# Structured Homotopy Theory
# from
# String Theory

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a couple of talks at

## Geometry, Topology & Physics

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Based on joint work with:

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**Abstract.**

*Homotopy theory is extremely rich, with various structured variants such as equivariant, graded, parameterized and stable homotopy theory. Powerful tools from differential-graded algebra, particularly in the rational approximation, serve for concrete computations.*

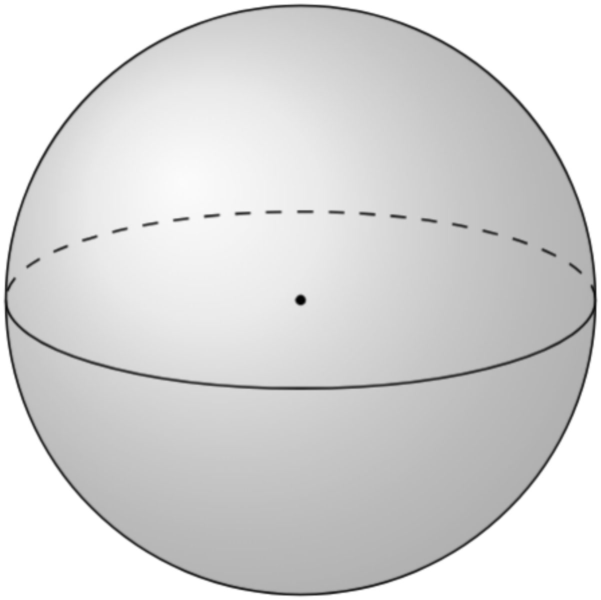
*Also string/M-theory is extremely rich, revealing a system of higher dimensional objects (branes) with subtle inter-relations.*

*In these talks I survey recent insights into a close relation between the two, which provides structured homotopy theory with curious new examples and sheds light on the elusive foundations of string/M-theory.*

*The 4-sphere will play a surprisingly central role, as initially realized by Hisham Sati.*

## The 4-sphere

$$S^4$$



We will discuss its incarnation
in various flavours of *homotopy theory*:

1)

**Super homotopy theory**

**and**

**fundamental M2/M5-branes**

2)

**Parameterized stable homotopy theory**

**and**

**gauge enhancement**

3)

**Equivariant homotopy theory**

**and**

**black M-branes**

# 1)

# Super homotopy theory
# and
# fundamental M2/M5-branes

based on FSS 13, FSS 15, FSS 16, HS 17

# Classical homotopy theory

$$\mathrm{Ho}(\mathrm{Spaces}) \;:=\; \mathrm{Spaces}\left[\{\text{Isos on all homotopy groups}\}^{-1}\right]$$

see ncatlab.org/nlab/show/Introduction+to+Homotopy+Theory

For example homotopy groups of spheres:

$$\pi_k(S^n) := \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Spaces})}\left(S^k, S^n\right)$$

These exhibit an endlessly rich pattern.
But most of them are torsion groups.

For example for $S^4$ two of them non-torsion:
generated by the *quaternionic Hopf fibration*:

# Quaternionic Hopf fibration

$$\begin{tikzcd}
S^7 \arrow[r, phantom, "\simeq"] \arrow[d, "\mathrm{Hopf}_{\mathbb{H}}"] & S(\mathbb{H} \oplus \mathbb{H}) \arrow[d] & (x, y) \arrow[d, mapsto] & {|x|^2 + |y|^2 = 2} \\
S^4 \arrow[r, phantom, "\simeq"] & S(\mathbb{R} \oplus \mathbb{H}) & \left( \begin{array}{l} t := 1 - |x|^2, \\ z := x \cdot y \end{array} \right) & t^2 + |z|^2 = 1
\end{tikzcd}$$

$$\pi_\bullet\left(S^4\right) \;\simeq\; \mathbb{Z}\underbrace{\langle \mathrm{id}_{S^4} \rangle}_{\mathrm{deg}=4} \;\oplus\; \mathbb{Z}\underbrace{\langle \mathrm{Hopf}_{\mathbb{H}} \rangle}_{\mathrm{deg}=7} \;\oplus\; \text{torsion}$$

## Rational homotopy theory

disregards all torsion information:

$$\mathrm{Ho}(\mathrm{Spaces}_{\mathbb{Q}}) \;:=\; \mathrm{Spaces}\left[\{\text{Isos on rationalized homotopy groups}\}^{-1}\right]$$

$$\pi_k(S^n) \otimes \mathbb{Q} = \mathrm{Hom}_{\mathrm{Ho}(\mathrm{Spaces}_{\mathbb{Q}})}\left(S^k, S^n\right)$$

$$\pi_\bullet\left(S^4\right) \otimes \mathbb{Q} \;=\; \mathbb{Q}\underbrace{\langle \mathrm{id}_{S^4} \rangle}_{\mathrm{deg}=4} \;\oplus\; \mathbb{Q}\underbrace{\langle \mathrm{Hopf}_{\mathbb{H}} \rangle}_{\mathrm{deg}=7}$$

## dg-Algebraic model for rational homotopy theory.

$$\mathrm{dgcAlg} \;:=\; \{\text{differential graded-commutative algebras over } \mathbb{R}\}$$

Homotopy theory of dg-algebras:

$$\begin{aligned} &\mathrm{Ho}\left(\mathrm{dgcAlg}^{\mathrm{op}}\right) \\ &:= \\ &\mathrm{dgcAlg}^{\mathrm{op}}\left[\{\text{Isos on all cohomology groups}\}^{-1}\right] \end{aligned}$$

Quillen-Sullivan equivalence:

$$\begin{tikzcd}
\mathrm{Ho}(\mathrm{Spaces}) \arrow[r, shift left=2, "\mathcal{O}"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}(\mathrm{dgcAlg}^{\mathrm{op}}) \arrow[l, shift left=2, "\mathcal{S}"] \\
\mathrm{Ho}\left(\mathrm{Spaces}_{\mathbb{Q},\mathrm{nil},\mathrm{fin}}\right) \arrow[u, hook] \arrow[r, shift left=2, "\mathcal{O}"] \arrow[r, phantom, "\simeq"] & \mathrm{Ho}\left(\mathrm{dgcAlg}^{\mathrm{op}}_{\mathrm{cn},\mathrm{fin}}\right) \arrow[u, hook] \arrow[l, shift left=2, "\mathcal{S}"]
\end{tikzcd}$$

## dg-Algebra model for the 4-sphere

Classical fact of rational homotopy theory:

$$\mathcal{O}(S^4) \;\simeq\; \mathbb{R}[\underbrace{\omega_4}_{\text{deg=4}}, \underbrace{\omega_7}_{\text{deg=7}}]/\begin{pmatrix} d\omega_4 = 0 \\ d\omega_7 = -\frac{1}{2}\omega_4 \wedge \omega_4 \end{pmatrix}$$

---

This equation

$$\begin{aligned} &dG_4 = 0 \\ &dG_7 + \tfrac{1}{2}G_4 \wedge G_4 \;=\; 0 \end{aligned}$$

is also the equations of motion in 11-dimensional supergravity for the M-brane flux.

**Conjecture** (Sati 13, Sect. 1.5):
*The cohomology theory classifying M-branes*
*is some flavour of degree-4 cohomotopy.*

## Super homotopy theory.

Via Quillen-Sullivan we may immediatly generalize homotopy theory to *superspaces*:

$$\text{sdgcAlgebras} \;:=\; \left\{ \begin{array}{c} \text{differential graded algebras} \\ \text{with} \\ \underbrace{\mathbb{Z}}_{\substack{\text{cohomological} \\ \text{grading}}} \;\times\; \underbrace{\mathbb{Z}/2}_{\substack{\text{super} \\ \text{grading}}}\text{-grading} \end{array} \right\}$$

In supergravity, cofibrant sdgc-algebras are known as “FDA”s.

We consider the corresponding homotopy theory:

$$\begin{array}{c} \mathrm{Ho}(\mathrm{SuperSpaces}_{\mathbb{Q},\mathrm{nil},\mathrm{fin}}) \\ := \\ \mathrm{sdgcAlg}^{\mathrm{op}}_{\mathrm{cn},\mathrm{fin}}\left[\{\text{Isos on all cohomology groups}\}^{-1}\right] \end{array}$$

see ncatlab.org/nlab/show/geometry+of+physics+–+superalgebra

# Example: Super-Minkowski spacetimes.

The nilpotency condition on the fundamental group allows precisely the mild non-abelianness that goes with super-Minkowski spacetimes:

$$\mathcal{O}\Big( \underbrace{\mathbb{T}^{p,1|\mathbf{N}}}_{\substack{\text{toroidal}\\ \text{super-Minkowski}\\ \text{spacetime}}} \Big) := \mathbb{R}\Big[ \underbrace{\underbrace{(e^a)_{a=0}^{p}}_{\text{deg=(1,even)}} , \underbrace{(\psi^\alpha)_{\alpha=1}^{N}}_{\text{(1,odd)}}}_{\text{super vielbein}} \Big] / \underbrace{\left( \begin{array}{rl} de^a &= \overline{\psi}\Gamma^a\psi \\ d\psi^\alpha &= 0 \end{array} \right)}_{\substack{\text{torsion}\\ \text{constraint}}}$$

see ncatlab.org/nlab/show/geometry+of+physics+–+supersymmetry

# Extension tower of super-Minkowski spacetime.

These appear in a sequence of iterated maximal R-symmetry invariant central extensions:

$$\begin{tikzcd}
 & \mathbb{R}^{10,1|\mathbf{32}} \arrow[dr] & \\
\mathbb{R}^{9,1|\mathbf{16}+\mathbf{16}} & \mathbb{R}^{9,1|\mathbf{16}} \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left] \arrow[r, shift right] \arrow[dr] & \mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \\
 & \mathbb{R}^{5,1|\mathbf{8}} \arrow[r, shift left] \arrow[r, shift right] \arrow[dl] & \mathbb{R}^{5,1|\mathbf{8}+\overline{\mathbf{8}}} \\
\mathbb{R}^{3,1|\mathbf{4}+\mathbf{4}} & \mathbb{R}^{3,1|\mathbf{4}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{2,1|\mathbf{2}+\mathbf{2}} & \mathbb{R}^{2,1|\mathbf{2}} \arrow[l, shift left] \arrow[l, shift right] \arrow[dl] & \\
\mathbb{R}^{0|\mathbf{1}+\mathbf{1}} & \mathbb{R}^{0|\mathbf{1}} \arrow[l, shift left] \arrow[l, shift right] & .
\end{tikzcd}$$

## The M2/M5-Brane cocycle.

On the top-dimensional 11d super Minkowski spacetime, there is a unique element in Spin(10, 1)-invariant rational cohomotopy in degree 4.

$$\boxed{\mathbb{T}^{10,1|\mathbf{32}} \xrightarrow{\mu_{M2/M5}} S^4}$$

$\in \mathrm{Ho}(\mathrm{SuperSpaces}_{\mathbb{R}})^{\mathrm{Spin}(10,1)}$

In string/M theory this statement
characterizes fundamental M2-brane and M5-branes
via the WZW terms of their Green-Schwarz-type sigma-models.

# 2)

# Parameterized stable homotopy theory

# and

# gauge enhacement

based on BSS 18

# Central extensions are homotopy fibers.

That 11d super-spacetime is an extension
of 10d super-spacetime
means that it is a *homotopy fiber* of a 2-cocyle:

$$\begin{tikzcd}
\mathrm{Ext}_{\overline{\psi}\Gamma_{10}\psi}\left(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}\right) \arrow[r, equal] & \mathbb{T}^{10,1|\mathbf{32}} \arrow[d, "\text{homotopy fiber}"] & \\
& \mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} \arrow[dr, "\text{2-cocyle}", "\overline{\psi}\Gamma_{10}\psi"'] & \\
& & BS^1
\end{tikzcd}$$

## The Ext/Cyc-Adjunction.

**Proposition.** The extension/homotopy fiber functor has a right adjoint

$$\begin{tikzcd} \mathrm{Ho}\,(\mathrm{Spaces}) \arrow[r, phantom, "\perp"] & \mathrm{Ho}\left(\mathrm{Spaces}_{/BS^1}\right) \arrow[l, bend right, "\mathrm{Ext}"'] \arrow[l, leftarrow, bend left, "\mathrm{Cyc}"'] \end{tikzcd}$$

given by forming *cyclic loop spaces*:

$$\mathrm{Cyc}(X) \;:=\; \mathrm{Maps}(S^1, X) /\!\!/ S^1$$

i.e. the homotopy quotient of the free loop space by the rigid rotation of loops.

## Example: cyclification of the 4-Sphere

The cyclification of the 4-sphere is

$$\mathcal{O}\left(\mathrm{Cyc}\left(S^4\right)\right) \;=\; \mathbb{Q}[h_3, h_7,\; \omega_2, \omega_4, \omega_6]\Big/ \left(\begin{array}{c} dh_7 = -\frac{1}{2}\omega_4 \wedge \omega_4 \\ \quad + \omega_2 \wedge \omega_6 \\ dh_3 = 0 \\ d\omega_{2p} = h_3 \wedge \omega_{2p-2} \end{array}\right)$$

Curiously, the terms in blue exhibit a truncation of rationalized twisted K-theory.

Below we will find also the rest of rationalized K-theory from the 4-sphere...

## The Ext/Cyc-adjunct of $\mu_{M2/M5}$

Hence the M2/M5-cocycle

$$\mathbb{T}^{10,1\vert\mathbf{32}} \xrightarrow{\mu_{M2/M5}} S^4$$

induces its Ext/Cyc-adjunct

$$\begin{tikzcd}
\mathbb{T}^{9,1\vert\mathbf{16}+\overline{\mathbf{16}}} \arrow[d, "\eta"'] \arrow[drr, bend left, "\widetilde{\mu_{M2/M5}}"] & & \\
\mathrm{CycExt}\left(\mathbb{T}^{9,1\vert\mathbf{16}+\overline{\mathbf{16}}}\right) \arrow[rr, "\mathrm{Cyc}\left(\mu_{M2/M5}\right)" description] \arrow[dr] & & \mathrm{Cyc}\left(S^4\right) \arrow[dl] \\
& B^2 S^1 &
\end{tikzcd}$$

see ncatlab.org/schreiber/show/Super+Lie+n-algebra+of+Super+p-branes

# **This is double dimensional reduction of M2/M5-cocycle**

to the F1/NS5/D0/D2/D4 branes of type IIA string theory:

| | |
|---|---|
| Sullivan algebra of cyclified 4-sphere | $dh_7 = -\frac{1}{2}\omega_4 \wedge \omega_4 + \omega_2 \wedge \omega_6$<br>$dh_3 = 0$<br>$d\omega_{2p} = h_3 \wedge \omega_{2p-2}$ |
| Bianchi identities of NS1/NS5-flux and D($p \leq 4$) RR-fluxes: | $dH_7 = -\frac{1}{2}F_4 \wedge F_4 + F_2 \wedge F_6$<br>$dH_3 = 0$<br>$dF_{2p} = H_3 \wedge F_{2p-2}$ |

$$2p \in \{0, 2, 4\}$$

## Extensions and actions

To make the double dimensional reduction more symmetric, we ask also $S^4$ to be an $S^1$-extension

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}} \arrow[d, "\text{homotopy fiber}"] & & S^4 \arrow[d, "\text{homotopy fiber}"] \\
\mathbb{R}^{9,1|\mathbf{16}+\overline{\mathbf{16}}} \arrow[dr] & & S^4 /\!\!/ S^1 \arrow[dl] \\
& BS^1 &
\end{tikzcd}$$

The base $S^4 /\!\!/ S^1$ of such an extension is necessarily the homotopy quotient by some $S^1$-action on $S^4$.

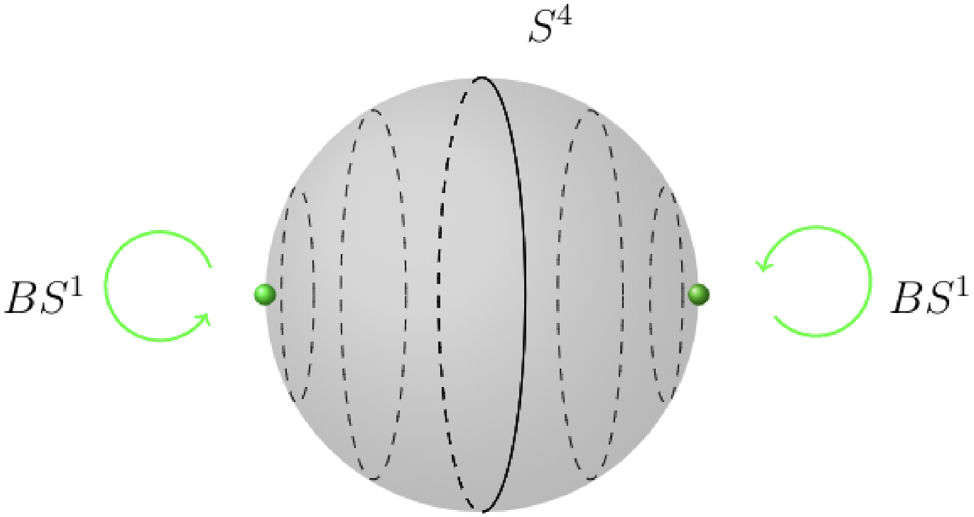
# Example: Suspended Hopf action on $S^4$

The identifications

$$S^1 \simeq U(1) \subset \mathrm{SU}(2) \simeq S(\mathbb{H})$$

$$S^4 \simeq S(\mathbb{R} \oplus \mathbb{H})$$

induce an $S^1$-action on $S^4$.



# Attempted lift in Super homotopy theory

For any choice of $S^1$-action on $S^4$
we may ask for a lift of
the double dimensional reduction of the M2/M5-cocyle:

$$\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} \arrow[rr, dashed, bend left, "\exists ?"] \arrow[d, "\eta"'] \arrow[drr, "\widetilde{\mu_{M2/M5}}"] & & S^4 \sslash S^1 \arrow[d, "\eta"] \\
\mathrm{CycExt}\left(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}\right) \arrow[rr, "\mathrm{Cyc}\left(\mu_{M2/M5}\right)"] \arrow[dr] & & \mathrm{CycExt}\left(S^4 \sslash S^1\right) \arrow[dl] \\
& B^2 S^1 &
\end{tikzcd}$$

# Attempted lift in Super homotopy theory

For any choice of $S^1$-action on $S^4$
we may ask for a lift of
the double dimensional reduction of the M2/M5-cocyle:

$$\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} \arrow[rr, dashed, bend left, "\nexists"] \arrow[d, "\eta"'] \arrow[drr, "\widetilde{\mu_{M2/M5}}"] & & S^4 \sslash S^1 \arrow[d, "\eta"] \\
\mathrm{CycExt}\left(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}\right) \arrow[rr, "\mathrm{Cyc}\left(\mu_{M2/M5}\right)"] \arrow[dr] & & \mathrm{CycExt}\left(S^4 \sslash S^1\right) \arrow[dl] \\
& B^2 S^1 &
\end{tikzcd}$$

**Prop.** *Such a lift does not exist, for any choice of action.*
But it may exist *to first linear order.*

# Stable homotopy theory

$$\mathrm{Ho}\,(\mathrm{Spectra}) \;:=\; \mathrm{Spectra}\left[\{\text{ Isos on all stable homotopy groups }\}^{-1}\right]$$

Spectra stabilize the operation of forming loop spaces:

$$\begin{tikzcd}
\mathrm{Ho}\left(\mathrm{Spaces}^{*/}\right) \arrow[r, shift right=2, "\Omega"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right=2, "\Sigma^\infty"'] & \mathrm{Ho}\left(\mathrm{Spaces}^{*/}\right) \arrow[l, shift right=2, "\Sigma"'] \arrow[d, shift right=2, "\Sigma^\infty"'] \\
\mathrm{Ho}\,(\mathrm{Spectra}) \arrow[u, shift right=2, "\Omega^\infty"'] \arrow[u, phantom, "\dashv"] \arrow[r, shift right=2, "\Omega"'] \arrow[r, phantom, "\simeq"] & \mathrm{Ho}\,(\mathrm{Spectra}) \arrow[l, shift right=2, "\Sigma"'] \arrow[u, shift right=2, "\Omega^\infty"'] \arrow[u, phantom, "\dashv"]
\end{tikzcd}$$

Loop spaces are the groups of homotopy theory.

Double loop space are the first-order commutative groups.

⋮

Hence: Spectra are the abelian groups of homotopy theory,

Hence: $\Sigma^\infty$ is linearization in homotopy theory.

see ncatlab.org/nlab/show/Introduction+to+Stable+Homotopy+Theory

# Parameterized stable homotopy theory

For $X \in \mathrm{Ho}\,(\mathrm{Spaces})$

$$\mathrm{Ho}\,(\mathrm{Spectra}_X) \;:=\; \mathrm{Spectra}_X \left[ \left\{ \begin{array}{c} \text{Isos on all stable homotopy groups} \\ \text{for all homotopy fibers over } X \end{array} \right\}^{-1} \right]$$

Parameterized spectra stabilize forming homotopy-fiber wise loop spaces:

$$\begin{tikzcd}
\mathrm{Ho}\left( \left( \mathrm{Spaces}_{/X} \right)^{X/} \right) \arrow[r, shift right, "\Omega_X"'] \arrow[r, phantom, "\perp"] \arrow[d, shift right, "\Sigma^\infty_X"'] \arrow[d, phantom, "\dashv"] & \mathrm{Ho}\left( (\mathrm{Spaces}_X)^{X/} \right) \arrow[l, shift right, "\Sigma_X"'] \arrow[d, shift right, "\Sigma^\infty_X"'] \arrow[d, phantom, "\dashv"] \\
\mathrm{Ho}\left( \mathrm{Spectra}_{/X} \right) \arrow[u, shift right, "\Omega^\infty_X"'] \arrow[r, shift right, "\Omega_X"'] \arrow[r, phantom, "\simeq"] & \mathrm{Ho}\,(\mathrm{Spectra}_X) \arrow[u, shift right, "\Omega^\infty_X"'] \arrow[l, shift right, "\Sigma_X"']
\end{tikzcd}$$

Hence: Parameterized spectra are the bundles of abelian groups in homotopy theory.

# Rational parameterized stable homotopy theory

**Theorem** (Braunack-Mayer 18) **:**

The Quillen-Sullivan dg-Model for rational homotopy theory generalizes to parameterized stable homotopy theory by modeling parameterized spectra by dg-modules:

$$\begin{tikzcd}
\mathrm{Ho}\left(\mathrm{Spectra}_{\mathcal{S}(A)}\right) \arrow[r, shift left=2, "\mathcal{O}"] \arrow[r, phantom, "\perp"] & \mathrm{Ho}\left(\mathrm{dgMod}_{A}\right)^{\mathrm{op}} \arrow[l, shift left=2, "\mathcal{S}"] \\
\mathrm{Ho}\left(\left(\mathrm{Spectra}_{\mathcal{S}(A)}\right)_{\mathbb{Q},\mathrm{fin}}\right) \arrow[u, hook] \arrow[r, shift left=2, "\mathcal{O}"] \arrow[r, phantom, "\simeq"] & \mathrm{Ho}\left(\mathrm{dgMod}_{A,\mathrm{bd}}\right)^{\mathrm{op}} \arrow[l, shift left=2, "\mathcal{S}"] \arrow[u, hook]
\end{tikzcd}$$

# Linearized lift of Ext/Cyc-adjunct of M2/M5-Cocycle.

Remember that we were asking for a full lift,
which, however, does not exist:

$$\begin{tikzcd}
\mathbb{T}^{9,1\vert\mathbf{16}+\mathbf{16}} \arrow[r, dashed, bend left, "\nexists"] \arrow[d, "\eta"'] \arrow[dr, "\widetilde{\mu_{M2/M5}}" description] & S^4 /\!\!/ S^1 \arrow[d, "\eta"] \\
\mathrm{CycExt}\left(\mathbb{T}^{9,1\vert\mathbf{16}+\overline{\mathbf{16}}}\right) \arrow[r, "\mathrm{Cyc}\left(\mu_{M2/M5}\right)"] \arrow[dr] & \mathrm{CycExt}\left(S^4 /\!\!/ S^1\right) \arrow[d] \\
 & B^2 S^1
\end{tikzcd}$$

$\in \mathrm{Ho}\left(\mathrm{SuperSpaces}_{\mathbb{R}}\right)$

# Linearized lift of Ext/Cyc-adjunct of M2/M5-Cocycle.

But now we may linearize the coefficients:

$$\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} \arrow[d, "\eta"'] \arrow[rr, dashed, bend left, "\nexists"] \arrow[rrd, "\widetilde{\mu_{M2/M5}}"] & & S^4 /\!\!/ S^1 \arrow[d, "\eta"] & \Omega^\infty_{B^2S^1}\Sigma^\infty_{B^2S^1}\left(S^4 /\!\!/ S^1\right) \arrow[d, "\Omega^\infty_{B^2S^1}\Sigma^\infty_{B^2S^1}(\eta)"] \\
\mathrm{CycExt}\left(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}\right) \arrow[rr, "\mathrm{Cyc}\left(\mu_{M2/M5}\right)"] \arrow[rrd] & & \mathrm{CycExt}\left(S^4 /\!\!/ S^1\right) \arrow[r] \arrow[d] & \Omega^\infty_{B^2S^1}\Sigma^\infty_{B^2S^1}\mathrm{CycExt}\left(S^4 /\!\!/ S^1\right) \arrow[ld] \\
& & B^2S^1 &
\end{tikzcd}$$

$\in \mathrm{Ho}\left(\mathrm{SuperSpaces}_{\mathbb{R}}\right)$

# Linearized lift of Ext/Cyc-adjunct of M2/M5-Cocycle.

**Theorem:** With the suspended Hopf $S^1$-action on $S^4$ the linearized lifting problem has a unique solution:

$$\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} \arrow[d, "\eta"'] \arrow[dr, "\widetilde{\mu_{M2/M5}}"] \arrow[rr, dashed, bend left, "\mu_{F1/Dp}"] & & \Omega^\infty_{B^2S^1}\Sigma^\infty_{B^2S^1}\left(S^4 /\!\!/ S^1\right) \arrow[d, "\Omega^\infty_{B^2S^1}\Sigma^\infty_{B^2S^1}(\eta)"] \\
\mathrm{CycExt}\left(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}\right) \arrow[r, "\mathrm{Cyc}\left(\mu_{M2/M5}\right)"] \arrow[dr] & \mathrm{CycExt}\left(S^4 /\!\!/ S^1\right) \arrow[r] \arrow[d] & \Omega^\infty_{B^2S^1}\Sigma^\infty_{B^2S^1}\mathrm{CycExt}\left(S^4 /\!\!/ S^1\right) \arrow[dl] \\
 & B^2S^1 &
\end{tikzcd}$$

$\in \mathrm{Ho}\left(\mathrm{SuperSpaces}_{\mathbb{R}}\right)$

# Linearized lift of Ext/Cyc-adjunct of M2/M5-Cocycle.

**Theorem:** This solution factors through a summand which is the (rationalized) twisted K-theory spectrum:

$$\begin{tikzcd}
\mathbb{T}^{9,1|\mathbf{16}+\mathbf{16}} \arrow[r, dashed, bend left, "\mu_{F1/Dp}"] \arrow[d, "\eta"'] \arrow[dr, "\widetilde{\mu_{M2/M5}}"] & \Omega^{\infty-2}_{B^2S^1}\left(\mathrm{ku}/\!\!/BS^1\right) \arrow[r, hook] & \Omega^{\infty}_{B^2S^1}\Sigma^{\infty}_{B^2S^1}(S^4 /\!\!/ S^1) \arrow[d, "\Omega^{\infty}_{B^2S^1}\Sigma^{\infty}_{B^2S^1}(\eta)"] \\
\mathrm{CycExt}\left(\mathbb{T}^{9,1|\mathbf{16}+\overline{\mathbf{16}}}\right) \arrow[r, "\mathrm{Cyc}\left(\mu_{M2/M5}\right)"] \arrow[dr] & \mathrm{CycExt}\left(S^4 /\!\!/ S^1\right) \arrow[r] \arrow[d] & \Omega^{\infty}_{B^2S^1}\Sigma^{\infty}_{B^2S^1}\mathrm{CycExt}\left(S^4 /\!\!/ S^1\right) \arrow[dl] \\
 & B^2S^1 &
\end{tikzcd}$$

$\in \mathrm{Ho}\left(\mathrm{SuperSpaces}_{\mathbb{R}}\right)$

## This is gauge enhancement of M2/M5-cocycle

to the full F1/D$p$ branes of type IIA string theory:

| | | |
|---|---|---|
| dg-module for fiberwise stabilized suspended Hopf quotient of $S^4$ $\Sigma^\infty_{B^2 S^1}\left(S^4 /\!\!/ S^1\right)$ | $dh_3 = 0$<br>$d\omega_{2p} = h_3 \wedge \omega_{2p-2}$ | $2p \in \mathbb{Z}$ |
| Bianchi identities of NS1-flux and D$p$ RR-fluxes: | $dH_3 = 0$<br>$dF_{2p} = H_3 \wedge F_{2p-2}$ | |

## Beyond rational approximation?

We would like to eventually
solve the open problem
of lifting this discussion
beyond the rational approximation.

As a first step
towards solving this problem
we now lift the $S^1 \subset \mathrm{SU}(2)$-action beyond rational
while keeping the spaces acted on rationally.

This hybrid approach
is
*equivariant rational homotopy theory.*

# 3)

# Equivariant homotopy theory
# and
# black M-branes

based on HSS18

## Equivariant homotopy theory

For $G$ a compact Lie group, the evident definition is:

$$\mathrm{Ho}(G\mathrm{Spaces}) :=$$

$$G\mathrm{CWComplexes}\left[\{G\text{-equivariant homotopy equivalences}\}^{-1}\right]$$

The equivariant Whitehead theorem relates this to fixed-point loci:

$$\mathrm{Ho}(G\mathrm{Spaces}) \simeq$$

$$G\mathrm{Spaces}\left[\left\{\begin{array}{c}\text{Isos on all homotopy groups}\\ \text{after restriction to } H\text{-fixed points}\\ \text{for all closed subgroups } H \subset G\end{array}\right\}^{-1}\right]$$

## Systems of fixed point loci

But the $H$-fixed points are equivalently
the $G$-equivariant maps out of the orbit space $G/H$:

$$X^H \;=\; \mathrm{Maps}(G/H, X)^G$$

Hence if we form the category of all possible $G$-orbit spaces

$$\mathrm{Orb}_G \;:=\; \{G/H\}_{\underset{\text{closed}}{H \subset G}} \subset G\mathrm{Spaces}$$

Then the *system of fixed point loci* is extracted as

$$G\mathrm{Spaces} \xrightarrow{Y} \qquad \mathrm{PSh}(\mathrm{Orb}_G, \mathrm{Spaces})$$

$$X \qquad \mapsto \qquad \left( \begin{tikzcd} G/H_1 \arrow[dd, "f"] & & X^{H_1} \\ & \mapsto & \\ G/H_2 & & X^{H_2} \arrow[uu, "\mathrm{Maps}(f{,}X)^G"'] \end{tikzcd} \right)$$

# Equivariant homotopy theory is about fixed point loci.

**Elmendorf's theorem:** Equivariant homotopy theory is plain homotopy theory of the systems of fixed point loci:

$$\mathrm{Ho}\left(G\mathrm{Spaces}\right) \xrightarrow[\simeq]{Y} \mathrm{Ho}\left(\mathrm{PSh}\left(\mathrm{Orb}_G, \mathrm{Spaces}\right)\right)$$

This induces in particular
equivariance for non-classical homotopy theories:
**Equivariant rational homotopy theory.**

$$\mathrm{Ho}\left(G\mathrm{Spaces}_{\mathbb{Q},\mathrm{nil},\mathrm{fin}}\right) \;\simeq\; \mathrm{Ho}\left(\mathrm{PSh}\left(\mathrm{Orb}_G\,, \mathrm{dgcAlg}^{\mathrm{op}}_{\mathrm{cn},\mathrm{fin}}\right)\right)$$

**Equivariant rational super homotopy theory.**

$$\mathrm{Ho}\left(G\mathrm{Super}\mathrm{Spaces}_{\mathbb{Q},\mathrm{nil},\mathrm{fin}}\right) \;\simeq\; \mathrm{Ho}\left(\mathrm{PSh}\left(\mathrm{Orb}_G\,, \mathrm{sdgcAlg}^{\mathrm{op}}_{\mathrm{cn},\mathrm{fin}}\right)\right)$$

## **Example:** $\mathrm{SU}(2)$**-Action on** $S^4_{\mathbb{Q}}$

**Prop.** The Hopf $S^1$-action on $S^4$ is trivial on the rational homotopy type of $S^4$, but in equivariant rational homotopy theory it is visible via its fixed point locus.

In fact this holds for the full $\mathrm{SU}(2)$-action:

$$\begin{tikzcd}
 & & \mathrm{SU}(2)/1 \arrow[d, "g"] & & S^4 \\
(S^4)^{(-)} & : & \mathrm{SU}(2)/1 \arrow[d] & \longmapsto & S^4 \arrow[u, "\mathrm{id}"'] \\
 & & \mathrm{SU}(2)/\mathrm{SU}(2) & & S^0 \arrow[u, hook]
\end{tikzcd}$$

$\in \mathrm{Spaces}_{\mathbb{Q}}$

## $G_{\mathrm{HW}}$-action on $S^4$

There is also a $\mathbb{Z}_2$-action on $S^4$ which does act non-trivially in rational homotopy theory, this is a reflection

$$\begin{tikzcd}
S^4 \arrow[r, "\text{reverse}"] & S^4 \\[-20pt]
(x_1, \cdots, x_4, x_5) \arrow[r, mapsto] & (x_1, \cdots, x_4, -x_5)
\end{tikzcd}$$

The corresponding system of fixed points:

$$\begin{tikzcd}
 & & \mathbb{Z}_2/1 \arrow[d, "\sigma"] & & S^4 \\
(S^4)^{(-)} & : & \mathbb{Z}_2/1 \arrow[d] & \longmapsto & S^4 \arrow[u, "\text{reverse}"] \\
 & & \mathbb{Z}_2/\mathbb{Z}_2 & & S^3 \arrow[u, hook]
\end{tikzcd}$$

We will denote this action by $G_{\mathrm{HW}}$.

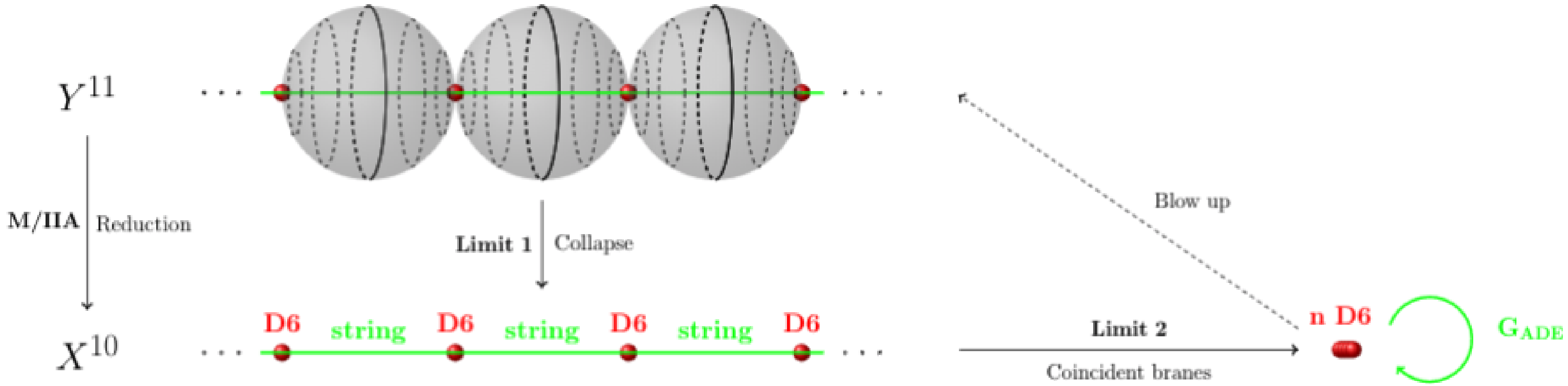
$\in \mathrm{Spaces}_{\mathbb{Q}}$

## ADE-classification of the finite subgroups of $\mathrm{SU}(2)$:

| **Dynkin label** | **Finite subgroup** $G_{\mathrm{ADE}} \subset \mathrm{SU}(2)$ | **Name of group** |
|---|---|---|
| $\mathbb{A}_n$ | $\mathbb{Z}_{n+1}$ | Cyclic |
| $\mathbb{D}_{n+4}$ | $2\mathrm{D}_{n+2}$ | Binary dihedral |
| $\mathbb{E}_6$ | $2\mathrm{T}$ | Binary tetrahedral |
| $\mathbb{E}_7$ | $2\mathrm{O}$ | Binary octahedral |
| $\mathbb{E}_8$ | $2\mathrm{I}$ | Binary icosahedral |

# Resolution of ADE-singularities

**Fact (du Val).** The blow-up of an orbifold singularity fixed by a finite subgroup of SU(2) is a system of spheres touching according to a Dynkin diagram:



Sen had suggested an interpretation in terms of M-brane physics. But the mathematical formulation of M-branes had remained an open problem.

# Equivariant enhancement?

Hence we have lifted the 4-sphere to an object in $G_{\mathrm{ADE}} \times G_{\mathrm{HW}}$-equivariant rational (super) homotopy theory.

Therefore we are now entitled to the following

**Question**: Does the M2/M5-cocyle have a corresponding equivariant enhancement?

$$\begin{tikzcd}
\mathbb{R}^{10,1|\mathbf{32}} \arrow[r, "\mu_{M2/M5}"] & S^4 & \in \mathrm{Ho}\left(G_{\mathrm{ADE,HW}}\mathrm{SuperSpaces}_{\mathbb{R}}\right) \arrow[d, "\text{forget equivariance}"] \\
\mathbb{R}^{10,1|\mathbf{32}} \arrow[r, "\mu_{M2/M5}"] & S^4 & \in \mathrm{Ho}\left(\mathrm{SuperSpaces}_{\mathbb{R}}\right)
\end{tikzcd}$$

First, this requires identifying $G_{\mathrm{ADE}} \times G_{\mathrm{HW}}$-actions on $\mathbb{R}^{10,1|\mathbf{32}}$ ...

# Theorem: ADE-Singularities in 11d super spacetime.

| **Group action on $\mathbb{R}^{10,1\vert\mathbf{32}}$** | **Possible singular locus** | | | | | **BPS** |
|---|---|---|---|---|---|---|
| type H | $\mathrm{NS1}_{\mathrm{H}}$ | M2 | $1/2\,\mathrm{M5}_{\mathrm{H}}$ | MKK6 | $\mathrm{M9}_{\mathrm{H}}$ | |
| type I | E1 | | $1/2\,\mathrm{M5}_{\mathrm{I}}$ | | $\mathrm{M9}_{\mathrm{I}}$ | |
| $G_{\mathrm{ADE}}$ | | | | $\mathbb{R}^{6,1\vert\mathbf{16}}$ | | $1/2$ |
| $G_{\mathrm{ADE}}$ | | $\mathbb{R}^{2,1\vert\geq 8\cdot\mathbf{2}}$ | | | | $\geq 1/2$ |
| $G_{\mathrm{HW}} = \mathbb{Z}_2$ | | | | | $\mathbb{R}^{9,1\vert\mathbf{16}}$ | $1/2$ |
| $G_{\mathrm{ADE,HW}} = \mathbb{Z}_2$ | $\mathbb{R}^{1,1\vert 16\cdot\mathbf{1}}$ | | $\mathbb{R}^{5,1\vert\mathbf{8}+\mathbf{8}}$ | | | $1/2$ |
| $G_{\mathrm{ADE}} \times G_{\mathrm{HW}}$ | $\mathbb{R}^{1,1\vert 8\cdot\mathbf{1}}$ | | $\mathbb{R}^{5,1\vert\mathbf{8}}$ | | | $1/4$ |

**Theorem:** There are two $G_{\mathrm{ADE,HW}}$-equivariant enhancements of the M2/M5-brane cocycle, exhibiting…

1) …the $1/2$-BPS black $1/2\mathbf{M5}$ of $D = 5+1$ and $N = (2,0)$
2) …the $1/2$-BPS black $\mathbf{NS1}_{\mathrm{H}}$ of $D = 1+1$ and $N = (16,0)$

$$\begin{tikzcd}
\overbrace{G_{\mathrm{ADE,HW}}/1}^{\text{shape of ADE-singularity}} \arrow[d, "g"] \\
G_{\mathrm{ADE,HW}}/1 \arrow[d] \\
G_{\mathrm{ADE,HW}}/G_{\mathrm{ADE,HW}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\overbrace{\mathbb{R}^{10,1|\mathbf{32}}}^{\text{11d super spacetime}} \arrow[rrr, "\overbrace{\mu_{M2/M5}}^{\text{WZW-terms of fundamental M2/M5-brane}}"] & & & S^4 \\
\mathbb{R}^{10,1|\mathbf{16}} \arrow[u, "\rho(g)"] \arrow[rr, "\mu_{M2/M5}"] \arrow[drr, Rightarrow, "\mathrm{vol}_{p+1}", "\text{NG-Lagrangian}"'] & & S^4 \arrow[r, equal] & S^4 \arrow[u, "\text{reverse}"] \\
\underbrace{\mathbb{R}^{p,1|16}}_{\text{black brane at ADE-singularity}} \arrow[u, hook] \arrow[rr] & & \ast \arrow[u, hook] \arrow[r] & S^0 \arrow[u, hook]
\end{tikzcd}$$

$\in \mathrm{Ho}\left(\mathrm{PSh}(\mathrm{Orb}_{G_{\mathrm{ADE,HW}}}, \mathrm{SuperSpaces}_{\mathbb{R}})\right)$

**Theorem:** There are two $G_{\mathrm{ADE}}$-equivariant enhancements of the M2/M5-brane cocycle restricted to $\mathrm{M9}_H$, exhibiting…

1) …the $1/4$-BPS black $1/2$**M5** of $D = 5+1$ and $N = (1,0)$
2) …the $1/4$-BPS black $\mathbf{NS1}_{\mathrm{H}}$ of $D = 1+1$ and $N = (8,0)$

$$\begin{tikzcd}
\overbrace{G_{\mathrm{ADE}}/1}^{\text{shape of ADE-singularity}} \arrow[d, "g"] \\
G_{\mathrm{ADE}}/1 \arrow[d] \\
G_{\mathrm{ADE}}/G_{\mathrm{ADE}}
\end{tikzcd}$$

$$\begin{tikzcd}
\overbrace{\mathbb{R}^{9,1|\mathbf{16}}}^{\text{10d super spacetime}} \arrow[rr, "\overbrace{\mu_{M2/M5}\vert_{\mathrm{M9}}}^{\text{WZW-terms of fundamental M2/M5-brane}}"] & & S^4 \\
\mathbb{R}^{9,1|\mathbf{32}} \arrow[u, "\rho(g)"] \arrow[r, "\mu_{M2/M5}\vert_{\mathrm{M9}}"] \arrow[dr, Rightarrow, "\mathrm{vol}_{p+1}", "\text{NG-Lagrangian}"'] & S^4 \arrow[r, equal] & S^4 \arrow[u, "\mathrm{id}"'] \\
\underbrace{\mathbb{R}^{p,1|16}}_{\text{black } p \text{ brane at ADE-singularity}} \arrow[u, hook] \arrow[r] & \ast \arrow[u, hook] \arrow[r] & S^0 \arrow[u, hook]
\end{tikzcd}$$

---

$\in \mathrm{Ho}\left(\mathrm{PSh}(\mathrm{Orb}_{G_{\mathrm{ADE}}}, \mathrm{SuperSpaces}_{\mathbb{R}})\right)$

## **Conclusion**:

A fair bit of
the expected structure of M-theory
emerges out of the superpoint $\mathbb{R}^{0|1}$
in rational equivariant super homotopy theory.

## **Evident Conjecture**:

The full theory emerges
once passing beyond the rational approximation
in full super-geometric homotopy theory.
(arXiv:1310.7930).

see also ncatlab.org/schreiber/show/StringMath2017

# Epilogue

In full super-geometric homotopy theory
the superpoint $\mathbb{R}^{0|1}$ itself
emerges from $\varnothing$

$$
\begin{array}{ccccccccc}
\mathrm{id} & \dashv & \mathrm{id} & & & & & & \\
\vee & & \vee & & & & & & \\
\rightrightarrows & \dashv & \rightsquigarrow & \dashv & \boxed{\mathbb{R}^{0|1}} & & & & \\
 & & \vee & & \vee & & & & \\
 & & \Re & \dashv & \boxed{\mathbb{D}} & \dashv & \text{Ét} & & \\
 & & & & \vee & & \vee & & \\
 & & & & \boxed{\mathbb{R}} & \dashv & \flat & \dashv & \sharp \\
 & & & & & & \vee & & \vee \\
 & & & & & & \varnothing & \dashv & *
\end{array}
$$

(Schreiber 16, FOMUS proceedings)