Equivariant Stable Cohomotopy and Branes

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talks at

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Part I

joint work with H. Sati

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[FSS13, FSS16a, FSS16b, HSS18, BSS18]

https://ncatlab.org/schreiber/show/Equivariant+Stable+Cohomotopy+and+Branes

What is... a brane?

Folklore offers patchy answers:

aspect of branes	mathematical formalization
1. Macroscopically: *solitonic* or *black* branes	singular solution to higher-dim. **super-gravity** equations
2. Microscopically: *fundamental* or *probe* branes	Green-Schwarz-type **sigma-model** on super-gravity background
3. Via brane charge: *bound states* or *reaction products* of brane/anti-brane creation/annihilation	cocycle in (twisted, equivariant, differential, ...) **generalized cohomology**
specifically for D-branes (only) there is one more:	
4. String theoretically: *endpoint* loci for open strings	boundary state in 2d SCFT

Open problem: *Unify this to one coherent theory of branes.*

Branes in point-particle physics:
(classical)

What is... an electric/magnetic monopole?

1. **Macroscopically**: extremally charged black hole:

$$\underbrace{(X,g) = \mathrm{AdS}_2 \times S^2}_{\substack{\text{metric} \\ \updownarrow \\ \text{field of gravity}}}, \qquad \underbrace{F = \mathrm{dvol}_{S^2}}_{\substack{\text{2-form} \\ \updownarrow \\ \text{electromagnetic field}}}$$

2. **Microscopically**: particle trajectory

$$\Sigma_1 \xrightarrow{\;\phi\;} X \xrightarrow{\;F\;} K(\mathbb{R},2) \qquad \text{subject to Lorentz force} \quad \underset{\substack{\text{tangent} \\ \text{vector}}}{\iota_v} F$$

3. **Via charge**: Dirac charge quantization

magnetic charge/first Chern class: $X \xrightarrow{c_1} \overset{\text{Eilenberg-MacLane spectrum}}{\Sigma^2 H\mathbb{Z}}$

$$\begin{array}{ccccc} c_1 & \longmapsto & (c_1)_{\mathbb{R}} \simeq [F] & \longleftarrow & F \\ H^2(X,\mathbb{Z}) & \xrightarrow[\text{realification}]{} & H^2(X,\mathbb{R}) & \xleftarrow[\substack{\text{de Rham} \\ \text{theorem}}]{} & \Omega^2(X) \end{array}$$

Branes in String Theory:

(plenty of folklore, few hard results, dust has not settled, rather a dust storm)

What is... a D-brane?

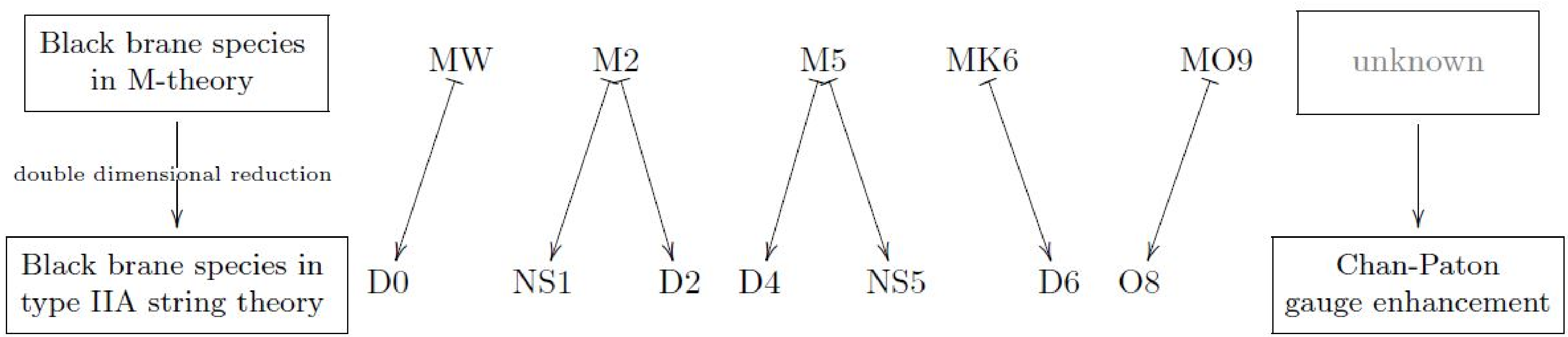
1. **Macroscopically**:
   BPS black D-brane $(X, g) \underset{\text{conf}}{\sim} \mathrm{AdS}_{p+2} \times S^{10-p-2}, \qquad F_{p+2} = \mathrm{dvol}_{S^{10-p+2}}$

2. **Microscopically**:
   GS sigma-model D-brane $\Sigma_{p+1} \xrightarrow{\ \phi\ } X \xrightarrow{e^{\mathcal{F}} \wedge (C_2 + C_4 + C_6 + \cdots)} \prod_k K(\mathbb{R}, 2k)$

3. **Via brane charge**
   brane/anti-brane reaction products: $X \longrightarrow \mathrm{KU}$ (K-theory spectrum) "tachyon condensation"

Unification$_{/\mathbb{Q}}$: [BSS18], presented in **V. Braunack-Mayer's talk**

Punchline: D-branes$_{/\mathbb{Q}}$ do derive from M-branes $\sim$ as suggested by folklore:



Therefore here we press on to discussion of M-branes $\longrightarrow$

Branes in M-theory

(some folklore, but remains key open problem of string theory, systematic attack in [FSS13, FSS16a, FSS16b, HSS18, BSS18])

1. **Macroscopically**:

   black BPS M-brane

$$(X, g) = \underbrace{\mathrm{AdS}_{p+2} \times S^{11-p-2}/G_{\mathrm{ADE}}}_{\text{spherical space form}} \xleftarrow{\ \ell_P \gg 1\ } \cdots \xrightarrow{\ \ell_P \ll 1\ } \underbrace{\mathbb{R}^{p,1} \times \mathbb{R}^{11-p-1} /\!\!/ G_{\mathrm{ADE}}}_{\substack{\text{orbifold} \\ \text{du Val singularity}}}$$

2. **Microscopically**

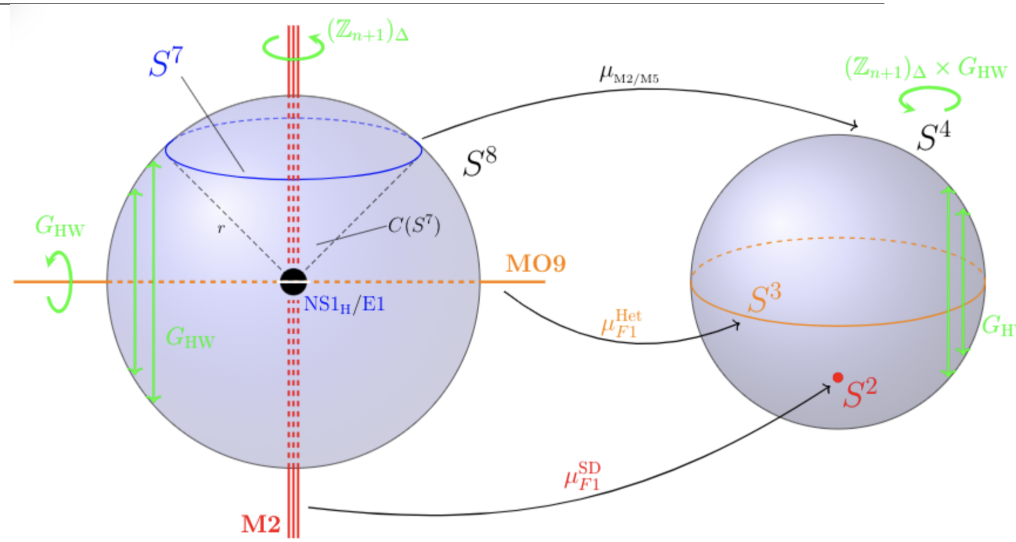
   GS sigma-model M-brane $\Sigma_{p+1} \xrightarrow{\ \phi\ } X \xrightarrow{\ \mu_{M2/M5}\ } S^4_{\mathbb{R}}$ ([FSS16a])

Unification$_{/\mathbb{Q}}$: [HSS18]

via equivariant super cohomotopy

$$\overset{G_{\mathrm{ADE}}}{\circlearrowright}\ \mathbb{R}^{10,1|\mathbf{32}} \xrightarrow{\ \mu_{M2/M5}\ } \overset{G_{\mathrm{ADE}}}{\circlearrowright}\ S^4_{\mathbb{R}}$$

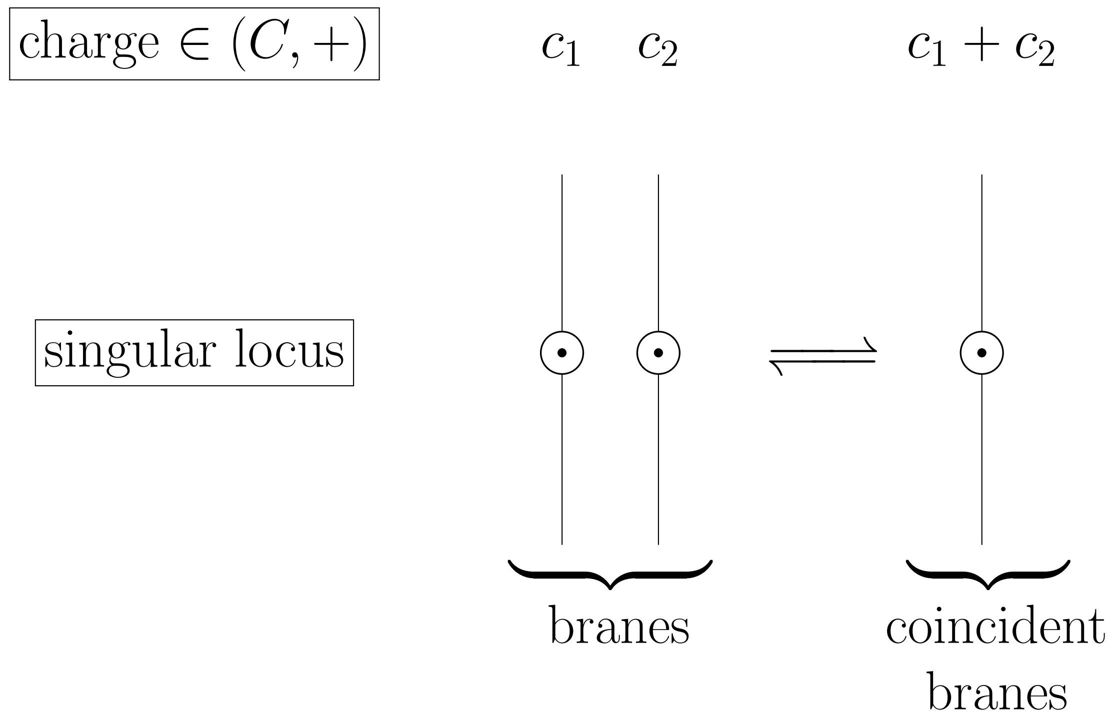
presented in **J. Huerta's talk**



3. **Via brane charge:** $X \longrightarrow ??$ (some spectrum) remaining open problem – discussed now$\longrightarrow$

Brane charge – 1st order approximation

Let $(C, +)$ be an abelian semigroup / a "commutative monoid" of charges.



Fundamental example: the natural numbers

$$(C, +) \;=\; (\mathbb{N}, +)$$

charge c = number of coincident branes $\Leftrightarrow$ each brane carries unit charge

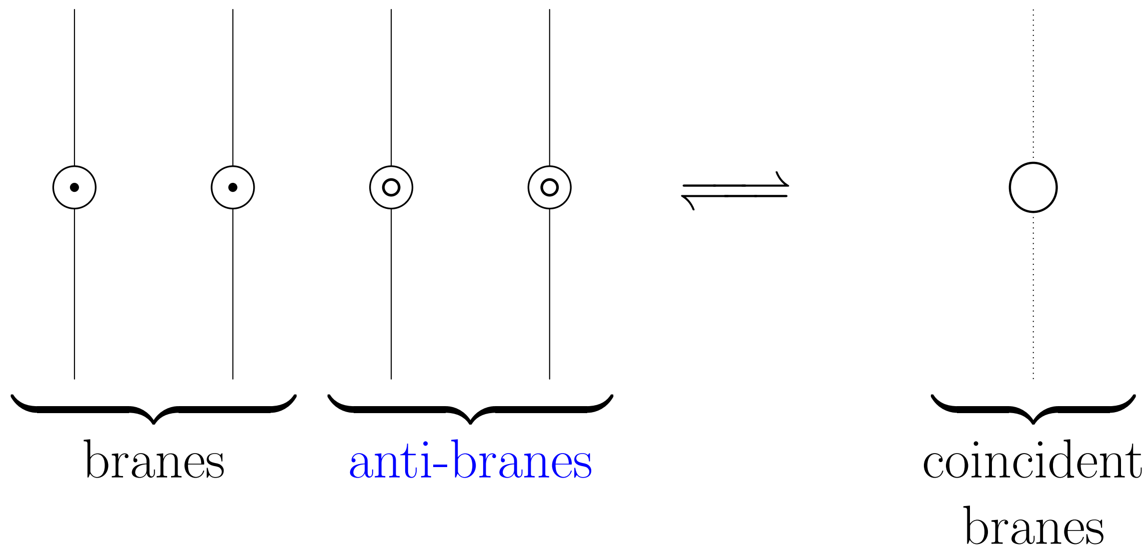
Brane charge – 2nd order approximation

Including anti-brane charges, hence negative brane charges, means to pass to the abelian group completion of the charge monoid:

$$K(C,+) \;\coloneqq\; \Big\{ (c^+, c^-) \mid c^\pm \in C \Big\} \Big/ \Big((c,c) \sim 0 \Big)$$

charge $\in K(C,+)$ $\quad (c_1, 0)\;\; (c_2, 0)\;\; (0, c_2)\;\; (0, c_1) \qquad (c_1 + c_2, c_1 + c_2) \sim 0$

singular locus



Fundamental example: the integers: $\quad K(\mathbb{N},+) \;=\; (\mathbb{Z},+)$

$(c^+, c^-) =$ number of coincident branes minus number of anti-branes $\;\Leftrightarrow\;$ each brane carries unit charge, each anti-brane carries negative unit charge

Brane charge – 3rd order approximation

The categorification of *commutative monoid* $(C,+)$ is *symmetric monoidal category* $(\mathcal{C},\oplus)$

$$(C,+) \;=\; \pi_0\,(\mathcal{C},\oplus)$$

- **Fundamental non-linear example**

  Finite pointed sets with disjoint unition $(\mathcal{C},\oplus) \;=\; \left(\mathrm{Set}^{*/}_{\mathrm{fin}}, \sqcup\right)$

  this categorifies the previous example: $\pi_0\left(\mathrm{Set}^{*/}_{\mathrm{fin}}, \sqcup\right) \;\simeq\; (\mathbb{N},+)$

- **Fundamental linear example** for $\mathbb{F}$ a field

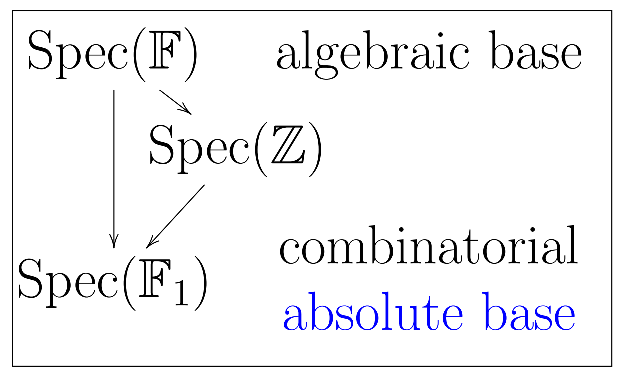
  finite-dim vector spaces with direct sum $(\mathcal{C},\oplus) = \left(\mathbb{F}\mathrm{Vect}_{\mathrm{fin}}, \oplus\right)$

  this *also* categorifies the previous example: $\pi_0\left(\mathbb{F}\mathrm{Vect}_{\mathrm{fin}}, \oplus\right) \;\simeq\; (\mathbb{N},+)$

Unified on a deeper level:

pointed sets may be regarded as the vector spaces over the "absolute ground-field with one element" $\mathbb{F}_1$

$$\left(\mathrm{Set}^{*/}_{\mathrm{fin}}, \sqcup\right) \;\simeq\; \left(\mathbb{F}_1\mathrm{Vect}_{\mathrm{fin}}, \oplus\right)$$



Brane charge in generalized cohomology

Brane/anti-brane annihilation may be varying over spacetime X

$\rightsquigarrow$ enhance discrete abelian group of charges to a *space* of charges

The homotopification of *abelian group* $(A,+)$ is *∞-loop space / spectrum* $\mathcal{A}$

$$(A,+) = \pi_0(\mathcal{A})$$

brane charge

locally constant	**locally varying**
$X \longrightarrow \underbrace{(A,+)}_{\text{discrete abelian group}}$	$X \longrightarrow \underbrace{\mathcal{A}}_{\infty\text{-loop space or spectrum}}$

Hence brane charge group on spacetime X is generalized cohomology group:

$$\mathcal{A}(X) \coloneqq \pi_0 \mathrm{Maps}(X, \mathcal{A})$$

Example: D-brane/anti-D-brane bound states

open string tachyon condensation profile:

$$X \longrightarrow \mathrm{KU}$$ (conjecturally, or similar)

↖ K-theory spectrum

in conclusion:

brane/anti-brane annihilation ⇝ abelian group completion of charge monoid	locally varying brane charge ⇝ spectral enhancement of charge group

combine:

The categorification / homotopification of *abelian group completion* $K\Big(\pi_0\big(\mathcal{C}, \oplus\big)\Big)$ is *algebraic K-theory spectrum* $\mathbb{K}\big(\mathcal{C}, \oplus\big) \coloneqq \Omega B_{\oplus} \mathcal{C}$

$$K\Big(\pi_0\big(\mathcal{C}, \oplus\big)\Big) \;=\; \pi_0\Big(\mathbb{K}\big(\mathcal{C}, \oplus\big)\Big)$$

Algebraic K-Theory – Examples

- **algebraic K-theory spectrum** of a field $\mathbb{F}$

$$K\mathbb{F} \;\simeq\; \mathbb{K}\big(\mathbb{F}\mathrm{Vect}_{\mathrm{fin}}\big)$$

- **complex algebraic K-theory** $K\mathbb{C}$

  is *flat* K-theory:

$$\underbrace{K\mathbb{C}}_{\substack{\text{algebraic} \\ \text{K-theory}}} \xrightarrow[\text{flat cocycles}]{\text{include}} \overbrace{\widehat{\mathrm{ku}}}^{\substack{\text{differential} \\ \text{K-theory}}} \xrightarrow[\text{topological class}]{\text{project out}} \mathrm{KU}$$

$$K\mathbb{C} \longrightarrow \mathrm{KU}$$

- **absolute algebraic K-theory** $K\mathbb{F}_1 \;\coloneqq\; \mathbb{K}\big(\mathbb{F}_1\mathrm{Vect}_{\mathrm{fin}}\big) \;\simeq\; \mathbb{K}\big(\mathrm{Set}^{*/}_{\mathrm{fin}}\big)$

  is stable Cohomotopy theory (Barrat-Priddy-Quillen theorem):

$$\underbrace{K\mathbb{F}_1}_{\substack{\text{absolute algebraic} \\ \text{K-theory spectrum}}} \;\simeq\; \underbrace{\mathbb{S}}_{\substack{\text{sphere} \\ \text{spectrum}}}$$

A *representation sphere* $\overset{G}{\curvearrowright} S^V$:= one-point compactification of linear representation $\overset{G}{\curvearrowright} V$

A G-equivariant spectrum $\mathcal{A}$ is
a spectrum of G-spaces indexed by representation spheres, hence

1. a system of pointed G-spaces $\left\{ \overset{G}{\curvearrowright} \mathcal{A}_V \;\middle|\; \overset{G}{\curvearrowright} V \text{ a linear } G\text{-representation} \right\}$
2. with equivariant suspension morphisms $S^V \wedge \mathcal{A}_W \xrightarrow{\sigma_{V,W}} \mathcal{A}_{V \oplus W}$

Examples

- The *equivariant suspension spectrum* of a G-space $\overset{G}{\curvearrowright} S^V$ is

$$\Sigma^\infty_G X \;:\; V \;\mapsto\; S^V \wedge X\,.$$

- The equivariant K-theory of a contractible space is the representation ring

$$\mathrm{KU}_G(\overset{G}{\curvearrowright}\mathbb{R}^{d,1}) \;\simeq\; \mathrm{KU}_G(\overset{G}{\curvearrowright} *) \;\simeq\; R_{\mathbb{C}}(G) \;\simeq\; \mathbb{Z}[\overbrace{\rho_1, \cdots, \rho_n}^{\text{irreps}}]$$ "fractional D-branes"

Summing it all up:

A compelling candidate for M-brane charge cohomology theory is...

by above discussion	non-linear brane/anti-brane bound states	fundamental & black M-branes	by [HSS18] (Huerta's talk)
absolute K-theory	$K\mathbb{F}_1$	$G_{\mathrm{ADE}} \curvearrowright S^4$	ADE-equivariant Cohomotopy
	$\wr\vert$	$\wr\vert$	
stable Cohomotopy	$\mathbb{S}$	$S^{\mathbb{H}}$	in RO-degree $\mathbb{H}$ (↖ quaternions)

$\mathbb{S} \xrightarrow{\text{equivariantize}} \mathbb{S}^{\mathbb{H}}_{G_{\mathrm{ADE}}}$; $S^{\mathbb{H}} \xrightarrow{\text{stabilize}} \Sigma^{\infty}_{G_{\mathrm{ADE}}} S^{\mathbb{H}}$

$$\mathbb{S}^{\mathbb{H}}_{G_{\mathrm{ADE}}} \simeq \Sigma^{\infty}_{G_{\mathrm{ADE}}} S^{\mathbb{H}}$$

a

ADE-equivariant stable Cohomotopy in RO-degree $\mathbb{H}$

Hypothesis H:

The **generalized cohomology theory** for **M-brane charge** is **ADE-equivariant stable Cohomotopy in RO-degree $\mathbb{H}$**

Hypothesis **H** predicts M-brane charge groups: $\mathbb{S}^{\mathbb{H}}_{G_{\mathrm{ADE}}}\Big(\underbrace{\overset{G_{\mathrm{ADE}}\ \curvearrowright}{X}}_{\substack{\text{11d spacetime}\\ \text{orbifold}}} \Big)$

How does this compare to / clarify folklore of perturbative string theory:

- intersecting MK6-branes $\rightsquigarrow$ fractional D-branes ?
- M-theoretic "discrete torsion"?
- GUT at E-type singularities ?
- $\cdots$

This is discussed in **part II**.

References

[FSS13] D. Fiorenza, H. Sati, U. Schreiber,
Super Lie n-algebras, higher WZW models, and super p-branes,
Intern. J. Geom. Meth. Mod. Phys. **12** (2015) 1550018 , [arXiv:1308.5264]

[FSS16a] D. Fiorenza, H. Sati, U. Schreiber,
Sphere valued supercocycles in M-theory and type IIA string theory,
J. Geom. Phys. **114** (2017) 91-108, [arXiv:1606.03206].

[FSS16b] D. Fiorenza, H. Sati, U. Schreiber,
T-Duality from super Lie n-algebra cocycles for super p-branes,
ATMP Volume 22 (2018) Number 5, [arXiv:1611.06536].

[HSS18] J. Huerta, H. Sati, U. Schreiber,
Real ADE-equivariant (co)homotpy of super M-branes,
[arXiv:1805.05987].

[BSS18] V. Braunack-Mayer, H. Sati, U. Schreiber
Gauge enhancement via Parameterized stable homotopy theory,
[arXiv:1806.01115]