}) \oplus H^2(S^2, \mathbb{C}) \cong \mathbb{C}^6.$$

Example 4.11 ($d = 2$ space group pm). $G = \mathbb{Z}_2 = \{1, s\}$ acts on $\mathbb{Z}^2$ by $(v_1, v_2) \mapsto (-v_1, v_2)$. The fixed points of s in $\mathbb{T}^2$ are $k = (0, k_2)$ and (π, k_2) for all $k_2 \in \mathbb{T}^1$, so that X^s is the disjoint union of two circles. Moreover, X/G is homeomorphic to a cylinder and hence homotopic to a circle. Thus

$$K^0_{\mathbb{Z}_2}(\mathbb{T}^2) \otimes \mathbb{C} \cong H^0(\mathbb{T}^1 \sqcup \mathbb{T}^1, \mathbb{C}) \oplus H^0(\mathbb{T}^1, \mathbb{C}) \cong \mathbb{C}^3.$$

Example 4.12 ($d = 2$ space group pmm). Consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ with generators s and r of the first respectively the second factor of G. Let r act on $\mathbb{Z}^2$ by $v \mapsto -v$ and let s act by $(v_1, v_2) \mapsto (-v_1, v_2)$. From the examples above, we know that X^r consists of the four points $(0,0),(0,\pi),(\pi,0)$ and (π,π) and X^s contains all points of the form $k=(0,k_2)$ and (π,k_2). Completely similarly, we have that X^{sr} consists of points of the form $(k_1,0)$ and (k_1,π), so again two circles. The remaining action on the two circles X^s (and similarly for X^{sr}) consists of simultaneous reflections $(0,k_2)\mapsto(0,-k_2)$ and $(\pi,-k_2)\mapsto(\pi,k_2)$. Therefore X^s/G and X^{sr}/G are homeomorphic to two disjoint intervals, hence homotopic to two points:



Finally, note that $(0,\pi)\times(0,\pi)$ is a fundamental domain of the action of G on X. Since there are no conditions on the boundary, the space

$$X/G \cong [0,\pi]\times[0,\pi]$$

is homeomorphic to a square and hence homotopic to a point. Also clearly $X^r/G = X^r$. Therefore we get

$$\begin{aligned} K^0_{\mathbb{Z}_2\times\mathbb{Z}_2}(\mathbb{T}^2)\otimes\mathbb{C} \cong & H^0(\mathrm{pt}\sqcup\mathrm{pt}\sqcup\mathrm{pt}\sqcup\mathrm{pt},\mathbb{C})\oplus \\ & H^0(\mathrm{pt}\sqcup\mathrm{pt},\mathbb{C})\oplus H^0(\mathrm{pt}\sqcup\mathrm{pt},\mathbb{C})\oplus H^0(\mathrm{pt},\mathbb{C}) \\ & \cong \mathbb{C}^9. \end{aligned}$$

Example 4.13 ($d=2$ space group $p4mm$). We take $G=D_4$, the group generated by a rotation r and a reflection s satisfying $s^2=r^4=1$ and $srs=r$. Let G act on the two-dimensional lattice by $r(v_1,v_2)=(-v_2,v_1)$ and $s(v_1,v_2)=(-v_1,v_2)$. A basic algebraic computation shows that the conjugacy classes of G are

$$\{1\},\{r,r^3\},\{s,sr^2\},\{sr,sr^3\},\{r^2\}$$

and that centralizers of representatives of the conjugacy classes are

$$\begin{gathered} Z(1)=G,\quad Z(s)=\{1,r^2,s,sr^2\},\quad Z(r)=\{1,r,r^2,r^3\},\\ Z(sr)=\{1,r^2,sr,sr^3\},\quad Z(r^2)=G. \end{gathered}$$

We now construct the homotopy classes of the five spaces of interest corresponding to the conjugacy classes.

- $g=1$: The triangle $\{(k_1,k_2)\in(0,\pi)\times(0,\pi)/\sim : k_1<k_2\}$ is a fundamental domain for the action of G on X and there are no boundary conditions. Hence X/G is homeomorphic to a triangle and hence contractible.

- $g = r$: The formula for the action of r gives $X^g = \{(0,0),(\pi,\pi)\}$ so that $X^g/Z(g)$ consists of two points.
- $g = s$: As has been argued in the examples above, X^s consists of two disjoint circles of the form $k = (0, k_2)$ and (π, k_2). Looking at $Z(s)$, we see that we only have to quotient by the action of r^2. Analogously to the last example, this gives two disjoint intervals.
- $g = sr$: Since this is a reflection over the diagonal $k_1 + k_2 = 0$, the invariant points are of the form $k = (k_1, -k_1)$ and comprise a circle. In order to get the desired space one still has to quotient by the action of r^2, resulting in $X^{sr}/Z(sr)$ being an interval.
- $g = r^2$: We have $X^{r^2} = \{(0,0),(0,\pi),(\pi,0),(\pi,\pi)\}$. In this set, $(\pi,0)$ is mapped to $(0,\pi)$ by r. The other points are fixed under G. Hence $X^{r^2}/Z(r^2)$ consists of three points.

One can now write down the K-theory:

$$\begin{aligned} K^0_{D_4}(X) \otimes \mathbb{C} \cong & H^0(\mathrm{pt}, \mathbb{C}) \oplus H^0(\mathrm{pt} \sqcup \mathrm{pt}, \mathbb{C}) \oplus H^0(\mathrm{pt} \sqcup \mathrm{pt}, \mathbb{C}) \\ & \oplus H^0(\mathrm{pt}, \mathbb{C}) \oplus H^0(\mathrm{pt} \sqcup \mathrm{pt} \sqcup \mathrm{pt}, \mathbb{C}) \\ & \cong \mathbb{C}^9. \end{aligned}$$

4.2. Class A: Equivariant Splitting and K-theory of Real Representations

To illustrate the different facets of equivariant K-theory, we will follow another possible route to compute equivariant K-theory. Moreover, this will show the fact that in many fortunate cases, there is no torsion in the nontwisted equivariant K-theory. This is based on the equivariant James splitting theorem of Section 3.10, and reduces our problem to the computation of equivariant K-theory of representation spheres S^V. Karoubi computed these K-theory groups in [39], see Theorem 4.16 below. Let G finite and let V be a real representation of G. Note that given a $g \in G$, the subspace $V^g := \{v \in V : gv = v\}$ forms a representation of $Z(G)$.

Definition 4.14. We say that an element $g \in G$ is

- *of degree $k \in \mathbb{Z}_2$ in V* if $\dim_{\mathbb{R}} V^g \equiv k \pmod 2$;
- *positively oriented in V* if all $h \in Z(G)$ act by linear maps $h : V^g \to V^g$ of positive determinant.

Lemma 4.15. *Let $g_1, g_2 \in G$. Then*

- *g_1 is of degree $k \in \mathbb{Z}_2$ in V if and only if $g_2 g_1 g_2^{-1}$ is of degree $k \in \mathbb{Z}_2$ in V;*
- *g_1 is positively oriented in V if and only if $g_2 g_1 g_2^{-1}$ is positively oriented in V.*

Proof. This is elementary. □

Theorem 4.16 (Karoubi)**.** *Let G be a finite group and let V be a real representation of G. Let k_i be the number of conjugacy classes of G of which the elements are positively oriented and of degree i. Then*

$$\widetilde{K}^0_G(S^V) \cong \mathbb{Z}^{k_0}, \quad \widetilde{K}^1_G(S^V) \cong \mathbb{Z}^{k_1}.$$

The splitting theorem (Theorem 3.77) now gives the following corollary:

Corollary 4.17. *If the action of G on the Brillouin zone $X = (S^1)^d$ can be realized as the restriction of an action of H^d, where H is some group acting on S^1, then $K_G(X)$ istorsion-free.*

Remark 4.18. For a counterexample of this corollary in case the action is not of this form, it is shown by Shiozaki et al. [62] that in the example of space group F222 torsion does occur.

Remark 4.19. Karoubi's formula for the K-theory of representation spheres can be used to illustrate the fact that there is no direct generalization of the complex equivariant Thom isomorphism 3.78 to real vector bundles (even for ordinary complex equivariant K-theory over a point). See the examples below.

We will now compute some examples using the equivariant splitting method and Karoubi's theorem. Note that this method only works if the action splits in the sense of the corollary above. Otherwise techniques from the last or the next section have to be addressed. Note that all results agree with (and generalize, since we could in principle also detect torsion here) the computations using Segal's formula of Section 4.1. Let us now first warm up with a computation of the classical nonequivariant nontwisted complex K-theory of the torus.

Example 4.20 (Ordinary complex K-theory of the torus). In this case splitting gives results quickly using the suspension theorem $\widetilde{K}^i(\Sigma X) \cong \widetilde{K}^{i-1}(X)$: (alternatively, one can of course still use Theorem 4.16)

$$\begin{aligned}
K^0(\mathbb{T}^2) &\cong K^0(S^1 \vee S^1 \vee S^2) \cong K^0(\mathrm{pt}) \oplus \widetilde{K}^0(S^1 \vee S^1 \vee S^2) \\
&\cong K^0(\mathrm{pt}) \oplus \widetilde{K}^0(S^1) \oplus \widetilde{K}^0(S^1) \oplus \widetilde{K}^0(S^2) \\
&\cong \mathbb{Z} \oplus \widetilde{K}^{-1}(S^0) \oplus \widetilde{K}^{-1}(S^0) \oplus \widetilde{K}^{-2}(S^0) \\
&\cong \mathbb{Z} \oplus K^{-1}(\mathrm{pt}) \oplus K^{-1}(\mathrm{pt}) \oplus K^{-2}(\mathrm{pt}) \\
&\cong \mathbb{Z}^2.
\end{aligned}$$

By a similar computation one can find that the K-theory of the 3-torus is $\mathbb{Z}^4$.

Example 4.21 ($d = 2$ space group $p2$). Here $G = \mathbb{Z}_2 = \{1, r\}$ acts by $k \mapsto -k$. We have to see how the action of G on the torus gives an action on spheres via the stable homotopy equivalence $\mathbb{T}^2 \sim_s S^1 \vee S^1 \vee S^2$. The action on both circles on the right hand side are one-point compactifications of the one-dimensional real representation $V_1 = \mathbb{R}$ given by $x \mapsto -x$. The action on the sphere is the one-point compactification of the real representation $V_2 = \mathbb{R}^2$ given by $x \mapsto -x$. In order to determine the K-theory groups $\widetilde{K}^i_G(S^{V_i})$, Karoubi tells us to determine the fixed point spaces V_i^g for g in a set of full representatives of the conjugacy classes of G, determine whether they have even or odd dimension and finally to determine whether central elements act by linear maps of positive or negative determinant.

Clearly $V_i^1 = V_i$ and $V_i^r = 0$. On V_2 the group G acts by elements of positive determinant, but on V_1 the element r has a negative determinant. To use the language of Definition 4.14, 1 and g are positively oriented of even degree with respect to V_2. On the other hand 1 is negatively oriented of odd degree with respect to V_1, while r is positively oriented of even degree. Hence by Karoubi's Theorem 4.16

$$\widetilde{K}^0_G(S^{V_1}) \cong \mathbb{Z}, \quad \widetilde{K}^0_G(S^{V_2}) \cong \mathbb{Z}^2.$$

One can now conclude that

$$\begin{aligned} K^0_G(\mathbb{T}^2) &\cong K^0_{\mathbb{Z}_2}(\mathrm{pt}) \oplus \widetilde{K}^0_G(S^{V_1}) \oplus \widetilde{K}^0_G(S^{V_1}) \oplus \widetilde{K}^0_G(S^{V_2}) \\ &\cong \mathbb{Z}^2 \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}^2 \\ &= \mathbb{Z}^4, \end{aligned}$$

since $K^0_{\mathbb{Z}_2}(\mathrm{pt})$ is the nontwisted complex representation ring of $\mathbb{Z}_2$.

Example 4.22 ($d = 2$ space group pm). The group $G = \mathbb{Z}_2 = \{1, s\}$ acts by $(k_1, k_2) \mapsto (-k_1, k_2)$ on the Brillouin zone. This time the splitting gives us three real representations of interest:

$$\begin{aligned} &V_0 = \mathbb{R} \quad \text{the trivial representation,} \\ &V_1 = \mathbb{R} \quad \text{with} \quad x \mapsto -x, \\ &V_2 = \mathbb{R}^2 \quad \text{with} \quad (x_1, x_2) \mapsto (-x_1, x_2). \end{aligned}$$

Clearly both elements are positively oriented of odd degree with respect to V_0. For V_1, we have that s has odd determinant on $(V_1)^1$, so 1 is negatively oriented with respect to V_1. Also s is positively oriented of even degree with respect to V_1. Finally, for V_2 note that again that s has odd determinant on $(V_2)^1$, so 1 is negatively oriented with respect to V_2. However, $(V_2)^s = \{(0, x_2) : x_2 \in \mathbb{R}\}$, so that s is positively oriented and odd on V_2. Hence Karoubi tells us that

$$\widetilde{K}^0_G(S^{V_0}) = 0, \quad \widetilde{K}^0_G(S^{V_1}) = \mathbb{Z}, \quad \widetilde{K}^0_G(S^{V_2}) = 0.$$

Using the equivariant splitting again, we get

$$\begin{aligned} K^0_G(\mathbb{T}^2) &\cong K^0_G(\mathrm{pt}) \oplus \widetilde{K}^0_G(S^{V_0}) \oplus \widetilde{K}^0_G(S^{V_1}) \oplus \widetilde{K}^0_G(S^{V_2}) \\ &\cong \mathbb{Z}^3. \end{aligned}$$

Note in particular that in the last example $K^0_G(S^{V_2}) \ncong K^0_G(\mathrm{pt})$. This is a priori unexpected, since one could be be tempted to write down the following (wrong!) argument: the space S^{V_2} is the one-point compactification of a one-dimensional complex G-vector bundle over a point, so that by the equivariant Thom isomorphism (Theorem 3.78), $K^0_G(S^{V_2})$ is isomorphic to $K^0_G(\mathrm{pt})$. However, s would not act complex linearly on such a bundle and therefore the argument is false.

4.3. Class A: Bredon Cohomological Computations and Spectral Sequences

In this section the spectral sequence of Section 3.11 will be considered in class A. Because the spectral sequence method will play a prominent role in class AI and AII (in which computations are also more involved), we will not provide any explicit computations for class A using the Atiyah-Hirzebruch spectral sequence. However, the interested reader could use Proposition 4.23 below to reproduce the results of earlier sections.

For class A the spectral sequence is well-known, see Dwyer [22] (or more generally [9] for twists that are not constant and [26] for Lie groupoids). Given that ϕ and c are trivial, the twisted group algebra A is a nongraded complex algebra. Hence Bott-periodicity is of order two and the Bredon coefficient functor $\mathcal{R}_G^{\tau+q}$ vanishes for q odd. Thus the second page is

$$E_2^{p,q} = \begin{cases} H_G^p(X, \mathcal{R}_G^\tau) & \text{if } q \text{ even,} \\ 0 & \text{if } q \text{ odd,} \end{cases}$$

where $\mathcal{R}_G^\tau$ is the twisted representation ring functor of G that maps G/H to the abelian group $R^\tau(H)$ of projective representations of H with cocycle $\tau|_H$. More precisely, $R^\tau(H)$ is the free abelian group generated by the irreducible modules of the complex algebra $\mathbb{C}^\tau H$. It maps a conjugation $\hat{a} : G/H \to G/(aHa^{-1})$ to the map $R^\tau(aHa^{-1}) \to R^\tau(H)$ which on the level of representations is $\rho \mapsto \rho^a$ where

$$\rho^a(h) := \rho(aha^{-1}).$$

Note that this map is an isomorphism of abelian groups and that we can identify conjugate subgroups under this functor. Finally under a quotient map $q : G/H \to G/K$ with $H \subseteq K$ we have that $\mathcal{R}_G^\tau(q) := r_H^K : R^\tau(K) \to R^\tau(H)$ is the map induced by restrictions of representations to the subgroup H.

Since differentials in a cohomological spectral sequence have bidegree $(r, 1-r)$, we see that the second differential vanishes. On the third page, the differential becomes of bidegree $(3, -2)$, of which only the map $d_3 : H_G^0(X, \mathcal{R}_G^\tau) \to H_G^3(X, \mathcal{R}_G^\tau)$ can be nontrivial. Hence the exact sequences that followed from the spectral sequence of Section 3.11 reduce to the single exact sequence

$$0 \to H_G^2(X, \mathcal{R}_G^\tau) \to K_G^\tau(X) \to \ker d_3 \to 0.$$

Since $R^\tau(H)$ is torsion-free for all subgroups $H \subseteq G$, all groups of cochains are torsion-free. As $H_G^0(X, \mathcal{R}_G^\tau)$ is the kernel of a homomorphism between cochains, it is torsion-free too and hence so is $\ker d_3$. By an elementary argument, this implies that the sequence splits and we get an isomorphism

$$K_G^\tau(X) \cong \ker d_3 \oplus H_G^2(X, \mathcal{R}_G^\tau).$$

If we make use of Remark 4.8, we can strengthen this argument and get rid of the third differential altogether. Indeed, note that we have that

$$K_G^\tau(X) \otimes_{\mathbb{Z}} \mathbb{Q} \cong (\ker d_3 \oplus H_G^2(X, \mathcal{R}_G^\tau)) \otimes_{\mathbb{Z}} \mathbb{Q} \cong (H_G^0(X, \mathcal{R}_G^\tau) \oplus H_G^2(X, \mathcal{R}_G^\tau)) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

However, both $\ker d_3$ and $H^0_G(X, \mathcal{R}^\tau_G)$ are torsion-free. Hence they are abstractly isomorphic[7].

The conclusion of the above discussion is the following:

Proposition 4.23. *The group of reduced topological phases in class A for a symmorphic crystal with anomaly τ is isomorphic to*

$$K^\tau_G(X) \cong H^0_G(X, \mathcal{R}^\tau_G) \oplus H^2_G(X, \mathcal{R}^\tau_G).$$

If one still happens to be interested in whenever the third differential vanishes, Theorem 3.79 gives us an abstract geometric condition. Indeed, let $f \in Z^0_G(X, \mathcal{R}^\tau_G)$ be a 0-cocycle representing a class $[f] \in H^0_G(X, \mathcal{R}^\tau_G)$. Let $[E] - [\theta_M] \in K^\tau_G(X^2)$ be an extension of the element of $K^\tau_G(X^0)$ induced by f to a bundle over X^2 (this element exists because $d_1 f$ and $d_2 f$ vanish and every element in nongraded K-theory can be written in this form by Lemma 3.34). Then by Theorem 3.79, $d_3 f = [\delta([E] - [\theta_M])]$. Hence $d_3 f$ vanishes if and only if

$$\delta([E]-[\theta_M]) = d_1([E_1]-[\theta_{M_1}]) = \delta k([E_1]-[\theta_{M_1}]) \iff \delta([E]-[\theta_M]-k([E_1]-[\theta_{M_1}])) = 0$$

for some class $[E_1] - [\theta_{M_1}] \in K^\tau_G(X^2, X^1) = \widetilde{K}^\tau_G(X^2/X^1)$, where

$$k : \widetilde{K}^\tau_G(X^2/X^1) \to K^\tau_G(X^2)$$

is the map given by extending bundles to X^2 with constant value on X^1. Note that restricting $k([E_1] - [\theta_{M_1}])$ to X^1 gives the trivial class, so by adapting our choice of lift $[E] - [\theta_M]$ we get rid of this term. By the long exact sequence of the pair (X^2, X^1) the vanishing of df is now equivalent to demanding the existence of $[E_2] - [\theta_{M_2}] \in K^\tau_G(X)$ such that

$$[E] - [\theta_M] = [E_2|_{X_2}] - [\theta_{M_2}|_{X_2}]$$

In other words, there exists M' such that

$$E \oplus \theta_{M_2} \oplus \theta_{M'} \cong E_2|_{X^2} \oplus \theta_M \oplus \theta_{M'}.$$

So E is stably isomorphic to a bundle coming from X. We conclude that the third differential d_3 vanishes on a cocycle $f \in C^0_G(X, \mathcal{R}^\tau_G)$ if and only if there is a class in $K^\tau_G(X)$ that restricts to f on X_0.

4.4. Class A: Comparisons with Methods Using Band Structure Combinatorics

The method using band structure combinatorics of Kruthoff et al. in [44] consists of two steps. Firstly, it is noted whenever an assignment of virtual representations to the

[7]Note that the above argument does not imply that they are equal, so that the third differential not necessarily vanishes.

cells of the Brillouin zone is consistent. Secondly, they include Chern numbers of 2-dimensional fixed subspaces of the Brillouin zone. The main claim of this section is that under some mild assumptions, zeroth degree Bredon cohomology is computed in step one, while second degree Bredon cohomology is computed in step two. Hence under these assumptions, it should follow by Proposition 4.23 that the method described in [44] reproduces the desired equivariant K-theory. For step one we provide a rigorous proof, while for step two we only sketch a proof for the free part of the K-theory in dimension two. Although we do not provide a proof, we expect the actual second degree Bredon cohomology group to agree with this method for $d = 2$. However, for $d = 3$, Shiozaki et al. [62] provide an example (F222) in which the second degree Bredon cohomology group has torsion. Since the Chern number count of Kruthoff et al. does not detect torsion, the method therefore does not fully reproduce the equivariant K-theory group in this case. It would definitely be interesting to compare the method of computing the second degree Bredon cohomology group with the Chern number count in general. In order to use the tools developed in the last chapter, we have to assume all space groups are symmorphic.

Consider the equivalence relation on k-cells $\sigma_1 \sim \sigma_2 \iff \sigma_1 = g\sigma_2$ for some $g \in G$. Note that if l is a 1-cell, the endpoints of its equivalence class are well-defined equivalence classes of points; the endpoints of gl are equivalent to the endpoints of l. Moreover, because the definition of Bredon cohomology asks us to choose the orientation such that group elements preserve it (see Definition C.8), the order of endpoints can even be preserved. In order to define the band structure combinatorics method, let us first make another assumption.

Definition 4.24. We say a full set of symmetry data for a topological insulator of type A protected by G with trivial twists (both anomalies and crystal twist) together with a G-CW structure satisfies *Assumption 1* if there exists a choice of a complete set of representatives $\mathcal{C}^k_G(X)$ of k-cells under the equivalence relation defined above such that whenever $l \in \mathcal{C}^1_G(X)$ is a line, its boundary points are again in the set of representatives: $\partial_0 l, \partial_1 l \in \mathcal{C}^0_G(X)$.

The notion of a consistent assignment of representations as in Kruthoff et al. [44] can be formalized as follows. It is stated in the work of Kruthoff et al. [44] that a consistent point assignment extends uniquely to higher-dimensional cells. By Proposition 4.23, we know that this consistent point assignment uniquely extends if and only if the third differential vanishes. It would be interesting to seek explicit expressions for the higher differentials in further research. However, in order to make sure we agree with the method of Kruthoff et al., we only assign virtual representations to points.

Definition 4.25. Suppose X satisfies Assumption 1. Take $\mathcal{C}^k_G(X)$ to be a complete set of representatives of k-cells under the equivalence relation of the desired form. We say a map

$$f : \mathcal{C}^0_G(X) \to \cup_{p \in \mathcal{C}^0_G(X)} R(G_p)$$

is a *consistent point assignment* if

1. $f(p) \in R(G_p)$ for all $p \in \mathcal{C}^0_G(X)$;

2. for any 1-cell $l \in \mathcal{C}^1_G(X)$ with boundary points p_1, p_2 we have

$$r^{G_{p_1}}_{G_l}(f(p_1)) = r^{G_{p_2}}_{G_l}(f(p_2)).$$

We consider point assignments as an abelian group $\mathcal{H}^0_G(X)$.

Proposition 4.26. *There is an isomorphism of abelian groups*

$$\mathcal{H}^0_G(X) \cong H^0_G(X, \mathcal{R}_G).$$

Proof. Consider the map $\psi : \mathcal{H}^0_G(X) \to H^0_G(X, \mathcal{R}_G)$ that sends a function to its equivariant extension. In other words, if $f : \mathcal{C}^0_G(X) \to \cup_{p \in \mathcal{C}^0_G(X)} R(G_p)$ is a consistent point assignment, p' is a 0-cell and $g \in G, p \in \mathcal{C}^0_G(X)$ are chosen such that $gp = p'$, then we define

$$\psi(f)(p') = \psi(f)(gp) = F(\hat{g})(f(p)) = \mathcal{R}_G(\hat{g})(f(p)),$$

where $F = \mathcal{R}_G$ is the coefficient functor of our Bredon cohomology (see C.3 for the definitions of Bredon cohomology). It has to be shown that ψ is a well-defined isomorphism of groups.

We first check that it is well-defined in the sense that whenever

$$f : \mathcal{C}^0_G(X) \to \cup_{p \in \mathcal{C}^0_G(X)} R(G_p)$$

is a consistent point assignment, $\psi(f)$ is a G-cochain. So let $p \in \mathcal{C}^0_G(X)$ and $g, h \in G$. Note that $\mathcal{R}_G(\hat{g})$ is a map $R(G_p) \to R(gG_pg^{-1}) = R(G_{gp})$[8] so that $\psi(f)(gp) \in R(G_{gp}) = F(G/G_{gp})$, since $f(p) \in R(G_p)$ by definition of a consistent point assignment. Hence we get the first requirement for being a G-cochain. For the second requirement, we compute

$$\psi(f)(ghp) = F(\widehat{gh})(f(p)) = F(\hat{h}\hat{g})(f(p)) = F(\hat{g}) \circ F(\hat{h})(f(p)) = F(\hat{g})(f(hp)),$$

so $\psi(f)$ is indeed a G-cochain.

We now check that f satisfies the second condition of being a consistent point assignment if and only if $\psi(f)$ is in the kernel of d. Let l' be a 1-cell with endpoints p'_1, p'_2. Take $g \in G$ and $l \in \mathcal{C}^1_G(X)$ such that $gl = l'$, so that also l' has endpoints gp_1 and gp_2 with $p_1, p_2 \in \mathcal{C}^0_G(X)$ respectively (the 'first' and 'second' end-point of an equivalence class of lines is well-defined). Then we compute for $g \in G$ that

$$\begin{aligned} d(\psi(f))(l') = d(\psi(f))(gl) &= r^{G_{gp_1}}_{G_{gl}}(\psi(f)(gp_1)) - r^{G_{gp_0}}_{G_{gl}}(\psi(f)(gp_0)) \\ &= r^{G_{gp_1}}_{G_{gl}}(F(\hat{g})(f(p_1))) - r^{G_{gp_0}}_{G_{gl}}(F(\hat{g})(f(p_0))) \\ &= F(\hat{g}) r^{G_{p_1}}_{G_l}(f(p_1)) - F(\hat{g}) r^{G_{p_0}}_{G_l}(f(p_0)) \\ &= F(\hat{g}) \left(r^{G_{p_1}}_{G_l}(f(p_1)) - r^{G_{p_0}}_{G_l}(f(p_0)) \right), \end{aligned}$$

where in the second-last equation we used the commutativity of the following diagram in the orbit category $\mathcal{O}_G$:

[8] It is easy to check that $G_{gp} = gG_pg^{-1}$.

$$\begin{array}{ccc} G/G & \xrightarrow{\hat{g}} & G/G \\ \downarrow q & & \downarrow q \\ G/G_p & \xrightarrow{\hat{g}} & G/(gG_pg^{-1}). \end{array}$$

By picking $g = 1$ in the above equation, we find that $d\psi(f) = 0$ would imply $f(p_1)|_{G_l}) = f(p_2)|_{G_l}$. Conversely, if this holds, it clearly follows that $d(\psi(f))(l') = 0$ for all 1-cells l'.

Now note that ψ is a map of abelian groups. Moreover, there is an obvious inverse map that is just restriction of G-cochains to the set of representatives $\mathcal{C}^0_G(X)$. Hence ψ is an isomorphism. □

Now the number of possible Chern numbers will be compared with the rank of the second Bredon cohomology. Since the case $d = 3$ is more complicated, we assume that $d = 2$.

Definition 4.27. If X is a topological insulator of class A of dimension 2, define the *group of Chern numbers* $\mathcal{H}^2_G(X)$ to be zero if the group contains an element reversing the orientation of X and $\mathbb{Z}$ otherwise.

Definition 4.28. We say that a topological insulator of class A of dimension 2 satisfies *Assumption 2* if for every nontrivial element $g \in G$, the fixed point set X^g is of dimension at most 1.

We sketch a proof of the desired isomorphism for the nontorsion part.

Lemma 4.29. *If X is a 2-torus for which Assumption 2 holds, then*

$$H^2_G(X, \mathcal{R}_G) \otimes_{\mathbb{Z}} \mathbb{C} = \begin{cases} 0 & \textit{if } G \textit{ contains an element reversing the orientation of } X, \\ \mathbb{C} & \textit{otherwise.} \end{cases}$$

Proof. Consider Theorem 4.5 together with the ordinary Chern isomorphism of complex K-theory (or equivalently, Remark 4.8). Since the only fixed point set that is two-dimensional is the one of the unit element $1 \in G$, the degree two part of the cohomology groups just encompasses $H^2(X/G, \mathbb{C})$. Because X/G is a two-dimensional orientable orbifold if and only if G contains no orientation-reversing elements, we have that

$$H^2(X/G, \mathbb{C}) = \begin{cases} 0 & \text{if } G \text{ contains an element reversing the orientation of } X, \\ \mathbb{C} & \text{otherwise.} \end{cases}$$

□

Note that this proof does not a priori exclude the possibility of torsion in $H^2_G(X, \mathcal{R}_G)$. If indeed it does not contain torsion, a proof of this fact would probably require the use of more properties of X.

Definition 4.30. The *combinatoric K-theory* is defined to be $\mathcal{H}^0_G(X) \oplus \mathcal{H}^2_G(X)$.

The conclusion of the above discussion is the following.

Corollary 4.31. *If X is a 2-dimensional topological insulator satisfying Assumption 1 and Assumption 2 such that $H^2_G(X, \mathcal{R}_G)$ is torsion-free, then combinatorial K-theory is isomorphic to the free part of equivariant K-theory.*

It would also be interesting to compare the computational methods of Kruthoff et al. for class AI and AII topological insulators [43]. This is especially so because they use a representation-theoretic technique analogous to the Frobenius-Schur indicator to compute step one, which is expected to equal $H^0_G(X, {}^{\phi}\mathcal{R}^{\tau}_G)$.

4.5. Class AI & AII

Now time reversal will be included, resulting in a significantly more difficult setting. Since from now on $\phi = pr_2 \neq 1$, we have to deal with anti-linear maps in our vector bundles, which gives us twisted versions of equivariant KR-theory. Consequently, torsion groups - which are physically relevant to detect - are no longer exceptions, since they can already appear for the K-theory of a point. Because this significantly reduces the value of an analogue of the equivariant Chern isomorphism or Segal's formula in the setting of time reversal, general computations will be done using the spectral sequence of Section 3.11, occasionally making use of the equivariant splitting method of Section 3.10.

As in the last subsection, we warm up with the case of no spatial symmetries. In that case, twists are classified by $\mathbb{Z}_2$ depending on whether $T^2 = \pm 1$. The K-theories that resemble reduced topological phases of free fermions are now $KR^0(\mathbb{T}^d)$ if $T^2 = 1$ and $KR^4(\mathbb{T}^d)$ if $T^2 = -1$ (see Example 3.24). Here the action on the Brillouin zone torus is given by $k \mapsto -k$ if we see the Brillouin zone as the usual quotient of $[-\pi, \pi]^d$. Next to the equivariant splitting method, we use two types of suspensions of KR-theory: the positive suspension $\Sigma^+ X$ of a pointed $\mathbb{Z}_2$-space X is the usual suspension of $\mathbb{Z}_2$-spaces ΣX. The negative suspension $\Sigma^- X$ is ΣX as a topological space, but with different $\mathbb{Z}_2$-action than Σ^+. It is the quotient of the action of $\mathbb{Z}_2$ on $X \times [0,1]$ given by simultaneous action on X and reflecting the interval: $T(x,t) = (T(x), 1-t)$. Then there is a suspension theorem in KR-theory that says that $\widetilde{KR}^i(\Sigma^+ X) \cong \widetilde{KR}^{i-1}(X)$ and $\widetilde{KR}^i(\Sigma^- X) \cong \widetilde{KR}^{i-1}(X)$. For details on the definitions and claims in this paragraph, see for example Atiyah's paper [3].

We can now compute $KR^i(\mathbb{T}^d)$ with the usual action of time reversal $k \mapsto -k$. Writing S^n_- for the n-sphere with $\mathbb{Z}_2$-action that is the one-point compactification of the n-dimensional real representation given by $k \mapsto -k$, we see that

$$\begin{aligned} KR^i(\mathbb{T}^2) &\cong KR^i(\mathrm{pt}) \oplus \widetilde{KR}^i(S^1_-) \oplus \widetilde{KR}^i(S^1_-) \oplus \widetilde{KR}^i(S^2_-) \\ &= KR^i(\mathrm{pt}) \oplus KR^{i+1}(\mathrm{pt}) \oplus KR^{i+1}(\mathrm{pt}) \oplus KR^{i+2}(\mathrm{pt}). \end{aligned}$$

Looking up the table for $KO^i(\mathrm{pt})$, for example as a special case of $KO^i_G(\mathrm{pt})$ as in the last page of [7] or the ten-fold way table in Appendix B.2, we see that

$$KR^0(\mathbb{T}^2) = \mathbb{Z}, \quad KR^4(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2.$$

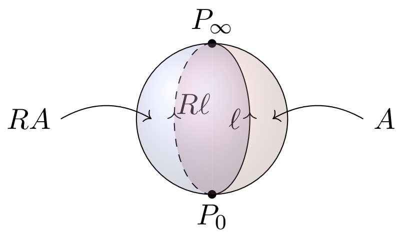


Figure 4.1.: A $\mathbb{Z}_2$-CW structure of the $\mathbb{Z}_2$-space S^2 that is the one-point compactification of the two-dimensional representation of $\mathbb{Z}_2$ given by $k \mapsto -k$.

Similarly,

$$KR^i(\mathbb{T}^3) \cong KR^i(\mathrm{pt}) \oplus KR^{i+1}(\mathrm{pt})^3 \oplus KR^{i+2}(\mathrm{pt})^3 \oplus KR^{i+3}(\mathrm{pt}),$$

so that

$$KR^0(\mathbb{T}^3) = \mathbb{Z}, \quad KR^4(\mathbb{T}^3) \cong \mathbb{Z} \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_2.$$

The first three factors of $\mathbb{Z}_2$ come from the two-dimensional subtori of $\mathbb{T}^3$ and are the so-called weak $\mathbb{Z}_2$-invariants. The last factor $\mathbb{Z}_2$ is fully three-dimensional and is the strong invariant. It can be realized as the orbital magnetoelectric polarizability or as the Fu-Kane-Mele invariant, see Freed & Moore Paragraph 11 for more mathematical details.

More general examples are significantly harder to compute in case time reversal is present, as the spectral sequence contains much more complicated information. First of all, one already has to be careful in computing the coefficient systems ${}^{\phi}\mathcal{R}_G^{\tau+n}$; periodicity is now 8-fold and the action on morphisms can be nontrivial, see Section 3.9. Secondly, the second differential in the spectral sequence can be nontrivial, giving more ambiguity for computing the last page. Thirdly and finally, because many torsion groups appear, the exact sequences at the end of Section 3.11 are often nonsplit, so that the filtration is not always uniquely determined by the spectral sequence. When this is the case, extra geometric constraints have to be found in order to determine the K-theory.

Example 4.32. To illustrate the subtleties involved in the first and the second problem, we recompute the KR-theory of the sphere $X = S^2$ with involution equal to the antipodal map $x \mapsto -x$. We expect to find $\mathbb{Z} \oplus \mathbb{Z}_2$ for the number of bands and the Fu-Kane-Mele invariant. Instead of using negative suspensions Σ^-, we make use of the spectral sequence method. So let us recall Example 3.75, in which we computed the relevant coefficients for Bredon cohomology for $G = \mathbb{Z}_2, \phi$ nontrivial and c trivial. This example exactly applies here and depending on whether we pick $\tau = \tau_0$ trivial ($T^2 = 1$) or $\tau = \tau_1$ nontrivial ($T^2 = -1$) we get $KR^0(S^2)$ or $KR^4(S^2)$.

Using the table of representation groups computed in that example as coefficients for Bredon cohomology, we compute the groups relevant for the spectral sequence. First choose an oriented G-CW structure as in Figure 4.1. Here p_0 is the origin and p_∞ is the point at infinity. Note that the action fixes p_0 and p_∞, but is free everywhere else. We now compute the relevant Bredon cohomology groups keeping in mind that the relevant part of the spectral sequence looks like Figure 3.2, but with vanishing cohomology in degree 3.

- $H^0_G(S^2, {}^\phi\mathcal{R}^\tau_G)$: Our equivariant 0-cochains are $C^0_G(S^2, {}^\phi\mathcal{R}^\tau_G) = \mathbb{Z}^2 = \langle \pi_0, \pi_\infty \rangle_\mathbb{Z}$, where
$$\pi_0 : \{p_0, p_\infty\} \to {}^\phi\mathcal{R}^\tau_G(G/G_{p_0}) = {}^\phi R^\tau(G_{p_0}) = {}^\phi R^\tau(\mathbb{Z}_2) \cong \mathbb{Z}$$
maps p_0 to 1 and p_∞ to 0, while π_∞ maps p_∞ to 1 and p_0 to 0. Note that π_0 and π_∞ are equivariant since the points are fixed by the action of the nontrivial element $T \in \mathbb{Z}_2$ and the action of T on the homogeneous space G/G is of course trivial. The equivariant 1-cochains are $C^1_G(S^2, {}^\phi\mathcal{R}^\tau_G) = \mathbb{Z} = \langle \lambda \rangle_\mathbb{Z}$, where
$$\lambda : \{l, Tl\} \to {}^\phi\mathcal{R}^\tau_G(G/G_l) = {}^\phi R^\tau(G_l) = {}^{\phi|_{G_l}} R^{\tau|_{G_l}}(\{1\}) \cong \mathbb{Z}$$
is uniquely specified by $\lambda(l) = 1$, since by equivariance
$$\lambda(Tl) = {}^\phi\mathcal{R}^\tau_G(T)(\lambda(l)) = \lambda(l) = 1,$$
because the action of T on the representation ring is trivial in degree zero.
We now compute the Bredon differential $d : \mathbb{Z}^2 = C^0_G(S^2, {}^\phi\mathcal{R}^\tau_G) \to C^1_G(S^2, {}^\phi\mathcal{R}^\tau_G) = \mathbb{Z}$:
$$d\pi_0(l) = {}^\phi\mathcal{R}^\tau_G(G/\{1\} \to G/G)(\pi_0(\partial l)) = {}^\phi\mathcal{R}^\tau_G(G/\{1\} \to G/G)(1))$$
Recalling the table from Example 3.75 again, we see that this restriction map is the identity if $T^2 = 1$ and multiplication by two if $T^2 = -1$. Hence we get
$$d\pi_0 = \begin{cases} \lambda & \text{if } \tau = \tau_0, \\ 2\lambda & \text{if } \tau = \tau_1. \end{cases}$$
Analogously we get that $d\pi_\infty = -d\pi_0$. In both cases, we see that
$$\ker d = H^0_G(S^2, {}^\phi\mathcal{R}^\tau_G) \cong \mathbb{Z}.$$

- $H^k_G(S^2, {}^\phi\mathcal{R}^{\tau-1}_G)$: Considering the table of Example 3.75 again, we see that if $T^2 = -1$ the coefficient functor is trivial, so that all cohomology groups vanish. If $T^2 = 1$ there is only the nontrivial value ${}^\phi\mathcal{R}^{\tau_0-1}_G(G/G) = \mathbb{Z}_2$, which is only relevant for 0-cochains. We see that there are no differentials and hence $H^0_G(S^2, {}^\phi\mathcal{R}^{\tau_0-1}_G) = \mathbb{Z}_2^2$, while it vanishes in other degrees.

- $H^2_G(S^2, {}^\phi\mathcal{R}^{\tau-2}_G)$: Since only degree one and two cochains are involved in this group, we only have to use
$${}^\phi\mathcal{R}^{\tau-2}_G(G) = R^{-2}(\mathbb{C}) = \mathbb{Z}$$
for both signs of T^2. So let $\alpha : \{A, TA\} \to \mathbb{Z}$ be the unique equivariant 2-cochain such that $\alpha(A) = 1$. Similarly define λ by $\lambda(l) = 1$. The reader should keep in mind that the action of T is multiplication by -1, so that $\lambda(Tl) = -1$, see Example 3.75. Therefore the Bredon differential on 1-cochains gives us
$$d\lambda(A) = \lambda(\partial A) = \lambda(l) - \lambda(Tl) = 2.$$
Hence $d\lambda = 2\alpha$. This gives us our final cohomology group $H^2_G(S^2, {}^\phi\mathcal{R}^{\tau-2}_G) = \mathbb{Z}_2$.

Summarizing the results by filling in the relevant part of the second page of the spectral sequence as in Figure 3.2, we get the following tables for $T^2 = 1$ and $T^2 = -1$ respectively:

	$q = 0$	$q = 1$	$q = 2$
$p = 0$	$H^0_G(S^2, {}^\phi\mathcal{R}^{\tau_0}_G) = \mathbb{Z}$		
$p = 1$	$H^0_G(S^2, {}^\phi\mathcal{R}^{\tau_0 - 1}_G) = \mathbb{Z}_2^2$	$H^1_G(S^2, {}^\phi\mathcal{R}^{\tau_0 - 1}_G) = 0$	$H^2_G(S^2, {}^\phi\mathcal{R}^{\tau_0 - 1}_G) = 0$
$p = 2$			$H^2_G(S^2, {}^\phi\mathcal{R}^{\tau_0 - 2}_G) = \mathbb{Z}_2$

	$q = 0$	$q = 1$	$q = 2$
$p = 0$	$H^0_G(S^2, {}^\phi\mathcal{R}^{\tau_1}_G) = \mathbb{Z}$		
$p = 1$	$H^0_G(S^2, {}^\phi\mathcal{R}^{\tau_1 - 1}_G) = 0$	$H^1_G(S^2, {}^\phi\mathcal{R}^{\tau_1 - 1}_G) = 0$	$H^2_G(S^2, {}^\phi\mathcal{R}^{\tau_1 - 1}_G) = 0$
$p = 2$			$H^2_G(S^2, {}^\phi\mathcal{R}^{\tau_1 - 2}_G) = \mathbb{Z}_2$

If $T^2 = -1$, we see immediately that all higher differentials vanish. Hence the K-theory fits in an exact sequence

$$0 \to \mathbb{Z}_2 \to KR^4(S^2) \to \mathbb{Z} \to 0.$$

Since $\mathbb{Z}$ is a free group, this gives us $KR^4(S^2) = \mathbb{Z} \oplus \mathbb{Z}_2$ as expected from using the suspension theorem for negative suspensions. Note the interesting fact that the torsion invariant $\mathbb{Z}_2$ managed to appear because of the nontrivial action of T on $R^{-2}\mathbb{C}$ induced by complex conjugation, not by the torsion in the KO-theory of a point as it did in the approach to computing $KR^4(S^2)$ at the beginning of this section.

For $T^2 = 1$ another lesson is to be learned from this example. Namely, note that as long as we do not know any expression for the second differential $d_2 : \mathbb{Z}_2^2 \to \mathbb{Z}_2$, we cannot uniquely determine the K-theory group by the spectral sequence method. However, by the computation in the beginning of this section, we know that $KR^0(S^2) = \mathbb{Z}$ so that this differential must be surjective. If in future research an explicit expression for the second differential for general G-CW complexes is found, it could therefore be interesting to compute it in this example.

Example 4.33 ($d = 2$ p2 with time reversal for spinful fermions). For a more exciting example, we now also include a rotation by π. So consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2, \phi = pr_2$ and denote the generators by R and T. Let τ be the anomaly for which $R^2 = T^2 = -1$ and $TR = RT$ (since this example represents spinful fermions we should demand $R^2 = -1$). So we have a torus $\mathbb{T}^2 = [-\pi, \pi]^2/\sim$ with action $Tk = -k$ and $Rk = -k$. The equivariant splitting technique of Corollary 3.77 gives a G-equivariant stable homotopy equivalence with $S^2 \vee S^1 \vee S^1$. Note that for the G-action on the sphere and the two circles, the same formulae can be used as above if we consider the spaces as one point compactifications of representations of G. In particular, as there are no group element permuting the circles, we have to compute

$${}^\phi K^\tau_G(\mathbb{T}^2) \cong {}^\phi K^\tau_G(\text{pt}) \oplus {}^\phi\widetilde{K}^\tau_G(S^2) \oplus {}^\phi\widetilde{K}^\tau_G(S^1) \oplus {}^\phi\widetilde{K}^\tau_G(S^1).$$

These groups will now be computed by the spectral sequence.

Before the topological computations, we first have to compute the twisted Bredon coefficients, i.e. the representation rings and the relevant maps between them. Note that the only stabilizers that occur are G and $H := \{1, TR\}$, so we only have to compute twisted representations for them. Because of this exceptional role played by TR it is useful to set $S := TR$ and forget T for the moment. Note that in the twisted group algebra, $Si = -iS$, $S^2 = 1$ and $SR = RS$. The twisted group algebras are

$$\begin{aligned} {}^{\phi}\mathbb{C}^{\tau}H &= \frac{\mathbb{R}[i, S]}{(i^2 = -S^2 = 1, iS = -Si)} \cong |Cl_{1,1}| \cong M_2(\mathbb{R}) \\ {}^{\phi}\mathbb{C}^{\tau}G &= \frac{\mathbb{R}[i, S, R]}{(i^2 = R^2 = -S^2 = 1, iS = -Si, RS = SR, iR = Ri)} \cong M_2(\mathbb{C}), \end{aligned}$$

where the last isomorphism follows because the twisted group algebra is ${}^{\phi}\mathbb{C}^{\tau}H \otimes_{\mathbb{R}} \mathbb{C}$. The representation rings are therefore

$$\begin{aligned} R^p({}^{\phi}\mathbb{C}^{\tau}H) &\cong R^p(M_2(\mathbb{R})) \cong R^p(\mathbb{R}) \cong KO^p(\mathrm{pt}) \\ R^p({}^{\phi}\mathbb{C}^{\tau}G) &\cong R^p(M_2(\mathbb{C})) \cong R^p(\mathbb{C}) \cong K^p(\mathrm{pt}), \end{aligned}$$

see Corollary B.24. The restriction map $\mathbb{Z} \cong R^0({}^{\phi}\mathbb{C}^{\tau}G) \to R^0({}^{\phi}\mathbb{C}^{\tau}H) \cong \mathbb{Z}$ is given by mapping the irreducible $M_2(\mathbb{C})$-module $\mathbb{C}^2$ to the reducible $M_2(\mathbb{R})$-module $\mathbb{R}^4 = \mathbb{R}^2 \oplus \mathbb{R}^2$. Therefore it is given by multiplication by two. for $p = -1$, and $p = -3$ the restriction map can only be zero. For $p = -2$ it can either be zero or reduction mod 2, but it turns out that we do not need to know which one it is in order to compute the K-theory. The only remaining map between representation rings is the K-theory functor applied to $\hat{R}$, i.e. conjugating modules over ${}^{\phi}\mathbb{C}^{\tau}H$ with R. Since R is in the center of ${}^{\phi}\mathbb{C}^{\tau}G$, the automorphism on $R^0({}^{\phi}\mathbb{C}^{\tau}H)$ resulting from this is trivial. Since $R^{-1}({}^{\phi}\mathbb{C}^{\tau}H) = R^{-2}({}^{\phi}\mathbb{C}^{\tau}H) = \mathbb{Z}_2$ and $\mathrm{Aut}\,\mathbb{Z}_2 = \{1\}$, the action of R is trivial here as well.

Secondly we have to decide on G-CW decompositions of S^2 and S^1. Since the action of R is the same as the action of T, we can reuse the G-CW structure of the last example, given in Figure 4.1, for S^2. For the circle we use the one-dimensional sub-G-CW complex of the G-CW structure on S^2.

We now compute the twisted equivariant K-theory of the circle by first calculating the relevant Bredon cohomology groups appearing on the second page.

- $H^0_G(S^1, {}^{\phi}\mathcal{R}^{\tau}_G)$: Just as in the last example, we can define a $\mathbb{Z}$-basis of equivariant 0-cochains π_0, π_∞ and 1-cochains λ. The restriction map is multiplication by two, so the differential is given by

$$d\pi_0(l) = d\pi_0(l) = {}^{\phi}\mathcal{R}^{\tau}_G(G/H \to G/G)(\pi_0(\partial l)) = 2 \implies d\pi_0 = 2\lambda.$$

  Similarly $d\pi_\infty = -2\lambda$. Hence $H^0_G(S^1, {}^{\phi}\mathcal{R}^{\tau}_G) = \mathbb{Z}$.

- $H^1_G(S^1, {}^{\phi}\mathcal{R}^{\tau,-1}_G)$: The twisted representation ring of G vanishes in this degree, so there are no 0-cochains. Hence this cohomology group is equal to the group of equivariant 1-cochains $\langle\lambda\rangle_{\mathbb{Z}_2}$ so that $H^1_G(S^1, {}^{\phi}\mathcal{R}^{\tau,-1}_G) \cong \mathbb{Z}_2$.

The spectral sequence now gives the exact sequence

$$0 \to \mathbb{Z}_2 \to {}^\phi K_G^\tau(S^1) \to \mathbb{Z} \to 0,$$

which is clearly split. Since ${}^\phi K_G^\tau(\mathrm{pt}) = R({}^\phi\mathbb{C}^\tau G) = \mathbb{Z}$, we see that ${}^\phi \widetilde{K}_G^\tau(S^1) = \mathbb{Z}_2$ for both circles in the splitting of the torus.

Now we turn to the computation of the twisted equivariant K-theory of the 2-sphere. We use the same bases of equivariant cochains $\{\pi_0, \pi_\infty\}, \{\lambda\}$ and $\{\alpha\}$ as in the last example.

- $H_G^0(S^2, {}^\phi\mathcal{R}_G^\tau)$: Exactly in the same way as for the circle, we see that $d\pi_0 = 2\lambda$ and $d\pi_\infty = -2\lambda$ so that
$$\ker d = H_G^0(S^2, {}^\phi\mathcal{R}_G^\tau) \cong \mathbb{Z}.$$
- $H_G^k(S^2, {}^\phi\mathcal{R}_G^{\tau,-1})$: Since $R^{-1}({}^\phi\mathbb{C}^\tau G) = 0$, there are no 0-cochains. The differential on 1-cochains is
$$\begin{aligned} d\lambda(A) &= {}^\phi\mathcal{R}_G^{\tau-1}(G/H \to G/G)(\lambda(\partial A)) = {}^\phi\mathcal{R}_G^{\tau-1}(G/H \to G/G)(\lambda(l) + \lambda(Rl)) \\ &= {}^\phi\mathcal{R}_G^{\tau-1}(G/H \to G/G)(\lambda(l) + {}^\phi\mathcal{R}_G^{\tau-1}(\hat{R})(\lambda(l))) \\ &= {}^\phi\mathcal{R}_G^{\tau-1}(G/H \to G/G)(2\lambda(l)) \\ &= 0, \end{aligned}$$
since we are working in $\mathbb{Z}_2$. Hence we get that
$$H_G^0(S^2, {}^\phi\mathcal{R}_G^{\tau,-1}) = 0, \quad H_G^1(S^2, {}^\phi\mathcal{R}_G^{\tau,-1}) = \mathbb{Z}_2, \quad H_G^2(S^2, {}^\phi\mathcal{R}_G^{\tau,-1}) = \mathbb{Z}_2.$$
- $H_G^2(S^2, {}^\phi\mathcal{R}_G^{\tau-2})$: The same computation as in the last point gives that the differential vanishes on 1-cochains, hence $H_G^2(S^2, {}^\phi\mathcal{R}_G^{\tau-2}) = \mathbb{Z}_2$.

As in the last example, we write down our spectral sequence table.

	$q=0$	$q=1$	$q=2$
$p=0$	$H_G^0(S^2, {}^\phi\mathcal{R}_G^\tau) = \mathbb{Z}$		
$p=1$	$H_G^0(S^2, {}^\phi\mathcal{R}_G^{\tau,-1}) = 0$	$H_G^1(S^2, {}^\phi\mathcal{R}_G^{\tau,-1}) = \mathbb{Z}_2$	$H_G^2(S^2, {}^\phi\mathcal{R}_G^{\tau,-1}) = \mathbb{Z}_2$
$p=2$			$H_G^2(S^2, {}^\phi\mathcal{R}_G^{\tau,-2}) = \mathbb{Z}_2$

The second differential $d_2 : H_G^0(S^2, {}^\phi\mathcal{R}_G^\tau) \to H_G^2(S^2, {}^\phi\mathcal{R}_G^{\tau,-1})$ is either zero or reduction modulo 2. Independent of this distinction, the kernel of d_2 is isomorphic to $\mathbb{Z}$. Hence the relevant part of the final page of the spectral sequence agrees with the antidiagonal $p+q=0$ in the table above.

The definition of the spectral sequence now gives exact sequences

$$0 \to \mathbb{Z}_2 \to F^{1,-1} \to \mathbb{Z}_2 \to 0,$$
$$0 \to F^{1,-1} \to {}^\phi K_G^\tau(S^2) \to \mathbb{Z} \to 0.$$

Note that the second sequence splits. Unfortunately, the first exact sequence implies only that $F^{1,-1} = \mathbb{Z}_2^2$ or $F^{1,-1} = \mathbb{Z}_4$. Hence the Atiyah-Hirzebruch spectral sequence gives that ${}^\phi K_G^\tau(S^2)$ is either $\mathbb{Z} \oplus \mathbb{Z}_2^2$ or $\mathbb{Z} \oplus \mathbb{Z}_4$, depending on whether the first exact sequence splits or not. We can conclude that

$$ {}^\phi K_G^\tau(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2^4 \quad \text{or} \quad {}^\phi K_G^\tau(\mathbb{T}^2) \cong \mathbb{Z} \oplus \mathbb{Z}_2^2 \oplus \mathbb{Z}_4. $$

Note that from the abstract data of the spectral sequence alone (not including the explicit expressions of mappings), we can never find which of the two answers is correct. Hence we have to find a geometric constraint in order to make this distinction. The details of a proof whether the K-theory is $\mathbb{Z} \oplus \mathbb{Z}_2^2$ or $\mathbb{Z} \oplus \mathbb{Z}_4$ will be left to further research, but a sketch of a geometric condition for the splitting of the exact sequence will now be derived.

Let $S^2 =: X \supseteq X^1 \supseteq X^0$ denote the G-CW structure on S^2. Unraveling the definitions of the spectral sequence results in the fact that the exact sequence looks like

$$ \begin{aligned} 0 \to \ker\left({}^\phi K_G^\tau(X) \to {}^\phi K_G^\tau(X^1)\right) &\to \ker\left({}^\phi K_G^\tau(X) \to {}^\phi K_G^\tau(X^0)\right) \\ &\xrightarrow{\pi} \ker\left({}^\phi K_G^\tau(X^1) \to {}^\phi K_G^\tau(X^0)\right) \to 0. \end{aligned} $$

The fact that the first term is $\mathbb{Z}_2$ can be formulated as follows. First of all, because the Bredon differential vanishes at this level, we have that

$$ \ker\left({}^\phi K_G^\tau(X) \to {}^\phi K_G^\tau(X^1)\right) \cong {}^\phi K_G^\tau(X, X^1). $$

Constructing isomorphisms with the Bredon cocycles in a similar way to the general case of Section 3.11, we get

$$ \begin{aligned} \ker\left({}^\phi K_G^\tau(X) \to {}^\phi K_G^\tau(X^1)\right) &\cong {}^\phi \widetilde{K}_G^\tau(X/X^1) \cong {}^\phi \widetilde{K}_G^\tau(\Sigma^2(\mathbb{Z}_2 \sqcup \mathrm{pt})) \\ &\cong {}^\phi \widetilde{K}_G^{\tau-2}(\mathbb{Z}_2 \sqcup \mathrm{pt}) \cong {}^\phi K_G^{\tau-2}(\mathbb{Z}_2) \\ &\cong {}^{\phi|_{\mathbb{Z}_2}} K_{\mathbb{Z}_2}^{\tau|_{\mathbb{Z}_2}-2}(\mathrm{pt}) \cong KO^{-2}(\mathrm{pt}) \\ &\cong \mathbb{Z}_2. \end{aligned} $$

Indeed the subgroup $\mathbb{Z}_2 \subseteq G$ in the formulae above is generated by TR, so that ϕ is nontrivial and τ is trivial, giving KO-theory. Completely analogously we get that $\ker\left({}^\phi K_G^\tau(X^1) \to {}^\phi K_G^\tau(X^0)\right) \cong KO^{-1}(\mathrm{pt}) \cong \mathbb{Z}_2$.

Now, what is the algebraic constraint on the exact sequence for the existence of a section of the map π that is also a homomorphism? This is nothing more than the existence of an element α in $\ker\left({}^\phi K_G^\tau(X) \to {}^\phi K_G^\tau(X^0)\right)$ such that $2\alpha = 0$, but $\pi(\alpha) \neq 0$. Note also that since for every class α, we necessarily have that $\pi(2\alpha) = 2\pi(\alpha) = 0$, it is immediate that $2\alpha \in \ker\left({}^\phi K_G^\tau(X) \to {}^\phi K_G^\tau(X^1)\right)$.

We now translate this into a geometric constraint. Write $\alpha = [E] - [\theta_M]$, where M is a module over the (ϕ, τ)-twisted group algebra and E is a (ϕ, τ)-twisted G-equivariant vector bundle over X such that $E_{p_0} \cong E_{p_\infty} \cong M$. The condition that $\pi(\alpha) \neq 0$ is equivalent to saying that on the circle X^1, E is not stably equivalent to a trivial bundle. Let us follow the (analogous versions of the) above isomorphisms around for $\ker\left({}^{\phi}K_G^{\tau}(X^1) \to {}^{\phi}K_G^{\tau}(X^0)\right) \cong \mathbb{Z}_2$ to get a representative of the nontrivial element. A real bundle representing the nontrivial class in $\widetilde{KO}(S^1)$ is the Möbius bundle. This gives a (ϕ, τ)-twisted G-equivariant bundle over X^1 by putting this Möbius bundle along one path from p_0 to p_∞ and then extending it uniquely to the other half of X^1 by requiring the bundle map R to be an isomorphism squaring to -1.

On the other hand, using the correspondence between (isomorphism classes of) n-dimensional real bundles over S^2 and (homotopy classes of) clutching functions $S^1 \to O(n)$, we get a two-dimensional real bundle over S^2 with clutching function

$$\theta \mapsto \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

This bundle represents the nontrivial class in $\widetilde{KO}(S^2)$. For proofs of the facts in this paragraph, see for example Theorem III.5.19 in Karoubi's book.

To get a nontrivial class in the (ϕ, τ)-twisted G-equivariant K-theory of the wedge of two spheres $\Sigma^2(\mathbb{Z}_2 \times \mathrm{pt})$ in which R flips the two spheres, we again just demand R to be a bundle isomorphism squaring to -1. To finally get a bundle over X itself, we enlarge the point where the two spheres are connected to a circle to get X and extend the bundle with constant fiber on X^1. We can now finally formulate the geometric condition for the exact sequence to split: does there exist (ϕ, τ)-twisted G-equivariant bundles E and F over X with the following properties?

- $E_{p_0} \cong E_{p_\infty}$ as a module over the twisted group algebra $M_2(\mathbb{C})$ (this is of course always true);
- the bundle F stably equivalent to $E \oplus E$ and $F|_{X^1}$ is a trivial bundle (an F with these properties actually exists for all E). Then the real bundle induced by F on one (hence both) copies of the wedge sum of two spheres resulting from pinching X^1 to a point and taking TR to be the real structure is stably trivial;
- the restriction $E|_{X^1}$ is a nontrivial bundle, i.e. the real bundle over S^1 coming from the restriction of $E|_{X^1}$ to one half-circle and imposing the isomorphism $E_{p_0} \cong E_{p_\infty}$ is nonorientable.

The author suggests that this problem can perhaps be solved using the theory of characteristic classes.

Several other examples in $d = 2$ were computed by this method, also in class AI. The results are summarized in the provided table. Note that in the cases where a reflection is included, we have to make a choice on whether it squares to ± 1 in the twisted group algebra. Note that this can change the isomorphism class and it can be shown that it actually changes the K-theory as well[9]. Just as in the work of Kruthoff et al. [43], we picked the positive sign in class AI and the negative sign in class AII.

[9]For example, class AII pm with $S^2 = 1$ was shown to give $\mathbb{Z}^3$ in K-theory, compare with the given table.

	p1	p2	pm	p4
class AI	$\mathbb{Z}$	$\mathbb{Z}^5 \oplus T_1$	$\mathbb{Z}^3$	$\mathbb{Z}^6 \oplus T_1'$
class AII	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z} \oplus T_2$	$\mathbb{Z} \oplus \mathbb{Z}_2^2$	$\mathbb{Z}^3 \oplus T_2'$

Here T_1, T_2, T_1' and T_2' are torsion groups constrained as follows:

- T_1 is either $\mathbb{Z}_2^3$, $\mathbb{Z}_2^4$ or $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$;
- T_2 is either $\mathbb{Z}_2^4$ or $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$;
- T_1' is either 0 or $\mathbb{Z}_2$;
- T_2' is either $\mathbb{Z}_2^3$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_4$.

Deriving further geometric constraints on these K-theory groups in order to determine the precise torsion groups would be an interesting further research direction.

Populaire Samenvatting

Topologische isolatoren zijn een recent ontdekte nieuwe klasse materialen. Ze komen niet voor in de natuur, maar kunnen wel gemaakt worden in het lab. Er wordt veel onderzoek gedaan naar topologische isolatoren vanwege hun mogelijke toepassingen in bijvoorbeeld kwantumcomputers, maar ook vanwege hun fascinerende eigenschappen. Topologische isolatoren geleiden namelijk geen elektriciteit van binnen, maar wel op het oppervlak. Dit is een intrinsieke eigenschap: als je een stuk topologische isolator in twee stukken zou snijden, zou het nieuw gecreëerde oppervlak plotseling gaan geleiden.

Wat is de oorsprong van zo'n gekke eigenschap? Natuurkundigen hebben zich hier jaren lang het hoofd over gebroken en er is zelfs in 2016 een Nobelprijs voor uitgereikt. Drie aanwijzingen voor het antwoord op deze vraag zouden kunnen zijn:

1. topologische isolatoren werken alleen goed bij lage temperaturen;
2. topologische isolatoren werken alleen goed als het rooster dat de atomen vormen op microscopisch niveau heel netjes symmetrisch is;
3. topologische isolatoren werken niet meer in een magneetveld.

Veel natuurkundige verschijnselen werken alleen goed bij lage temperaturen. De reden hiervoor is vaker wel dan niet dat kwantummechanica een rol speelt en die suggestie is hier dan ook juist. Het tweede punt komt ook vaker voor in natuurkunde en betekent dat er een symmetriegroep G is, waaronder het systeem invariant is; in dit geval is $G = \mathbb{Z}^3$ de translatiesymmetrie van het atoomrooster. Het derde punt is iets mysterieuzer, maar het blijkt dezelfde reden te hebben als punt twee. Magneetvelden breken namelijk tijdsomkeersymmetrie $\mathbb{Z}/2\mathbb{Z}$; de symmetrie die komt van de transformatie $t \mapsto -t$. We kunnen dus concluderen dat de theorie van topologische isolatoren zich afspeelt in kwantummechanische systemen die invariant zijn onder een symmetriegroep $G = \mathbb{Z}^3 \times \mathbb{Z}/2\mathbb{Z}$ bij minimale temperatuur $T = 0K$.

De meeste van zulke kwantummechanische systemen kun je in elkaar 'omvormen'. Iets preciezer: je kan kijken of twee zulke systemen met een pad te verbinden zijn in 'de ruimte van alle mogelijke kwantummechanische systemen'. Het blijkt dat je dit wiskundig precies kan maken en kan uitrekenen hoe veel 'componenten' er zijn in deze ruimte, dus typen systemen die je niet met elkaar kan verbinden. Men noemt zulke componenten ook wel *topologische fasen beschermd door de symmetrie* G. Het blijkt dat er voor $G = \mathbb{Z}^3 \times \mathbb{Z}/2\mathbb{Z}$ zeker twee topologische fasen zijn beschermd door G. Deze twee sytemen kunnen gerepresenteerd worden door een topologische isolator en een gebruikelijk stuk materiaal (of het vacuum). De reden dat er zoiets geks gebeurt op de rand van een topologische isolator is dus dat de topologische fase in het materiaal anders is dan de topologische fase van het vacuum; er vindt een *topologische faseverandering* plaats.

In deze scriptie wordt een theorie ontwikkeld voor het classificeren van topologische fasen voor algemenere symmetrieën G. In het bijzonder wordt er gekeken naar de mogelijkheid dat G meer kristalsymmetrieën bevat, zoals draaiïngen en spiegelingen van het atoomrooster. Er wordt rekengereedschap geconstrueerd om de verzameling van topologische fasen te berekenen en deze berekeningen worden uitgevoerd in een paar simpele gevallen.

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A. Group Theory: Extensions and Cohomology

The problem of classifying extensions of discrete groups is classical and a main motivation for the development of group cohomology. In quantum mechanics, extensions naturally appear as anomalies via the Wigner extension. Multiple complications arise because of the relevant topological structure of the groups involved. However, since the main focus of this document is on finite groups, these complications are mostly ignored.

A.1. Group Extensions and Cohomology

To briefly let the reader become acquainted with the influence of topology on the general setting, we start in the most general context.

Definition A.1. A *topological group* G is a Hausdorff topological space, which is also a group such that the multiplication and inversion map are continuous. Given a topological group G, a *topological G-module* is an abelian topological group A which is also a G-module such that the action map is continuous.

The analogue of a group extension in the topological world could be argued to be a fibration. This motivates the following definition.

Definition A.2. If G and A are topological groups, then a *group extension* of G by A is a topological group E together with a short exact sequence

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

of continuous group homomorphisms such that π is a fibration. Group extensions of a given G by a given A form a category; a morphism from E to E' is a continuous group homomorphism such that the obvious diagram commutes. We say an extension is *split* if there exists a section of π that is a continuous group homomorphism.

Compared to the discrete case, the theory of extensions of general topological groups is subtle; a large number of different assumptions on group extensions can imposed, yielding classifications using different cohomology theories. For discrete groups, results simplify considerably.

For example, because $i(A) = \ker \pi$, we see that $i(A) \subseteq E$ is a closed normal subgroup. Hence π induces an isomorphism $G \cong E/A$ of topological groups. If E is a discrete group and A an abelian normal subgroup, then

$$1 \to A \to E \to E/A \to 1$$

is a group extension. Therefore, for discrete groups, the question of classifying extensions of G by A is equivalent to classifying groups E with a subgroup isomorphic to A such that $E/A \cong G$ (modulo suitable isomorphism). For more general groups, this equivalence does not hold.

Note however that the fibration assumption does imply the desirable property that an extension can also be seen as a principal A-bundle of topological spaces $\pi : E \to G$. In particular, if π has a continuous section (which is not necessarily assumed to be a group homomorphism), then E is a trivial principal bundle, hence homeomorphic to $A \times G$. Moreover, if such a continuous section exists and A is abelian, the group extension can be represented by a cocyle in continuous group cohomology. Unfortunately, not all exact sequences in this document have continuous sections and so we are bound to sometimes loosen our definition of cocycle somewhat by only assuming a measurable section to exist. We now state all relevant definitions.

Definition A.3. Let $C^k(G, A) := C(G^k, A)$ denote the abelian group of all continuous maps from G^k to A, which will be called *continuous (group) k-cochains*. Similarly, let $C^k_m(G, A)$ be the abelian group of all measurable maps $G^k \to A$, called *measurable (group) k-cochains*. Then $C^k(G, A)$ and $C^k_m(G, A)$ become a cochain complex by[1]

$$\begin{aligned} d^{n+1}\tau(g_1, \dots, g_{n+1}) &= g_1\tau(g_2, \dots, g_{n+1}) + \\ &\sum_{i=1}^{n}(-1)^i \tau\left(g_1, \dots, g_{i-1}, g_i g_{i+1}, g_{i+2}, \dots, g_{n+1}\right) + (-1)^{n+1}\tau(g_1, \dots, g_n). \end{aligned}$$

The corresponding cocycles $Z^k(G, A)$ and $Z^k_m(G, A)$ are called *continuous (group) k-cocycles* and *measurable (group) k-cocycles* respectively. The corresponding cohomology theories are called *continuous group cohomology* $H^\bullet(G, A)$ and *measurable group cohomology* $H^\bullet_m(G, A)$ respectively.[2] Note that if G is discrete, $H^\bullet(G, A)$ and $H^\bullet_m(G, A)$ both are equal to algebraic group cohomology (in which $C^k(G, A) = \mathrm{Maps}(G^k, A)$).

Remark A.4. This definition of group cohomology may be the first reasonable thing that comes to mind to the reader that already knows algebraic group cohomology, but unfortunately it is not the 'right' definition. This has to do with the fact that not every group extension admits a continuous or measurable section. The agreement is that the right definition of group cohomology of topological groups is the one of Segal, developed in [58] (see [59] for a few basic properties of Segal's group cohomology). However, since we prefer to work with explicit formulae, we avoid the most general definition and work with (preferably continuous) cocycles instead.

Remark A.5. In this document it is often useful to see (algebraic, topological or measurable) group k-cocycles as a category in the following way:

- objects are cocycles $Z^k(G, A)$;

[1] For readability purposes, we will write G multiplicatively and A additively. However, the reader should be aware that most G-modules in the main text and below are written multiplicatively.

[2] Meaurable group cohomology is often also referred to as Borel group cohomology, but since this language can be confused with Borel equivariant cohomology, we do not use this term here.

- morphisms between two cocycles $\tau_1, \tau_2 \in Z^k(G, A)$ are $(k-1)$-cochains whose boundaries connect them, so $\lambda \in Z^{k-1}(G, A)$ such that $d\lambda + \tau_1 = \tau_2$.

In that seemingly artificial way one can easily adopt a categorical point of view within the set of cocycles in such a way that one can work with explicit cocycles, but isomorphism classes of cocycles correspond to cohomology classes.

Now let

$$1 \to A \xrightarrow{i} E \xrightarrow{\pi} G \to 1$$

be a group extension of topological groups with A abelian (denoted multiplicatively), for which there exists a continuous section $s : G \to E$. By multiplying s with a fixed element of E, we can assume without loss of generality that $s(1) = 1$. If s is a homomorphism, then the extension is split. So the expression

$$\tau(g_1, g_2) := s(g_1)s(g_2)s(g_1 g_2)^{-1} \quad g_1, g_2 \in G$$

is a reasonable measure for the failure of the sequence to be split (and the split exact sequences are expected to correspond to trivial cocycles). Note that $\pi(\tau(g_1, g_2)) = 1$ and therefore the above expression gives a continuous map $\tau : G \times G \to A$. Hence s is a homomorphism 'up to factors in A' in the sense that $\tau(g_1, g_2)s(g_1 g_2) = s(g_1)s(g_2)$. We expect τ to be the obstruction, living in a continuous cohomology group classifying group extensions. This will now be made precise. Note that by the choice $s(1) = 1$, τ is unital in the following sense.

Definition A.6. We say a continuous group cochain $\tau \in C^k(G, A)$ is *unital* if $\tau(g_1, \ldots g_k) = 0$ whenever $g_i = 1$ for some i.

With what action is τ a group cocycle? It is the same action as the one corresponding to the semi-direct product structure of a split extension: consider the continuous map $\alpha : G \times A \to A$ given by $\alpha_g(a) = s(g)as(g)^{-1}$. Note that because $A \subseteq E$ is a normal subgroup, α is well-defined. The reader should be aware that although A is abelian, in general α is nontrivial since E can be nonabelian.

Lemma A.7. *The map α makes A into a G-module, written A_α and $\tau \in Z^2(G, A_\alpha)$ is a unital 2-cocycle. A different choice of section gives the same action and the new τ differs by a coboundary.*

Proof. The only nontrivial property to check in order for A to be a topological module is associativity of α:

$$\begin{aligned}
\alpha_{g_1 g_2}(a) &= s(g_1 g_2)as(g_1 g_2)^{-1} \\
\alpha_{g_1}(\alpha_{g_2}(a)) &= s(g_1)s(g_2)as(g_2)^{-1}s(g_1)^{-1} \\
&\implies s(g_1 g_2)^{-1}\alpha_{g_1}(\alpha_{g_2}(a)))s(g_1)s(g_2) = \tau(g_1, g_2)a\tau(g_1, g_2)^{-1} = a \\
&\implies \alpha_{g_1 g_2}(a) = \alpha_{g_1}(\alpha_{g_2}(a)).
\end{aligned}$$

An easy computation now shows that τ is a cocycle with corresponding action α:

$$\begin{aligned}\alpha_{g_1}(\tau(g_2,g_3))\tau(g_1,g_2g_3)\tau(g_1g_2,g_3)^{-1}\tau(g_1,g_2)^{-1} = \\ s(g_1)s(g_2)s(g_3)s(g_2g_3)^{-1}s(g_1)^{-1}s(g_1)s(g_2g_3)s(g_1g_2g_3)^{-1} \\ s(g_1g_2g_3)s(g_3)^{-1}s(g_1g_2)^{-1}s(g_1g_2)s(g_2)^{-1}s(g_1)^{-1} = 1\end{aligned}$$

Next it will be shown that the cohomology class $[\tau] \in H^2(G, A)$ is independent of the choice of section s. So let s' be another section with corresponding cocycle τ. First note that $\pi(s(g)) = \pi(s'(g))$ implies that $\eta(g) := s'(g)s(g)^{-1} \in A$ for all $g \in G$. Since A is abelian, this implies that the actions are equal:

$$s'(g)as'(g)^{-1} = s(g)s'(g)^{-1}s'(g)as'(g)^{-1}s'(g)s(g)^{-1} = s(g)as(g)^{-1}.$$

Now note that τ and τ' differ by $d\eta$:

$$\begin{aligned}d\eta(g_1,g_2)\tau(g_1,g_2) = \\ \eta(g_1)\alpha_{g_1}(\eta(g_2))\tau(g_1,g_2)\eta(g_1g_2)^{-1} = \\ s'(g_1)s(g_1)^{-1}s(g_1)s'(g_2)s(g_2)^{-1}s(g_1)^{-1} \\ s(g_1)s(g_2)s(g_1g_2)^{-1}s(g_1g_2)s'(g_1g_2)^{-1} = \\ s'(g_1)s'(g_2)s(g_2)^{-1}s(g_2)s'(g_1g_2)^{-1} = \tau'(g_1,g_2).\end{aligned}$$

□

By the above lemma, a group extension of G by abelian A gives a well defined class $[\tau] \in H^2(G, A)$ if it has a continuous section. A stronger statement is presented now.

Theorem A.8. *Let G be a topological group and A a topological G-module. Then the above construction of a group cocycle induces a one-to-one correspondence between $H^2(G, A)$ and isomorphism classes of topological group extensions of G by the topological group A, for which*

- *there exists a continuous section s;*
- *the induced action equals the given action: $g \cdot a = s(g)as(g)^{-1}$.*

Moreover, under this correspondence, the pull-back extension corresponds to the pull-back in cohomology and the push-forward extension corresponds to the push-forward in cohomology. More generally, if G and A are both locally compact then $H^2_m(G, A)$ classifies extensions that do not necessarily possess a continuous section.

Proof. See the survey of Stasheff [63] and references therein. □

Pull-backs and push-forwards of extensions are defined similarly as in the algebraic setting, see for example Appendix A of Freed & Moore [27]. Pull-backs and push-forwards in continuous cohomology are defined as follows. Given topological groups G and H, a topological G-module A and a continuous homomorphism $f : H \to G$, the module A

becomes a topological H-module by $h \cdot a := f(g)a$. It is easy to check that using this module structure, precomposition induces a map $f^* : H^\bullet(G, A) \to H^\bullet(H, A)$, which is the pull-back. On the other hand, if A' is another topological G-module and $\phi : A \to A'$ is a continuous G-module homomorphism, there is a push-forward mapping $\phi_* : H^\bullet(G, A) \to H^\bullet(G, A')$.

In order to be able to work with continuous cocycles as much as possible, but still use short exact sequences that do not admit continuous sections, we state the following comparison result.

Theorem A.9. *If G is locally compact, σ-compact and zero-dimensional*[3] *and A is metrizable, complete and second-countable, then $H^n(G, A) \cong H^n_m(G, A)$;*

Proof. See Wigner's paper[69] for a proof. □

The following lemma is easy to prove.

Lemma A.10. *Let G, H be topological groups with H abelian. Consider H as a trivial G-module. Then*

$$H^0(G, H) = H$$
$$H^1(G, H) = \mathrm{Hom}_{cts}(G, H),$$

where $\mathrm{Hom}_{cts}(G, A)$ denotes continuous group homomorphisms.

For proofs of the following statements, see the references in [63].

Proposition A.11. *Let G be a topological group and let*

$$0 \to A_1 \to A_2 \xrightarrow{\phi} A_3 \to 0$$

be an exact sequence of topological G-modules. If ϕ has a continuous section, then there is a long exact sequence

$$0 \to H^0(G, A_1) \to H^0(G, A_2) \to H^0(G, A_3) \to H^1(G, A_1) \to \dots.$$

Similarly, if ϕ has a measurable section, then this long exact sequence exists in measurable cohomology.

Proposition A.12. *Let G be a finite group and let A, A' be topological G-modules. A continuous family of continuous module homomorphisms $\phi_t : A \to A'$ leaves the map on cohomology constant:*

$$(\phi_0)_* = (\phi_1)_* : H^\bullet(G, A) \to H^\bullet(G, A').$$

[3] In the sense of Lebesgue covering dimension. So this means that every open covering U_i of G has a refinement $V_i \subseteq U_i$ of disjoint opens that still cover G.

Corollary A.13. *If A, A' are topological G-modules that are homotopic in the sense that there exist continuous G-maps $f : A \to A'$ and $g : A' \to A$ and continuous families $\phi_t : A \to A$ and $\psi_t : A' \to A'$ of continuous module homomorphisms such that*

$$\psi_0 = fg, \quad \psi_1 = id_{A'}, \quad \phi_0 = gf, \quad \phi_1 = id_A,$$

then $f_* H^\bullet(G, A) \to H^\bullet(G, A')$ is an isomorphism with inverse g_*.

The following proposition is the essential tool in computations of the groups of quantum anomalies in case the symmetry group is a point group.

Proposition A.14 (Künneth Formula)**.** *Let G, G' be finite groups, M a G-module and M' a G'-module (both discrete) such that at least one of M and M' is torsion-free. For $n \geq 0$, there is a split short exact sequence of abelian groups*

$$\begin{aligned} 0 \to \bigoplus_{p=0}^{n} H^p(G, M) \otimes H^{n-p}(G', M') \to H^n(G \times G', M \otimes M') \\ \to \bigoplus_{p=0}^{n+1} \operatorname{Tor}(H^p(G, M), H^{n+1-p}(G', M')) \to 0, \end{aligned}$$

where the tensor products and Tor *functor are over $\mathbb{Z}$.*

Proof. This is an application of a standard theorem in homological algebra, see for example Proposition I.0.6 in Brown's book [15]. Thetorsion-free assumption is needed, because at least one of the two complexes involved must be dimension-wise free. ☐

Corollary A.15. *Let G be a group, M a discrete G-module and A a discretetorsion-free abelian group. Then there is a split short exact sequence*

$$0 \to H^n(G, M) \otimes A \to H^n(G, M \otimes A) \to \operatorname{Tor}(H^{n+1}(G, M), A) \to 0.$$

A.2. 2-Cocycles as Twists and Anomalies

Let G be a finite group and A a topological G-module.

Lemma A.16. *Every degree 2 cohomology class $[\tau] \in H^2(G, A)$ has a unital representative.*

Proof. First note that for all $x \in G$

$$\tau(x, 1) = x \cdot \tau(1, 1) \quad \text{and} \quad \tau(1, x) = \tau(1, 1).$$

Indeed, in the cocycle relation set $g_1 = x, g_2 = g_3 = 1$ respectively $g_1 = g_2 = 1, g_3 = x$ to get

$$\begin{aligned} x \cdot \tau(1,1)\tau(x,1)^{-1}\tau(x,1)\tau(x,1)^{-1} = 1 \\ \tau(1,x)\tau(1,x)^{-1}\tau(1,x)\tau(1,1)^{-1} = 1, \end{aligned}$$

respectively. Now let $\eta : G \to A$ be constantly equal to $\tau(1,1)^{-1}$. Then for all $x \in G$

$$\tau(x,1)d\eta(x,1) = \tau(x,1)x \cdot \eta(1)\eta(x)^{-1}\eta(x) = x \cdot \tau(1,1)x \cdot (\tau(1,1))^{-1} = 1$$
$$\tau(1,x)d\eta(1,x) = \tau(1,x)\eta(x)\eta(x)^{-1}\eta(1) = \tau(1,1)\tau(1,1)^{-1} = 1$$

so $\tau d\eta$ is the desired representative of $[\tau]$. □

Now suppose $\phi : G \to \mathbb{Z}_2$ is a homomorphism and $A := U(1)_\phi$ is the topological G-module $U(1)$ where G acts by $g \cdot z = z^{\phi(g)}$. For applications to twisted group algebras, we prove some basic properties of continuous 2-cocycles with values in A. Recall the definition of the twisted group algebra of Section 1.6. In the following lemma, we use the language of superalgebras of Appendix B.

Lemma A.17. *Let $\tau, \tau' \in Z^2(G, U(1)_\phi)$ be unital 2-cocyles and $\lambda : G \to U(1)$ a unital cochain such that $\tau' = \tau \cdot d\lambda$. Write x_g and $\tilde{x}_g$ for the usual complex bases of ${}^\phi\mathbb{C}^{\tau,c}G$ and ${}^\phi\mathbb{C}^{\tau',c}G$ respectively. Then for any grading $c : G \to \mathbb{Z}_2$, the canonical $\mathbb{C}$-linear map*

$$f : {}^\phi\mathbb{C}^{\tau,c}G \to {}^\phi\mathbb{C}^{\tau',c}G \quad x_g \mapsto \lambda(g)\tilde{x}_g$$

is an isomorphism of superalgebras over $\mathbb{R}$.

Proof. By definition, f is a bijective linear map that preserves the grading. Therefore it only has to be shown that f is an algebra homomorphism. Because $\tau(g,1) = \tau(1,g) = \tau'(g,1) = \tau'(1,g) = 1$ we have that x_1 and $\tilde{x_1}$ are the units of the two algebras. Since $\lambda(1) = 1$, we have $f(1) = 1$. Now consider the following computation:

$$\begin{aligned} f(z_1 x_{g_1} z_2 x_{g_2}) &= f(z_1 z_2^{\phi(g_1)} \tau(g_1,g_2) x_{g_1 g_2}) \\ &= z_1 z_2^{\phi(g_1)} \tau(g_1,g_2)\lambda(g_1 g_2)\tilde{x}_{g_1 g_2}, \end{aligned}$$

and compare it to

$$\begin{aligned} f(z_1 x_{g_1}) f(z_2 x_{g_2}) &= z_1 \lambda(g_1)\tilde{x}_{g_1} z_2 \lambda(g_2)\tilde{x}_{g_2} \\ &= z_1 z_2^{\phi(g_1)} \lambda(g_1)\lambda(g_2)^{\phi(g_1)} \tau(g_1,g_2) d\lambda(g_1,g_2)\tilde{x}_{g_1 g_2}. \end{aligned}$$

Exactly because of the definition of $d\lambda$, these two elements are equal. □

Lemma A.18. *Every cohomology class in $H^2(G, U(1)_\phi)$ has a representative $\tau \in Z^2(G, U(1)_\phi)$ such that for all $g \in G$*

$$\tau(g,g^{-1}) = \tau(g^{-1},g)^{\phi(g)} = \begin{cases} 1 & \text{if either } \phi(g) = 1 \text{ or } g^{-1} \neq g, \\ \pm 1 & \text{if both } \phi(g) = -1 \text{ and } g^{-1} = g. \end{cases}$$

Proof. Let $\tau \in Z^2(G, U(1)_\phi)$ be a representative such that $\tau(x,1) = \tau(1,x) = 1$. First note that we could as well work in the twisted group algebra, because the cocycle property of τ is equivalent to the associativity in the twisted group algebra (and associativity is easier to handle). Hence we compute

$$\tau(g^{-1},g)x_{g^{-1}} = x_{g^{-1}} x_g x_{g^{-1}} = x_{g^{-1}} \tau(g,g^{-1}) \implies \tau(g,g^{-1}) = \tau(g^{-1},g)^{\phi(g)}$$

so that the first equality holds even for all 2-cocycles. Now we define our candidate $\lambda : G \to U(1)$ such that $d\lambda\tau$ satisfies the conditions of the lemma, first on $g \in G$ such that $\phi(g) = 1$. Take for $\lambda(g)$ any square root of $\tau(g, g^{-1})^{-1}$ making sure that it remains constant on inverses and that $\lambda(1) = 1$. In other words,

$$\lambda(g)^2 = \tau(g, g^{-1})^{-1} \quad \lambda(g) = \lambda(g^{-1}) \quad \text{for } g \in G \text{ such that } \phi(g) = 1.$$

This choice can be made consistently since $\phi(g) = 1$ implies

$$\lambda(g^{-1})^2 = \tau(g, g^{-1})^{-1} = \tau(g^{-1}, g)^{-1} = \lambda(g)^2.$$

Secondly we define λ on $g \in G$ with $\phi(g) = -1$ such that $g \neq g^{-1}$. Then instead we pick $\lambda(g)$ such that

$$\lambda(g)^2 = \tau(g, g^{-1})^{-1} \quad \lambda(g)^{-1} = \lambda(g^{-1}) \quad \text{for } g \in G \text{ such that } \phi(g) = -1 \text{ and } g \neq g^{-1}.$$

This can be done consistently since $g \neq g^{-1}$ and

$$\lambda(g^{-1})^2 = \tau(g, g^{-1})^{-1} = \tau(g^{-1}, g) = (\lambda(g)^2)^{-1} = (\lambda(g)^{-1})^2.$$

Thirdly, for the element with $g = g^{-1}$ and $\phi(g) = -1$, we put $\lambda(g) = 1$. We now check that $\tau' = \tau d\lambda$ satisfies the desired properties. Note that since $\lambda(1) = 1$,

$$d\lambda(g, g^{-1}) = \lambda(g^{-1})^{\phi(g)}\lambda(g) \implies \tau'(g, g^{-1}) = \tau(g, g^{-1})\lambda(g^{-1})^{\phi(g)}\lambda(g).$$

We now distinguish the cases. If $\phi(g) = 1$, then

$$\begin{aligned}\tau'(g, g^{-1}) &= \tau(g, g^{-1})\lambda(g^{-1})\lambda(g) = \tau(g, g^{-1})\lambda(g)^2 \\ &= \tau(g, g^{-1})\tau(g, g^{-1})^{-1} = 1.\end{aligned}$$

If $\phi(g) = -1$and $g \neq g^{-1}$, then

$$\begin{aligned}\tau'(g, g^{-1}) &= \tau(g, g^{-1})\lambda(g^{-1})^{-1}\lambda(g) = \tau(g, g^{-1})\lambda(g)^2 \\ &= \tau(g, g^{-1})\tau(g, g^{-1})^{-1} = 1.\end{aligned}$$

Finally if $\phi(g) = -1$ and $g^{-1} = g$, then

$$\tau'(g, g^{-1}) = \tau(g, g^{-1}) = \tau(g^{-1}, g)^{\phi(g)} = \tau(g, g^{-1})^{\phi(g)} = \tau'(g, g^{-1})^{-1}.$$

Hence $\tau'(g, g^{-1}) = \pm 1$. □

Remark A.19. To illustrate that both signs can occur in the situation where $\phi(g) = -1$ and $g^{-1} = g$, consider $G = \mathbb{Z}_2$ with ϕ the isomorphism. Let τ be a cocycle representing a class in $H^2(G, U(1)_\phi)$, which in Lemma A.21 was shown to be $\mathbb{Z}_2$. By Lemma A.16 we can assume that $\tau(1, 1) = \tau(-1, 1) = \tau(1, -1) = 1$. By Lemma A.18 we can also assume $\tau(-1, -1) = \pm 1$. These two cases must hence distinguish the trivial from the nontrivial class.

A.3. Anomaly Twist Computations

Anomalies as defined in this document are classified by the cohomology group $H^2(G, U(1)_\phi)$. In the applications that will be studied in this section, symmetry groups G are a point group or a magnetic point group of some crystal (i.e. $G = P \times \mathbb{Z}_2$ for a point group P and ϕ is projection onto the second term). Point groups are classified as certain finite subgroups of the orthogonal group, see Section 2.3. In this section, the group of isomorphism classes of anomaly twists will be computed for these examples using the Künneth formula. In order to simplify computations to algebraic cohomology, we first prove the following.

Lemma A.20. *There is an isomorphism $H^2(G, U(1)_\phi) \cong H^3(G, \mathbb{Z}_\phi)$, where $\mathbb{Z}_\phi$ is the G-module $g \cdot n := \phi(g)n$.*

Proof. Consider the exact sequence of topological G-modules

$$0 \to \mathbb{Z}_\phi \to \mathbb{C}_\phi \to \mathbb{C}^*_\phi \to 0,$$

where $\mathbb{C}_\phi$ becomes a G-module by $g \cdot z = \phi(g)z$ and $\mathbb{C}^*_\phi$ becomes a G-module by $g \cdot w = w^{\phi(g)}$. This exact sequence does not have a continuous section, but it does have a piecewise continuous section. Hence by Proposition A.11, there is a corresponding long exact sequence in measurable group cohomology.

Now we apply Theorem A.9 to all cohomology groups in the sequence to replace them by continuous cohomology groups. Note that $\mathbb{C}^*_\phi$ is not complete in its usual metric, but we can give it a metric such that it becomes complete (for example, $\mathbb{C}^*$ is homeomorphic to an infinite cylinder). Also, G is finite, so the conditions on G are clearly satisfied.

Now consider the following part of the sequence:

$$\cdots \to H^2(G, \mathbb{C}_\phi) \to H^2(G, \mathbb{C}^*_\phi) \to H^3(G, \mathbb{Z}_\phi) \to H^3(G, \mathbb{C}_\phi) \to \dots$$

Since $\mathbb{C}_\phi$ is homotopic to the zero-module in the category of G-modules (see Proposition A.12 and the corresponding corollary) we have that

$$H^i(G, \mathbb{C}_\phi) \cong H^i(G, 0) = 0.$$

Similarly we have that $\mathbb{C}^*_\phi$ is homotopic to $U(1)_\phi$. Hence

$$H^i(G, \mathbb{C}_\phi) \cong H^i(G, U(1)_\phi).$$

The above exact sequence therefore gives the desired isomorphism. □

We continue by summarizing some typical algebraic cohomology groups of point groups. In low order (e.g. for applications of this chapter) the following result can also be found using computer algebra programmes.

Lemma A.21. *Let D_n denote the dihedral group of order $2n$ and $\mathbb{Z}_\phi$ the $\mathbb{Z}_2$-module whose underlying abelian group is $\mathbb{Z}$ and the action is given by $n \mapsto -n$.*

$$H^p(\mathbb{Z}_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0, \\ \mathbb{Z}_n & p > 0, p \text{ even}, \\ 0 & p \text{ odd}. \end{cases}$$

$$H^p(\mathbb{Z}_2, \mathbb{Z}_\phi) = \begin{cases} \mathbb{Z} & p = 0, \\ \mathbb{Z}_2 & p \text{ odd}, \\ 0 & p > 0, p \text{ even}. \end{cases}$$

If n is odd then

$$H^p(D_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0, \\ \mathbb{Z}_{2n} & p > 0, p \equiv 0 \pmod 4, \\ \mathbb{Z}_n & p \equiv 2 \pmod 4, \\ 0 & p \text{ odd}. \end{cases}$$

If n is even then

$$H^p(D_n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & p = 0, \\ \mathbb{Z}_n \times (\mathbb{Z}_2)^{p/2} & p > 0, p \equiv 0 \pmod 4, \\ (\mathbb{Z}_2)^{(p+2)/2} & p \equiv 2 \pmod 4, \\ (\mathbb{Z}_2)^{(p-1)/2} & p \text{ odd}. \end{cases}$$

Remark A.22. Since the group $H^2(\mathbb{Z}_2, U(1)_\phi) \cong H^3(\mathbb{Z}_2, \mathbb{Z}_\phi)$ plays an important role in the main text, we briefly sketch how to compute $H^p(\mathbb{Z}_2, \mathbb{Z}_\phi)$, assuming the reader is acquainted with more advanced homological techniques in algebraic group cohomology. For this, let R denote the group ring of $\mathbb{Z}_2$ over $\mathbb{Z}$, so

$$R = \frac{Z[T]}{(T^2 - 1)}.$$

Then there is an obvious resolution

$$\ldots \overset{T-1}{\to} R \overset{T+1}{\to} R \overset{T-1}{\to} R \to \mathbb{Z}.$$

Using this resolution and a definition of algebraic cohomology in terms of free resolutions, it is easy to see that

$$H^k(\mathbb{Z}_2, \mathbb{Z}_\phi) = \begin{cases} \frac{\ker(1+T)}{\mathrm{Im}(1-T)} & \text{if } k \text{ odd}, \\ \frac{\ker(1-T)}{\mathrm{Im}(1+T)} & \text{if } k \text{ even and positive}. \end{cases}$$

Plugging in our module $\mathbb{Z}_\phi$ then immediately gives $H^3(\mathbb{Z}_2, \mathbb{Z}_\phi) \cong \mathbb{Z}_2$.

Our main tool to classify anomalies is the application of the following formula.

Lemma A.23. *For a finite group G, let $\phi : G \times \mathbb{Z}_2 \to \mathbb{Z}_2$ be projection onto the second factor. Then*

$$H^3(G \times \mathbb{Z}_2, \mathbb{Z}_\phi) \cong \mathbb{Z}_2 \oplus (\mathbb{Z}_2 \otimes H^2(G, \mathbb{Z})) \oplus \mathrm{Tor}(\mathbb{Z}_2, H^1(G, \mathbb{Z})).$$

Proof. By the characterization of H^1 given by Lemma A.10, $H^1(G, \mathbb{Z}) = 0$ because G is finite. Using the fact that we know $H^p(\mathbb{Z}_2, \mathbb{Z}_\phi)$ by Lemma A.21, the desired formula follows by the Künneth formula, Proposition A.14. □

Using this lemma and the computer algebra programme GAP, we classified anomaly twists for finite rotation groups. See the following table for a summary of our results. Recall that the symmetry group of the tetrahedron is S_4, the symmetry group of the cube and octahedron is $S_4 \times \mathbb{Z}_2$ and the symmetry group of the icosahedron and dodecahedron is $A_5 \times \mathbb{Z}_2$.

Abstract group	C_{2n+1}	C_{2n}	D_2	D_3	D_4	D_6	S_4	$S_4 \times \mathbb{Z}_2$	$A_5 \times \mathbb{Z}_2$
Without time reversal	0	0	0	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^2$
With time reversal	$\mathbb{Z}_2$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^2$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^4$	$\mathbb{Z}_2^3$	$\mathbb{Z}_2^5$	$\mathbb{Z}_2^5$

The quantum anomalies in the table above often have physical interpretations. The following statement illustrates this fact.

Proposition A.24. *Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ with ϕ projection on the second factor. Denote the generator of the first factor by R and the second by T. The four cohomology classes $\tau \in H^2(G, U(1)_\phi) \cong \mathbb{Z}_2^2$ have representative group extensions*

$$0 \to U(1) \to G^\tau \to G \to 0$$

which can be distinguished by choosing whether lifts $\tilde{T}$ and $\tilde{R}$ of T and R satisfy $\tilde{R}^2 = \pm 1$ and $T^2 = \pm 1$. In other words, quantum anomalies of the group $\mathbb{Z}_2$ (which the reader can imagine to represent rotation by π) with time reversal T are classified by $\mathbb{Z}_2^2$ depending on whether $T^2 = \pm 1$ and $R^2 = \pm 1$.

Proof. An equivalent statement is formulated Proposition 6.4(v) of Freed & Moore [27] with a direct algebraic proof (also see the remarks under that proposition). □

A.4. Splitting of Twists into Anomalies and Crystal Twists

The main conjecture written in this section is that general twists split into anomalies and crystal twists. The idea is to generalize Lemma 3.9 of [30] to $\phi \neq 1$ in the case of a torus.

Conjecture A.25. *Suppose we are given a full set of symmetry data, see Definition 4.1. Then there is a split exact sequence of topological G-modules*

$$0 \to C(X, \mathbb{R}_\phi)/(2\pi\mathbb{Z}) \xrightarrow{i} C(X, U(1)_\phi) \xrightarrow{\pi} \mathbb{Z}_\beta^d \to 0.$$

There is also an isomorphism of topological G-modules $C(X, \mathbb{R}_\phi)/(2\pi\mathbb{Z}) \cong U(1)_\phi \oplus C(X, 0, \mathbb{R}_\phi)$*, where*

$$C(X, 0, \mathbb{R}_\phi) = \{f \in C(X, \mathbb{R})_\phi : f(0) = 0\}.$$

Moreover, $C(X, 0, \mathbb{R}_\phi)$ *is G-contractible and hence* $H^2(G, C(X, U(1)_\phi)) \cong H^2(G, U(1)_\phi) \oplus H^2(G, \mathbb{Z}_\beta)$*.*

We will now sketch an idea of a possible proof. A proof that π is a surjective G-module map is postponed to later work. Let $i : C(X, \mathbb{R}_\phi)/(2\pi\mathbb{Z}) \to C(X, U(1)_\phi)$ be the map $i([f])(\lambda) = e^{if(\lambda)}$, where $\lambda \in X$. This is a well-defined map of abelian topological groups. It is also a G-module homomorphism since for all $g \in G, f \in C(X, \mathbb{R}_\phi), \lambda \in X$

$$i([g \cdot f])(\lambda) = e^{i(g \cdot f)(\lambda)} = e^{i\phi(g) f(g^{-1}\lambda)} = (e^{if(g^{-1}\lambda)})^{\phi(g)} = (g \cdot i([f]))(\lambda).$$

We now show that i is injective. Let $f \in C(X, \mathbb{R}_\phi)$ be such that $e^{if(\lambda)} = 1$ for all $\lambda \in X$. Then f is a continuous map $f : X \to 2\pi\mathbb{Z}$, so since X is connected, f is constant in $2\pi\mathbb{Z}$.

The splitting $\psi : \mathbb{Z}^d_\beta \to C(X, U(1)_\phi)$ of the sequence is defined by $(\psi(v))(\lambda) = \lambda(v)$. This is a map of G-modules since

$$(g \cdot \psi(v))(\lambda) = (\psi(v))(g^{-1}\lambda)^{\phi(g)} = (g^{-1}\lambda)(v)^{\phi(g)} = \lambda(\alpha(v)).$$

A possible idea for the map π is the following. Identify X with $S^1 \times \cdots \times S^1$ and $U(1)$ with S^1. Then an element $f \in C(X, U(1)_\phi)$ is a map $S^1 \times \cdots \times S^1 \to S^1$. By restricting to different copies of S^1 we get d mappings $f_i : S^1 \to S^1$, where $i = 1, \ldots, d$. Define $\pi(f) = (\deg f_1, \ldots, \deg f_d)$. It is easy to see that ψ is a section of this map.

For the final statements, note that the isomorphism

$$C(X, \mathbb{R}_\phi)/(2\pi\mathbb{Z}) \cong U(1)_\phi \oplus C(X, 0, \mathbb{R}_\phi)$$

is obvious; it is given by $[f] \mapsto (f(0), f - f(0))$. Moreover, $C(X, 0, \mathbb{R}_\phi)$ is a real vector space on which the G-action is given my linear maps. Therefore it is G-contractible. Then using Corollary A.13, we get that $H^2(G, C(X, 0, \mathbb{R}_\phi)) = 0$. By the split exact sequence, the desired isomorphism in cohomology should follow in case π can be shown to be a surjective G-module map.

B. Superalgebras and Clifford Algebra

B.1. Superalgebras

Various mathematical structures appearing in physics naturally come with a $\mathbb{Z}_2$-grading. Often this is because of the boson-fermion distinction, especially in the case of supersymmetry. Mathematical approaches to this kind of physics result in structures which are therefore given the prefix 'super', such as superalgebras, Lie superalgebras, supermanifolds and superevi. The physical motivation for the appearance of superstructures in this thesis is the particle-hole distinction. Here the basics of superalgebras will be reviewed quickly. For a thorough treatment consider [67], [19] and [18]. Let $\mathbb{F}$ denote either $\mathbb{R}$ or $\mathbb{C}$. Throughout this chapter all structures are finite-dimensional, i.e. finite-dimensional vector spaces, algebras, modules over algebras etcetera.

Definition B.1. We start with a small dictionary for reference purposes.

- A *super vector space* over $\mathbb{F}$ is a vector space with direct sum decomposition $V = V^0 \oplus V^1$.
- A *superalgebra* over $\mathbb{F}$ is an algebra over $\mathbb{F}$ with direct sum decomposition $A = A_0 \oplus A_1$ such that $A_i A_j \subseteq A_{i+j}$ (where the index is taken mod 2). We write $|A|$ for the underlying ungraded algebra.
- A *supersubalgebra* of A is a subalgebra B such that $B = (B \cap A_0) \oplus (B \cap A_1)$.
- A *superideal* of A is an ideal I such that $I = (I \cap A_0) \oplus (I \cap A_1)$.
- We say A is *simple* if its only superideals are 0 and A.
- The *supercommutator* of homogeneous elements $a_1, a_2 \in A$ is

  $$[a_1, a_2] := (-1)^{|a_1||a_2|}(a_1 a_2 - a_2 a_1),$$

  where $|a|$ is written for the *degree* of an element $a \in A$. The supercommutator is extended bilinearly to A.
- The *supercenter* $Z(A)$ is the subset of A consisting of all elements (including non-homogeneous ones) that supercommute with all elements of A.
- We say A is *central* if $Z(A) = \mathbb{F}$.
- We say A is a *superdivision algebra* if all nonzero homogeneous elements are invertible.

- Given A, B superalgebras, their *direct sum* is the usual direct sum algebra $A \oplus B$ with grading

$$(A \oplus B)_0 = A_0 \oplus B_0$$
$$(A \oplus B)_1 = A_1 \oplus B_1.$$

- A *superalgebra homomorphism* $A \to B$ is an algebra homomorphism $A \to B$ that maps A_0 into B_0 and A_1 into B_1.
- A *supermodule homomorphism* $M \to N$ of degree $i \in \mathbb{Z}_2$ is a module homomorphism such that M^j is mapped into N^{j+i} for both $j \in \mathbb{Z}_2$.
- A *(left) supermodule* over A is a finite dimensional super vector space $M = M_0 \oplus M_1$ over $\mathbb{F}$ with a linear (left-)action of A such that $A^i M^j \subseteq M^{i+j}$.
- An (A, B)-*superbimodule* is a super vector space M with a left A-supermodule structure and a right B-supermodule structure such that $(am)b = a(mb)$ for all $a \in A, b \in B$ and $m \in M$.
- A *supersubmodule* of a supermodule M is a submodule $N \subseteq M$ such that $N = (N \cap M_0) \oplus (N \oplus M_0)$.
- A supermodule M is called *irreducible* if the only supersubmodules $N \subseteq M$ are 0 and M.
- A supermodule M is called *semisimple* if there is a direct sum decomposition $M = N_1 \oplus \cdots \oplus N_k$ into irreducible supersubmodules;
- A superalgebra A is called *semisimple* if every supermodule over A is semisimple.

Example B.2. Let $W = W^0 \oplus W^1$ be a $\mathbb{Z}_2$-graded vector space over $\mathbb{F}$. Then $\operatorname{End} W$ becomes a superalgebra by composition; the even elements in $\operatorname{End} W$ are the maps of even degree and odd elements are maps of odd degree. Because it is simple as an ungraded algebra, the only superideals of $\operatorname{End} W$ are 0 and $\operatorname{End} W$. It can also be shown that $\operatorname{End} W$ is central as a superalgebra. Write $M_{r|s}(\mathbb{F})$ for the endomorphism superalgebra of $W = \mathbb{F}^k \oplus \mathbb{F}^s$.

Definition B.3. Given superalgebras A, B over $\mathbb{F}$, the *graded tensor product* $A \hat{\otimes}_{\mathbb{F}} B$ is defined using typical Koszul sign rules. As a vector space it is $A \hat{\otimes}_{\mathbb{F}} B$ with grading given by

$$(A \hat{\otimes}_{\mathbb{F}} B)_0 = (A_0 \otimes_{\mathbb{F}} B_0) \oplus (A_1 \otimes_{\mathbb{F}} B_1)$$
$$(A \hat{\otimes}_{\mathbb{F}} B)_1 = (A_0 \otimes_{\mathbb{F}} B_1) \oplus (A_1 \otimes_{\mathbb{F}} B_0)$$

and its product is given by

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{|b_1||a_2|}(a_1 a_2 \otimes b_1 b_2).$$

Also, given an (A, B)-superbimodule M and a (B, C)-superbimodule N, the usual (A, C)-bimodule $M \otimes_B N$ can be given a superbimodule structure by a grading similar to the above.

Because a notion of Morita equivalence in the context of superalgebras will be necessary, we record our definition using the right notion of matrix algebra.

Definition B.4. Superalgebras A, B are said to be *Morita equivalent* if there exist $p, p', q, q' \geq 0$ not all equal to zero such that

$$M_{p|q}(A) \cong M_{p'|q'}(B).$$

Since superalgebras in this document are considered exclusively because of the appearance of graded twisted group algebras, which turn out to be semisimple, we describe a structure theorem for real semisimple superalgebras. In particular, we study their behavior under restrictions and tensor products, in particular for their simple components. We start by classifying simple superalgebras over $\mathbb{R}$ and supermodules over them.

Lemma B.5. *Let D be a superdivison algebra over the real numbers. Then all supermodules over D are isomorphic to $D\hat{\otimes}_{\mathbb{R}}\mathbb{R}^{p|q}$ for some integers $p, q \geq 0$. Moreover, these isomorphism classes of D-supermodules are determined by*

- *the ordered pair (p, q) if $D_1 = 0$;*
- *the sum $p + q$ if $D_1 \neq 0$.*

Proof. We start by noting that D_0 is an ordinary division algebra. If $D_1 = 0$, a supermodule M over D is just a (nongraded) D_0-module with a direct sum decomposition. Since every module over a division algebra is free, we get that $M \cong \mathbb{R}^{p|q}\hat{\otimes}D$. The even part of M has real dimension $p \dim D$, while the odd part has dimension $q \dim D$. Hence for different p, q they can never be isomorphic.

Now suppose $D_1 \neq 0$. Fix a nontrivial element $d \in D_1$ throughout the proof. Since D_0 is a division ring, M_0 is a free module over D_0. Let $e_1, \ldots, e_k \in M_0$ be a basis of M_0 over D_0. In order to show that the real linear even map

$$\phi : D\hat{\otimes}_{\mathbb{R}}\mathbb{R}^{k|0} \to M, \quad \phi(d' \otimes e_i) \mapsto d' \cdot e_i$$

is an isomorphism of D-modules, it is sufficient to show that $e_1, \ldots, e_k$ form a basis of M over D.

1. *Spanning:* let $m = m_0 + m_1 \in M$ with $m_0 \in M_0$ and $m_1 \in M_1$. Write

$$m_0 = \sum_{i=1}^{k} a_0^i e_i \quad d \cdot m_1 = \sum_{i=1}^{k} b_0^i e_i$$

for some $a_0^i, b_0^i \in D_0$. Then

$$m = \sum_{i=1}^{k} a_0^i e_i + \sum_{i=1}^{k} d^{-1} b_0^i e_i.$$

2. *Linear independence:* suppose

$$\sum_{i=1}^{k}(a_0^i + b_1^i)e_i = 0,$$

for some $a_0^i \in D_0$ and $b_1^i \in D_1$. Then by the direct sum decomposition $M = M_0 \oplus M_1$ we first of all get

$$\sum_{i=1}^{k} a_0^i e_i = 0 \implies a_0^i = 0 \quad \forall i.$$

But also

$$\sum_{i=1}^{k} b_1^i e_i = 0 \implies \sum_{i=1}^{k} d \cdot b_1^i e_i = 0 \implies d \cdot b_1^i = 0 \quad \forall i \implies b_1^i = 0 \quad \forall i.$$

So we see that $M \cong D \hat{\otimes}_{\mathbb{R}} \mathbb{R}^{k|0}$. Since the real dimension of M equals $k \dim D$, clearly M cannot be isomorphic to $D \hat{\otimes}_{\mathbb{R}} \mathbb{R}^{k'|0}$ for $k' \neq k$. □

Proposition B.6. *Let A be a superalgebra over $\mathbb{R}$.*

- *If A is semisimple, then it is a direct sum of simple supersubalgebras;*
- *A superalgebra is simple if and only if it is of the form $M_{p|q}(D) := End(D \hat{\otimes}_{\mathbb{R}} \mathbb{R}^{p|q}) = M_{p|q} \hat{\otimes} D$ for some superdivision algebra D and natural numbers p, q;*
- *Isomorphism classes of algebras of the form $M_{p|q}(D)$ are determined by*
  - *the unordered pair $\{p, q\}$ if $D_1 = 0$;*
  - *the sum $p + q$ if $D_1 \neq 0$.*

Proof. The first point is an easy consequence of Proposition 2.4 in [37]. An outline of a proof is given there as well. For the second point, we refer the reader to Theorem 2.33 of [16]. We now prove the last point. First of all note that by Lemma B.5 above, $M_{p|q}(D) \cong M_{p+q|0}(D)$ if $D_1 \neq 0$. Moreover, in that case we have that the dimension of $D \hat{\otimes}_{\mathbb{R}} M_{n|0}$ is $n^2 \dim D$, so that for different n they can never be isomorphic. Now suppose instead $D_1 = 0$. Firstly, there is an obvious isomorphism $M_{p|q}(\mathbb{R}) \to M_{q|p}(\mathbb{R})$ of supervector spaces, giving an isomorphism $M_{p|q}(D) \cong M_{q|p}(D)$. Conversely, suppose $M_{p|q}(D) \cong M_{p'|q'}(D)$. Note that

$$\dim M_{p|q}(D) = (p+q)^2 \dim D, \quad \dim M_{p|q}(D)_1 = 2pq \dim D.$$

Hence, comparing dimensions gives $p + q = p' + q'$ and $pq = p'q'$. An elementary argument now shows that either $p = p'$ and $q = q'$ or $p = q'$ and $q = p'$. □

Proposition B.6 immediately implies the following.

Corollary B.7. *Every semisimple superalgebra over $\mathbb{R}$ is a direct sum over supermatrix rings with coefficients in superdivision algebras over $\mathbb{R}$.*

Proposition B.8. *The (ϕ, τ)-twisted c-graded group algebra ${}^{\phi}\mathbb{C}^{\tau,c}(G)$ is semisimple.*

Proof. We follow the proof of Maschke's theorem as in page 43 of [17], making few adjustments. Write A for ${}^{\phi}\mathbb{C}^{\tau,c}(G)$. Let M be a supermodule over A and let $N \subseteq M$ be a supersubmodule. By induction it is sufficient to show that there exists another supersubmodule $N' \subseteq M$ such that $N \oplus N' = M$. This is equivalent to finding a map of supermodules $T : M \to N$ (of degree zero) that is a projection, so $T(n) = n$ if $n \in N$. Indeed, given such a map, set $N' := (id - T)(M)$. Then N' is a supersubmodule and given $m \in M$ we have

$$m = (id - T)(m) + T(m) \in N + N'.$$

To show the sum is direct, let $m \in (id - T)(M) \cap T(M)$ and write $m = m' - T(m')$ for some $m' \in M$. Since $m \in T(M)$ and $T(m') \in T(M)$, we have $m' \in T(M)$, so that

$$m = m' - T(m') = m' - m' = 0.$$

So indeed the sum is direct.

Now, to construct such a T, first let $P : M \to N$ be any complex linear projector of even degree. This can always be constructed as the sum of a complex linear projector $M_0 \to N_0$ and $M_1 \to N_1$. Let $\rho(g) : M \to M$ be the map given by multiplication by x_g. We construct T by averaging:

$$T := \frac{1}{2|G|} \sum_{g \in G} \left(\rho(g) P \rho(g)^{-1} + i\rho(g) P (i\rho(g))^{-1} \right).$$

Note that T is a complex linear map of even degree. Since $\rho(g)(N) \subseteq N$ and $i(N) \subseteq N$, we clearly have $T(M) \subseteq N$. Moreover, if $n \in N$, then $\rho(g)^{-1}n \in N$ so that $P\rho(g)^{-1}n = \rho(g)^{-1}n$. Analogously, $Pi\rho(g)^{-1}n = i\rho(g)^{-1}n$ and hence $Tn = n$. Finally, for invariance, let $h \in G$. Then

$$\begin{aligned}
\rho(h) T \rho(h)^{-1} &= \frac{1}{2|G|} \sum_{g \in G} \big(\tau(h,g)\rho(hg) P \rho(hg)^{-1} \tau(h,g)^{-1} \\
&\quad + \phi(h)\tau(h,g) i\rho(hg) P (i\rho(hg))^{-1} \tau(h,g)^{-1} \phi(h)^{-1} \big) \\
&= \frac{1}{2|G|} \sum_{g \in G} \left(\rho(hg) P \rho(hg)^{-1} + i\rho(hg) P (i\rho(hg))^{-1} \right) \\
&= T.
\end{aligned}$$

Also completely similarly, we get

$$\begin{aligned}
i\rho(h) T (i\rho(h))^{-1} &= \frac{1}{2|G|} \sum_{g \in G} \big(\tau(h,g) i\rho(hg) P (i\rho(hg))^{-1} \tau(h,g)^{-1} \\
&\quad - \phi(h)\tau(h,g)\rho(hg) P \cdot (-\phi(h)\tau(h,g)\rho(hg))^{-1} \big) \\
&= T.
\end{aligned}$$

□

This proposition together with Corollary B.7 gives us the following.

Corollary B.9. *The (ϕ, τ)-twisted c-graded group algebra ${}^{\phi}\mathbb{C}^{\tau,c}(G)$ is a direct sum over supermatrix rings over superdivision algebras.*

B.2. Clifford Algebras

Very important examples of superalgebras, both from a mathematical and physical point of view, are Clifford algebras. They can be described abstractly on a vector space with a bilinear form, but we choose the more concrete approach.

Definition B.10. The n^{th} *complex Clifford algebra* $\mathbb{C}l_n$ is the complex superalgebra generated by n odd elements $\gamma_1, \dots, \gamma_n$ with relations

$$\begin{aligned} \gamma_i\gamma_j + \gamma_j\gamma_i &= 0 \quad \text{if } i \neq j, \\ (\gamma_i)^2 &= 1 \quad \forall i. \end{aligned}$$

The $(p, q)^{\text{th}}$ *real Clifford algebra* $Cl_{p,q}$ is the real superalgebra generated by $p + q$ odd elements $\gamma_1, \dots, \gamma_{p+q}$ with relations

$$\begin{aligned} \gamma_i\gamma_j + \gamma_j\gamma_i &= 0 \quad \text{if } i \neq j, \\ (\gamma_i)^2 &= 1 \quad \forall i = 1, \dots p, \\ (\gamma_i)^2 &= -1 \quad \forall i = p+1, \dots p+q. \end{aligned}$$

Example B.11. The first complex Clifford algebra is

$$\mathbb{C}l_1 = \frac{\mathbb{C}[e]}{(e^2 - 1)},$$

where e is odd. Clearly every homogeneous element is invertible, so that $\mathbb{C}l_1$ is a superdivision algebra. Note however that the underlying algebra $|\mathbb{C}l_1| \cong \mathbb{C} \oplus \mathbb{C}$ is not even a domain.

Proposition B.12. *Some properties of Clifford algebras are the following:*

1. *As a vector space, the Clifford algebras $\mathbb{C}l_{p+q}$ and $Cl_{p,q}$ are 2^{p+q}-dimensional with basis consisting of*
$$\gamma_{i_1} \dots \gamma_{i_k} \quad \text{with } i_1 < \dots < i_k.$$
The elements of odd length have odd grading and the elements of even length have even grading;

2. *the Clifford algebras can be constructed using graded tensor products:*
$$\begin{aligned} \mathbb{C}l_n &= \underbrace{\mathbb{C}l_1 \hat{\otimes}_{\mathbb{C}} \dots \hat{\otimes}_{\mathbb{C}} \mathbb{C}l_1}_{n \text{ times}} \\ Cl_{p,q} &= \underbrace{Cl_{1,0} \hat{\otimes}_{\mathbb{R}} \dots \hat{\otimes}_{\mathbb{R}} Cl_{1,0}}_{p \text{ times}} \hat{\otimes}_{\mathbb{R}} \underbrace{Cl_{0,1} \hat{\otimes}_{\mathbb{R}} \dots \hat{\otimes}_{\mathbb{R}} Cl_{0,1}}_{q \text{ times}}; \end{aligned}$$

3. *all Clifford algebras are simple;*

4. *the real Clifford algebras have supercenter* $\mathbb{R}$*, the complex Clifford algebras have supercenter* $\mathbb{C}$*;*

5. $\mathbb{C}l_2 \cong M_{1|1}(\mathbb{C})$*;*

6. $Cl_{0,8} \cong M_{8|8}(\mathbb{R}) \cong Cl_{8,0}$*;*

7. $Cl_{0,1}\hat{\otimes}_{\mathbb{R}}Cl_{1,0} \cong M_{1|1}(\mathbb{R})$*.*

Clifford algebras appear naturally in certain mathematical problems. For example, they solve the classification problem of central simple superalgebras over $\mathbb{R}$ and $\mathbb{C}$.

Definition B.13. The *super Brauer group* $\mathrm{sBr}(\mathbb{F})$ is the set of Morita equivalence classes of central simple superalgebras over $\mathbb{F}$, together with the operation of taking graded tensor products over $\mathbb{F}$.

Proposition B.14. *There are isomorphisms*

$$\mathrm{sBr}(\mathbb{R}) \cong \mathbb{Z}_8, \quad \mathrm{sBr}(\mathbb{C}) \cong \mathbb{Z}_2.$$

A generator for $\mathrm{sBr}(\mathbb{R})$ *is* $Cl_{1,0}$ *and a generator for* $\mathrm{sBr}(\mathbb{C})$ *is* $\mathbb{C}l_1$*. Every element of* $\mathrm{sBr}(\mathbb{R})$ *and* $\mathrm{sBr}(\mathbb{C})$ *has a unique representative that is a superdivision algebra. Hence there are* 10 *isomorphism classes of superdivision algebras over* $\mathbb{R}$*, of which two have center* $\mathbb{C}$ *and two have center* $\mathbb{R}$*. Explicit forms for the superdivision algebras are given in the table below.*

Proof. Originally shown by Wall [67]. Also see [18], especially for the explicit expressions for the superdivision algebras given in the table below. □

The ten isomorphism classes have connections with different concepts in mathematics and physics. These connections and some properties are summarized in the following table, of which the details will not be explained. Considering notation: e denotes an odd element squaring to one and commuting with the even part, f denotes an odd element squaring to minus one and commuting with the even part, $\tilde{e}$ denotes an odd element squaring to one satisfying $z\tilde{e} = \tilde{e}\bar{z}$, $\tilde{f}$ denotes an odd element squaring to minus one satisfying $z\tilde{f} = \tilde{f}\bar{z}$.

B.3. Representation Rings

In K-theory, and therefore in particular in the theory of topological phases, the local algebraic theory is determined by the representation rings of some (twisted graded) group algebra. More generally, the K-theory of spheres are conjectured to have a relation with higher representation rings (see Conjecture 3.71), which are defined as follows.

Superdivision Algebra D	$\mathbb{C}$	$\mathbb{C}[e]$	$\mathbb{R}$	$\mathbb{R}[e]$	$\mathbb{C}[\tilde{e}]$	$\mathbb{H}[f]$	$\mathbb{H}$	$\mathbb{H}[e]$	$\mathbb{C}[\tilde{f}]$	$\mathbb{R}[f]$
Clifford Algebra	$\mathbb{C}l_0$	$\mathbb{C}l_1$	$Cl_{0,0}$	$Cl_{1,0}$	$Cl_{2,0}$	$Cl_{3,0}$	-	$Cl_{0,3}$	$Cl_{0,2}$	$Cl_{0,1}$
Odd element, square 1	$\times$	✓	$\times$	✓	✓	✓	$\times$	✓	$\times$	$\times$
Odd element, square -1	$\times$	✓	$\times$	$\times$	$\times$	✓	$\times$	✓	✓	✓
Representation ring	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D_0	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$
$Z(D_0)$	$\mathbb{C}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$
$\|D\|$	$\mathbb{C}$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}$	$M_2(\mathbb{C})$	$M_2(\mathbb{R})$	$\mathbb{R}\oplus\mathbb{R}$
$\|D\|$ simple	✓	$\times$	✓	✓	✓	$\times$	✓	✓	✓	$\times$
$Z(\|D\|)$	$\mathbb{C}$	$\mathbb{C}\oplus\mathbb{C}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$
$Z(\|D\|)\cap D_1$	$\emptyset$	2d	$\emptyset$	1d	$\emptyset$	1d	$\emptyset$	1d	$\emptyset$	1d
Dyson type	$\mathbb{C}\mathbb{C}_1$	$\mathbb{C}\mathbb{C}_2$	$\mathbb{R}\mathbb{C}$	$\mathbb{R}\mathbb{H}$	$\mathbb{C}\mathbb{H}$	$\mathbb{H}\mathbb{H}$	$\mathbb{H}\mathbb{C}$	$\mathbb{H}\mathbb{R}$	$\mathbb{C}\mathbb{R}$	$\mathbb{R}\mathbb{R}$
Cartan label	A	AIII	AI	CI	C	CII	AII	DIII	D	BDI
CT-group	1×1	diagonal	$\mathbb{Z}_2\times 1$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$1\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$\mathbb{Z}_2\times 1$	$\mathbb{Z}_2\times\mathbb{Z}_2$	$1\times\mathbb{Z}_2$	$\mathbb{Z}_2\times\mathbb{Z}_2$
CT-twist	00	00	$+0$	$+-$	$0-$	$--$	-0	$-+$	$0+$	$++$

Definition B.15. Let A be a superalgebra and fix integers $p,q \geq 0$. Write $\mathrm{Mod}_s\,A$ for the monoid consisting of isomorphism classes of supermodules over A with direct sum as the operation. The inclusion $Cl_{p,q} \hookrightarrow Cl_{p+1,q}$ induces a restriction mapping $r_{p,q}$: $\mathrm{Mod}_s(A\hat{\otimes}Cl_{p+1,q}) \to \mathrm{Mod}_s(A\hat{\otimes}Cl_{p,q})$, which moreover preserves the monoid structure. The *(higher) representation ring $R^{p,q}A$ of A of degree p,q* is the cokernel of $r_{p,q}$, so

$$R^{p,q}A := \frac{\mathrm{Mod}_s(A\hat{\otimes}Cl_{p,q})}{\mathrm{Mod}_s(A\hat{\otimes}Cl_{p+1,q})}.$$

Note that unlike its name suggests, the higher representation ring of an algebra is only an abelian group.

Remark B.16. By Proposition B.12 $R^{p,q}A = RA^{p,q}$, where $A^{p,q} := A\hat{\otimes}Cl_{p,q}$ and $RB := R^{0,0}B$. Therefore it is sufficient to prove all properties just for $p,q = 0$. Note that an element $M \in \mathrm{Mod}_s(A)$ is in $\operatorname{Im} r$ (with $r := r_{0,0}$) if and only if M admits an odd automorphism squaring to one.

Lemma B.17. *The set $R^{p,q}A$ is an abelian group under direct sum.*

Proof. Without loss of generality, assume $p = q = 0$. Note that since r preserves the monoid structure, $\operatorname{Im} r$ is a submonoid of $\mathrm{Mod}_s(A)$. Now, given a supermodule M over a superalgebra B, define the inverted supermodule $\widehat{M}$ over B by $(\widehat{M})_0 = M_1$ and $(\widehat{M})_1 = M_0$ with the module structure induced by that of M. Then

- $M \oplus \widehat{M}$ has an odd automorphism of supermodules squaring to one induced by the identity map on M, hence $M \oplus \widehat{M} \in \operatorname{Im} r$;
- the map $I(M) = \widehat{M}$ preserves the monoid structure on $\mathrm{Mod}_s(A\hat{\otimes}Cl_{p,q})$;
- if M admits an odd automorphism squaring to one, then so does $I(M)$.

The statement now follows by the last part of the next lemma. □

Lemma B.18. *Let M be an abelian monoid and let $N \subseteq M$ be a submonoid, i.e. a subset containing* 0 *that is closed under addition. Then*

1. *the monoid structure of M induces a monoid structure on M/N (equivalence classes under $m_1 \sim m_2 \iff \exists n_1, n_2 \in N$ with $m_1 + n_1 = m_2 + n_2$) such that the quotient map becomes a monoid homomorphism;*

2. *let M' be another abelian monoid and let $\psi : M \to M'$ be a monoid map such that $\psi(n) = 0$ for all $n \in N$. Then there is a unique induced monoid homomorphism $\bar{\psi} : M/N \to M'$ such that the obvious diagram commutes;*

3. *if $I : M \to M$ is a map of monoids such that $I(N) = N$ and $m + I(m) \in N$ for all N, then M/N is an abelian group in which the inverse of $[m] \in M/N$ is $[I(m)] \in M/N$.*

Proof. The first two points have elementary proofs. The last point is taken from Gomi's work [31], Lemma C.1. □

We now determine the representation ring of semisimple superalgebras in terms of their decomposition.

Proposition B.19. *If A, B are real superalgebras, then $R(A \oplus B) \cong R(A) \oplus R(B)$.*

Proof. Consider the map $\psi_{A,B} : \mathrm{Mod}_s(A) \oplus \mathrm{Mod}_s(B) \to \mathrm{Mod}_s(A \oplus B)$ given by $\psi(M_1, M_2) = M_1 \oplus M_2$ with action $(a,b)(m_1, m_2) = (am_1, bm_2)$. If $\mathrm{Mod}_s(A) \oplus \mathrm{Mod}_s(B)$ is equipped with the direct sum monoid structure, this becomes a monoid map. The map is clearly injective. Moreover, if $M \in \mathrm{Mod}_s(A \oplus B)$, define the sets $M_1 := (1,0)M$ and $M_2 := (0,1)M_2$. They carry obvious A- and B-supermodule structures respectively. Moreover, it is easy to check that $M_1 \oplus M_2 \cong M$. So ψ is a bijective monoid map[1].

Now consider the diagram of monoids

$$\begin{array}{ccc} \mathrm{Mod}_s(A^{1,0} \oplus B^{1,0}) & \xrightarrow{\psi_{A^{1,0},B^{1,0}}} & \mathrm{Mod}_s(A^{1,0}) \oplus \mathrm{Mod}_s(B^{1,0}) \\ \downarrow{\scriptstyle r_{A \oplus B}} & & \downarrow{\scriptstyle r_A \oplus r_B} \\ \mathrm{Mod}_s(A \oplus B) & \xrightarrow{\psi_{A,B}} & \mathrm{Mod}_s(A) \oplus \mathrm{Mod}_s(B), \end{array}$$

where we identified

$$\begin{aligned} \mathrm{Mod}_s((A \oplus B)^{1,0}) &= \mathrm{Mod}_s((A \oplus B)\hat{\otimes}_{\mathbb{R}} Cl_{1,0}) \cong \mathrm{Mod}_s((A\hat{\otimes}_{\mathbb{R}} Cl_{1,0}) \oplus (B\hat{\otimes}_{\mathbb{R}} Cl_{1,0})) \\ &= \mathrm{Mod}_s(A^{1,0} \oplus B^{1,0}). \end{aligned}$$

It is easy to see that the diagram is commutative. Now, $\psi_{A,B}$ induces a map of monoids $\operatorname{coker} r_{A \oplus B} \to \operatorname{coker}(r_A \oplus r_B)$; if $M \in \operatorname{Im} r_{A \oplus B}$, then by commutativity of the diagram, $\psi_{A,B}(M) \in \operatorname{Im} r_A \oplus r_B$. Also, if $M \in \mathrm{Mod}_s(A \oplus B)$ is such that $\psi_{A,B}(M) \in \operatorname{Im} r_A \times r_B$, then (using $(\psi)^{-1}$) we get that $M \in \operatorname{Im} r_{A \oplus B}$. Hence the map $R(A \oplus B) = \operatorname{coker} r_{A \oplus B} \to \operatorname{coker}(r_A \oplus r_B) = R(A) \oplus R(B)$ is an isomorphism of abelian groups. □

In order to find the representation ring of simple superalgebras, we first prove a few lemmas. Considering the definition of the representation ring, it should be determined in what cases supermodules admit odd automorphisms squaring to 1.

[1] It can actually be made into an equivalence of categories.

Lemma B.20. *Let M be a supermodule over a superalgebra A.*

1. *If M admits an odd automorphism, then $M \oplus M$ admits an odd isomorphism squaring to 1;*

2. *If A is a superdivision algebra, then the left A-module A admits an odd automorphism (squaring to one) if and only if A has an odd element (squaring to one).*

Proof. 1. Let $\phi : M \to M$ be an odd automorphism. Write M^0 and M^1 for two copies of M. Define $\psi : M^0 \oplus M^1 \to M^0 \oplus M^1$ by $\phi_0 \oplus (\phi^{-1})_0$ on $(M^0 \oplus M^1)_0 = M^0_0 \oplus M^1_0$ and $(\phi^{-1})_1 \oplus \phi_1$ on $(M^0 \oplus M^1)_1 = M^0_1 \oplus M^1_1$. Then ψ is an odd automorphism of $M^0 \oplus M^1$. Moreover, on the even part ψ^2 acts as

$$\psi^2|_{M^0_0 \oplus M^1_0} = (\phi_1 \oplus (\phi^{-1})_1)((\phi^{-1})_0 \oplus \phi_0) = id_{M^0_0} \oplus id_{M^1_0}.$$

Similarly it is the identity on the odd part.

2. If $a \in A_1$, right multiplication by a defines an odd left A-supermodule homomorphism $A \to A$. Because A is a superdivision ring and a is homogeneous, this map is invertible if $a \neq 0$. Clearly if $a^2 = 1$, this map squares to the identity. Conversely, if $\psi : A \to A$ is an odd left A-supermodule homomorphism, we know that $\psi(1) \in A_1$. Since $\psi(0) = 0$ and ψ is bijective, $\psi(1)$ is nonzero. If $\psi \circ \psi = id_A$, then

$$\psi(1)^2 = \psi(1)\psi(1) = \psi(\psi(1)1) = \psi(\psi(1)) = 1,$$

so A has an odd element squaring to one.

□

Lemma B.21. *If A is a real superalgebra and $p, q \geq 0$ are integers of which at least one is positive, then*

$$R(A) \cong R(M_{p|q}(A)).$$

Proof. Consider the map $\psi : \mathrm{Mod}_s\, A \to \mathrm{Mod}_s\, M_{p|q}(A)$ given by $M \mapsto M^{p|q} := M \hat{\otimes}_{\mathbb{R}} \mathbb{R}^{p|q}$. There is an inverse $\psi^{-1} : \mathrm{Mod}_s\, M_{p|q}(A) \to \mathrm{Mod}_s\, A$ given by $\psi^{-1}(M') = E_{11} M|_A$, where $E_{11} \in M_{p|q}(\mathbb{R})$ is the matrix with the $(1,1)^{\text{th}}$-entry equal to one and all other entries equal to zero. It is easy to see that these are well-defined monoid maps. The inverse property amounts to the isomorphisms

$$E_{11} \cdot M \hat{\otimes}_{\mathbb{R}} \mathbb{R}^{p|q}|_A \cong M, \quad \mathbb{R}^{p|q} \hat{\otimes}_{\mathbb{R}} (E_{11} \cdot M'|_A) \cong M'.$$

This gives an isomorphism of monoids $\mathrm{Mod}_s\, A \cong \mathrm{Mod}_s\, M_{p|q}(A)$. It is easy to show that this isomorphism preserves the submonoid consisting of modules coming from $A \hat{\otimes}_{\mathbb{R}} Cl_{1,0}$-modules and $M_{p|q}(A) \hat{\otimes}_{\mathbb{R}} Cl_{1,0}$-modules respectively. □

Corollary B.22. *If A is a real superalgebra, then*

$$R^{p+1,q+1}(A) \cong R^{p,q}(A), \quad R^{p+8,q}(A) \cong R^{p,q}(A) \cong R^{p,q+8}(A).$$

If A is a complex superalgebra, then

$$R^{p+2,q}(A) \cong R^{p,q}(A) \cong R^{p,q+2}(A).$$

Proof. Follows by the last lemma and properties 5, 6 and 7 in Proposition B.12. □

Using this corollary, we often sloppily denote $R^{p,q}(A)$ by $R^{p-q}(A)$. Now let $D'_0 = \mathbb{C}, D'_1 = \mathbb{C}l_1$ be the representatives of the two nonisomorphic real superdivision algebras with center $\mathbb{C}$ and let $D_0, \ldots, D_7$ be the real superdivision algebras with center $\mathbb{R}$, chosen such that $D_0 = \mathbb{R}$ and such that $D_i \hat{\otimes}_{\mathbb{R}} Cl_{1,0}$ is Morita equivalent to D_{i+1}. Also note that $D'_i \hat{\otimes}_{\mathbb{R}} D'_j$ is Morita equivalent to D'_{i+j} (indices mod 2).

Proposition B.23. *Given $p \in \mathbb{Z}$ we have*

$$R^p(D_i) \cong KO^{p+i}(\mathrm{pt}), \quad R^p(D'_i) \cong K^{p+i}(\mathrm{pt})$$

Proof. First note that by Lemma B.5, $R^p(D_i) = R(D_i \hat{\otimes}_{\mathbb{R}} Cl_{p,0}) \cong R(D_{i+p})$ with the index taken modulo 8. A similar formula holds for D'_i. Hence it is sufficient to determine $R(D)$ for all ten superdivision rings, i.e. to determine in what cases supermodules over D admit an odd automorphism squaring to unity.

To this extent, let D be a superdivision ring over the real numbers. Suppose first that D is purely even. Then the isomorphism classes of supermodules are given by $\mathbb{N} \times \mathbb{N} = \{D \hat{\otimes}_{\mathbb{R}} \mathbb{R}^{p|q} : p, q \geq 0\}$ by Lemma B.5. For dimensional reasons, $D \hat{\otimes}_{\mathbb{R}} \mathbb{R}^{p|q}$ can only admit an odd automorphism if $p = q$. In that case there is an obvious odd automorphism given by

$$a \otimes (x, y) \mapsto a \otimes (y, x),$$

which squares to the identity. Hence in that case

$$R(D) = \frac{\mathbb{N} \times \mathbb{N}}{\{(n, n) : n \in \mathbb{N}\}} \cong \mathbb{Z}.$$

Now assume that $D_1 \neq 0$. In that case Lemma B.5 determines the isomorphism classes of supermodules as $\mathbb{N} = \{D \hat{\otimes}_{\mathbb{R}} \mathbb{R}^{n|0} : n \geq 0\}$. Taking a look at part 2 of Lemma B.20, we see that D as a left D-supermodule admits an odd automorphism. By part 1 of the same lemma, the supermodule $D \oplus D \cong D \hat{\otimes} \mathbb{R}^{2|0}$ has an odd automorphism squaring to one. Hence $2\mathbb{N}$ is contained in the image of r_A. It only remains to be determined in which cases D itself has an odd automorphism squaring to the identity. It follows by simple computation in what cases D has an odd element with square one, finishing the analysis. This results in the relevant row of the ten-fold way table, which exactly agrees with the well known values of $K^n(\mathrm{pt})$ and $KO^n(\mathrm{pt})$, for example given in [40] Theorem III.5.19. □

Combining the last few propositions now immediately gives the following result.

Corollary B.24. *Let A be a semisimple real superalgebra. Use Corollary B.7 to get an isomorphism*

$$A \cong \bigoplus_{i=0}^{7} \bigoplus_{k_i=1}^{m_i} M_{p_{k_i}|q_{k_i}}(D_i) \oplus \bigoplus_{j=0}^{1} \bigoplus_{l_j=1}^{n_j} M_{r_{l_j}|s_{l_j}}(D'_j).$$

Then

$$R^p A \cong \bigoplus_{i=0}^{7} \bigoplus_{k_i=1}^{m_i} KO^{p+i}(\mathrm{pt}) \oplus \bigoplus_{j=0}^{1} \bigoplus_{l_j=1}^{n_j} K^{p+j}(\mathrm{pt}).$$

To illustrate this corollary in a familiar case, let $A = \mathbb{R}G$ be the real group algebra of a finite group G. Using the real representation theory of finite groups, we can efficiently arrive at the decomposition

$$A \cong \bigoplus_{i=1}^{l} M_{p_i}(\mathbb{R}) \oplus \bigoplus_{j=1}^{m} M_{q_j}(\mathbb{C}) \oplus \bigoplus_{k=1}^{n} M_{r_k}(\mathbb{H})$$

as follows. First of all, note that $A \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}G$ is the complex group algebra. By basic representation theory, we know that the number of (necessarily complex) matrix rings $k \in \mathbb{N}$ occurring in the decomposition of $A \otimes_{\mathbb{R}} \mathbb{C}$ equals the number of irreducible representations of G, which again equals the number of conjugation classes in G. Hence

$$R^p(\mathbb{C}G) = \begin{cases} \mathbb{Z}^k & q \text{ even}, \\ 0 & q \text{ odd}, \end{cases}$$

by the corollary.

Now recall the theory of real representations, which tells us how we can refine this theory to find out the decomposition of A into matrix rings. Suppose χ is a character of an irreducible complex representation of G corresponding to some matrix ring $M_n(\mathbb{C})$ in the decomposition of $A \otimes_{\mathbb{R}} \mathbb{C}$. We want to determine over which real division ring our copy $M_n(\mathbb{C})$ was before we tensored with $\mathbb{C}$. Note that it was $M_n(\mathbb{R}), M_{n/2}(\mathbb{H})$ or $M_n(\mathbb{C})$ and if it came from $M_n(\mathbb{C})$, then there must be another copy of $M_n(\mathbb{C})$ occurring in $A \otimes_{\mathbb{R}} \mathbb{C}$ that is somehow dual to this copy under the isomorphism $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ (this turns out to correspond to the dual representation on the level of characters). The *Frobenius-Schur indicator*

$$\iota_\chi := \frac{1}{|G|} \sum_{g \in G} \chi(g^2)$$

distinguishes between these possibilities: (see Section II.13.2 of Serre's book [61] for more details)

Theorem B.25 (Frobenius-Schur)**.** *The Frobenius Schur indicator always lies in the set* $\{1, -1, 0\}$. *Moreover,*

- *it is equal to* 1 *if the matrix ring corresponding to* χ *comes from a real matrix ring in* A*;*
- *it is equal to* 0 *if the matrix ring corresponding to* χ *comes from a complex matrix ring in* A*;*
- *it is equal to* -1 *if the matrix ring corresponding to* χ *comes from a quaternionic matrix ring in* A*.*

If one knows the representation theory of G, the Frobenius-Schur indicator gives an efficient way to find the decomposition of $\mathbb{R}G$ into matrix rings over division rings. Then one can compute the higher representation rings of $\mathbb{R}G$ using the corollary above, giving an efficient way to compute the K-theory of a point in class A and AI. Moreover, it can be shown that this reproduces the formulae given in the last section of Atiyah and Segal's work on the completion theorem [7]. Since this theory does not apply to the twisted version, a generalization of the Frobenius-Schur index to twisted group algebras would be of serious interest to compute the twisted equivariant K-theory groups in class AII. Such formulae are used in [43] and [62], but rigorous proofs in the mathematical literature seem to be lacking.

C. Equivariant Algebraic Topology

C.1. Spectral Sequences

Spectral sequences are a major algebraic tool in algebraic topology. They form a way to compute complicated cohomological objects by splitting them into bite size pieces, allowing approximations by succesive computations of cohomology, often starting with an ordinary cohomology group. In equivariant algebraic topology, these ordinary cohomology groups should be substituted by equivariant Bredon cohomology, subject of Section C.3. Here the purely algebraic language of spectral sequences will be briefly reviewed. Throughout the cohomological convention is used.

Definition C.1. Let $R = \oplus_{n\in\mathbb{Z}} R^n$ be $\mathbb{Z}$-graded abelian group. For every $r \in \mathbb{Z}$ let $E_r = (E_r^{p,q}, d_r)$ be a chain complex of $\mathbb{Z}^2$-graded abelian groups such that the differential d_r has bidegree $(r, 1-r)$. Here E_r is called *the rth page.* We say E_r is a *spectral sequence converging to* R and write

$$E_r^{p,q} \implies R^{p+q}$$

if the following properties hold:

- as an abelian group $E_{r+1}^{p,q}$ is the cohomology of $(E_r^{p,q}, d_r)$ at position (p,q);
- the sequence *converges* in the sense that for every $p, q \in \mathbb{Z}$ there is an $r \in \mathbb{Z}$ such that if $r' > r$, then $E_r^{p,q} \cong E_{r'}^{p,q}$. Hence we can define the *limit* $E_\infty^{p,q}$ to be $E_r^{p,q}$ for $r \in \mathbb{Z}$ sufficiently large;
- there exists a filtration

  $$\cdots \subseteq F^{p+1,q-1} \subseteq F^{p,q} \subseteq F^{p-1,q+1} \subseteq \cdots \subseteq R^{p+q}$$

  such that

  $$E_\infty^{p,q} \cong \frac{F^{p,q}}{F^{p+1,q-1}}.$$

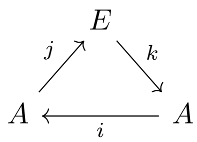
The definition of a spectral sequence may seem confusing as well as intimidating at first, but the reader should be reminder that it is just a useful summary of all algebraic information that can be extracted from certain given homological data, such as long exact sequences. A common source of spectral sequence comes in the form of exact couples.

Definition C.2. An *exact couple* is a pair (A, E) of $\mathbb{Z}^2$-graded abelian groups together with three maps:

- $i : A \to A$;

- $j : A \to E$;
- $k : E \to A$;

that together form an exact triangle as in the following diagram.



The *differential* of an exact couple is $d := jk$. The *derived exact couple* of an exact couple is defined as follows:

- $E' = \ker d / \operatorname{Im} d$;
- $A' = i(A)$;
- $i' = i|A'$;
- $j'(ia) = [ja]$;
- $k'[e] = ke$.

The idea is that an exact couple (A, E) will give a spectral sequence in which the pages are given by iterated derived exact couples $E, E', E'', \ldots$ together with their differential $d = jk$. In order to make this idea into a proposition, we have to make exact couples agree with our grading conventions of spectral sequences. To this extend, we say an exact couple is *graded* of degree r if A and E are $\mathbb{Z}^2$-graded such that i is of degree $(-1, 1)$, j is of degree $(r, 1-r)$ and k is of degree $(0, 0)$.

Proposition C.3. *Derived couples induce spectral sequences in the following sense.*

1. *The derived exact couple (A', E') of an exact couple (A, E) is again an exact couple. If (A, E) is graded of degree r, then (A', E') is graded of degree $r+1$.*
2. *Let (A, E) be a graded exact couple of degree 1. Assume that for all $n \in \mathbb{Z}$ the 'A-columns'*
$$C_n = \bigoplus_{(p,q):p+q=n} A^{p,q}$$
are bounded in the sense that there exist p_- and p_+ such that
   - *for $p < p_-$, $A^{p,n-p} = 0$;*
   - *for $p > p_+$, the maps $i : A^{p,n-p} \to A^{p-1,n-p+1}$ are isomorphisms.*

   Let R^n denote $A^{p_+,n-p_+}$. Then the derived exact couple construction $E_{r+1} := (E_r)'$ makes E_r into a spectral sequence converging to R. Moreover, the filtration F is given by
$$F^{p,q} = \ker(R^{p+q} \to A^{p,q}),$$
where the map is given by repeatedly applying i.

Proof. 1. See the draft of Hatcher's book on spectral sequences [33] for the nongraded case (Lemma 5.1). The degrees are easily extracted from the explicit formulae of the derived exact couple.

2. This is a reformulation of the first part of Proposition 5.2 in Hatcher [33] into our grading conventions.

□

C.2. Basic Equivariant Algebraic Topology

In this section we briefly review some constructions from equivariant algebraic topology, restricting to only the absolutely necessary notions. For a thorough treatment, see for example [29]. Since our interest lies in the torus with a finite group action, we let G be a fixed finite group and call X a *G-space* if it is compact Hausdorff and equipped with a continuous G-action. We sometimes also consider noncompact G-spaces[1]. Some basic notions concerning G-spaces are the following.

The *fixed point space*, written X^G, is the subspace of X consisting of the points x such that $gx = x$ for all $g \in G$. If $K \subseteq X$ is a subset (or a point), the *stabilizer group at K* is defined to be

$$G_K := \{g \in G : gx = x \ \forall x \in K\}.$$

If X and Y are G-spaces, then $C(X, Y)$ with the compact-open topology is a (possibly noncompact) G-space by

$$(g \times f)(x) = gf(g^{-1}y).$$

If $f \in C(X, Y)^G$, then we say f is *G-equivariant* or a *G-map*. The category $\mathbf{Top_G}$ of G-spaces has G-maps as morphisms. Two G-maps $f, g : X \to Y$ are *G-homotopic* if there exists a G-equivariant homotopy $H : X \times I \to Y$ from f to g (here G acts trivially on the second component of $X \times I$). This gives a naive homotopy category $\mathbf{hTop_G}$ of G-spaces in which a morphism is an equivalence class of G-maps under G-homotopy. A *G-subspace* $A \subseteq X$ is a subspace which is closed under the action of G. A *pair of G-spaces* (X, A) consists of a G-space X and a G-subspace $A \subseteq X$. In particular, a *pointed G-space* is a pair (X, x_0) of a G-space X and an element $x_0 \in X^G$. The category $\mathbf{Pairs}_G$ of pairs of G-spaces has as morphisms $(X, A) \to (Y, B)$ G-maps $f : X \to Y$ such that $f(A) \subseteq B$. There are also *triples* (X, A, a) of G-spaces, defined such that (X, A) is a pair of G-spaces and (A, a) is a pointed G-space.

Most constructions from ordinary algebraic topology generalize to the equivariant setting. If X is a G-space, then

- the *cone of X* is the G-space $CX := \frac{X \times [0,1]}{X \times 0}$ with action $g \cdot (x, t) = (gx, t)$;
- the *suspension of X* is the G-space $SX := \frac{CX}{X \times 1}$ with action $g \cdot (x, t) = (gx, t)$.

If (X, x_0) is a pointed G-space, then

[1] Most results should generalize to compactly generated X with compact group action, but we do not pursue this setting here.

- the *(reduced) cone of* (X, x_0) is the G-space $KX := \frac{X\times[0,1]}{(X\times 0)\cup(x_0\times[0,1])}$ with action $g\cdot(x,t) = (gx,t)$;
- the *(reduced) suspension of* (X, x_0) is the G-space $\Sigma X := \frac{CX}{X\times 1}$ with action $g\cdot(x,t) = (gx,t)$.

Just as in the nonequivariant case, a very useful computational tool is the structure of a CW complex. For the equivariant case, one usually studies G-CW complexes, which are constructed from the 'equivariant points' G/H (where $H \subseteq G$ is a subgroup) instead.

Definition C.4. A *G-CW complex* is a filtered G-space $X \supseteq \cdots \supseteq X^p \supseteq \cdots \supseteq X^0$, where X^0 is a disjoint union of orbit spaces G/H and X^{p+1} is built from X^p by attaching G-spaces of the form $G/H \times D^{p+1}$ by G-maps $f : G/H \times S^p \to X^p$. Denote the category of G-CW complexes by $\mathbf{CW_G}$ and the corresponding naive G-homotopy category by $\mathbf{hCW_G}$.

Using the fact that G is finite, one can prove that a G-CW complex X is exactly a G-space that is also a CW complex, such that

1. the action is by cellular maps;
2. for every $g \in G$ the set $\{x \in X : gx = x\}$ is a subcomplex [68].

A *G-subcomplex* in this language is a subcomplex, which is also a G-subspace. We get a category $\mathbf{CW_G pairs}$ of *G-CW pairs* (X, A) with a G-CW complex and a G-subcomplex A, in which morphisms are G-maps that preserve the corresponding subspace. Similarly, we can define pointed G-CW complexes and triples of G-CW complexes. This gives naive homotopy categories for pointed G-CW complexes, pairs and triples as well.

The *orbit category* $\mathcal{O}_G$ of G is the full subcategory of $\mathbf{Top_G}$ consisting of cosets G/H for all subgroups $H \subseteq G$. Note that it is naturally contained in $\mathbf{CW_G}$. For reference in the main text, we state some basic facts about the orbit category here.

Proposition C.5. *Let $H, K \subseteq G$ be subgroups. Then the morphisms $G/H \to G/K$ in $\mathcal{O}_G$ (i.e. the equivariant maps) are the maps of the form $\hat{a}(gH) := gaK$ for some $a \in G$ such that $H \subseteq a^{-1}Ka$. Two such maps $\hat{a}, \hat{b}$ are equal if and only if $a^{-1}b \in K$.*

Proof. Easy, see for example [14]. □

Corollary C.6. *The map $a \mapsto \hat{a}$ induces an isomorphism*

$$\frac{N_G(H)}{C_G(H)} \cong \mathrm{Aut}_{\mathcal{O}_G}(G/H),$$

where $N_G(H)$ is the normalizer of H and $C_G(H)$ the centralizer.

C.3. Equivariant Generalized Cohomology & Bredon Cohomology

Segal's complex equivariant K-theory, which is a well-studied special case of the K-theory studied in this document, is a (generalized) cohomology theory on the category of G-spaces. By the methods of Segal developed in [56] (see [57] for the complete argument), this implies that there is a spectral sequence converging to equivariant K-theory with the second page equal to an equivariant version of ordinary cohomology. This ordinary cohomology turns out to be Bredon's equivariant cohomology as in [14]. Here the well-known theory will be briefly reviewed as a warm-up for the twisted equivariant case. An example of a formulation of the axioms in the equivariant context is the following (taken from Matsumoto [51]).

Definition C.7. An *equivariant cohomology theory* is a family $h^n_G : \mathbf{hCW_G\text{-}pairs} \to \mathbf{Ab}$ of contravariant functors for $n \in \mathbb{Z}$, together with a natural transformation $\delta : h^n_G(X, A) \to h^{n+1}_G(A, \emptyset)$ such that

- (Long Exactness) A G-CW pair (X, A) gives a long exact sequence

$$\dots \xrightarrow{\delta} h^n_G(A, \emptyset) \to h^n_G(X, \emptyset) \to h^n_G(X, A) \xrightarrow{\delta} h^{n+1}_G(A, \emptyset) \to \dots$$

- (Excision) If $X = A \cup B$ for some G-subcomplexes $A, B \subseteq X$, then the inclusion map $(A, A \cap B) \to (X, B)$ induces an isomorphism

$$h^n_G(X, B) \cong h^n_G(A, A \cap B)$$

  for every $n \in \mathbb{Z}$.

An equivariant cohomology theory is called *additive* if it also satisfies the following axiom: if a G-CW pair (X, A) is the disjoint union of (X_i, A_i) as G-spaces, then the inclusion maps $(X_i, A_i) \to (X, A)$ induce an isomorphism

$$h^n_G(X, A) \cong \bigoplus_i h^n_G(X_i, A_i).$$

It is called *finitely additive* if this holds for finite disjoint unions. An equivariant cohomology theory satisfies the *dimension axiom* if there exists a functor $F : \mathcal{O}_G \to \mathbf{Ab}$ such that for every subgroup $H \subseteq G$

$$h^n_G(G/H) = \begin{cases} 0 & \text{if } n \neq 0, \\ F(G/H) & \text{if } n = 0. \end{cases}$$

We say h is an *ordinary equivariant cohomology theory with coefficient system* $F : \mathcal{O}_G \to$ ***Ab*** if it is additive and satisfies the dimension axiom with F as in the above.

Only the dimension axiom requires additional comments. In the equivariant case, the coset spaces G/H should be seen as the 'equivariant points'; we need to specify the theory on all such spaces for the theory to be fixed.

As in the non-equivariant case, there are various ways to explicitely construct ordinary equivariant cohomology theories. For computational purposes, cellular equivariant cohomology will now be constructed following Bredon's work [14].

Definition C.8. Let G be a finite group, $F : \mathcal{O}_X \to \mathbf{Ab}$ a functor and X a G-CW complex. Pick an orientation of the cells of X such that all $g \in G$ preserve it. Let $C^q(X)$ denote the set of (non-equivariant) q-cells of X. A mapping

$$f : C^q(X) \to \bigsqcup_{\substack{H \subseteq G \\ \text{subgroup}}} F(G/H)$$

is called a *q-dimensional G-cochain* if

- $f(\sigma) \in F(G/G_\sigma)$;
- f is *equivariant* in the sense that

  $$F(\hat{g})(f(\sigma)) = f(g\sigma)$$

  for all $g \in G$ and $\sigma \in C^q(X)$.

Write $C^q_G(X)$ for the set of G-cochains. There is a differential on the G-cochains given by

$$(\delta f)(\sigma) = \sum_{\tau \in C^q(X)} [\tau : \sigma] F(G/G_\sigma \to G/G_\tau)(f(\tau)),$$

where $G/G_\sigma \to G/G_\tau$ is the quotient map.[2] The cohomology theory corresponding to this chain complex is called the *(Bredon) equivariant cellular cohomology* of X with coefficients F.

It is shown in [14] that an equivariant cohomology theory is ordinary with coefficient system F if and only if it is naturally isomorphic to equivariant cellular cohomology with coefficients F.

[2]Note that $G_\sigma \subseteq G_\tau$ if $\tau \subseteq \sigma$.