KAROUBI'S RELATIVE CHERN CHARACTER AND BEILINSON'S REGULATOR

by

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Abstract. — We construct a variant of Karoubi's relative Chern character for smooth varieties over **C** and prove a comparison result with Beilinson's regulator with values in Deligne-Beilinson cohomology. As a corollary we obtain a new proof of Burgos' Theorem that for number fields Borel's regulator is twice Beilinson's regulator.

***Résumé* (Le caractère de Chern relatif de Karoubi et le régulateur de Beilinson)**

Nous construisons une variante du caractère de Chern relatif de Karoubi pour les variétés lisses sur **C** et prouvons un résultat de comparaison avec le régulateur de Beilinson à valeurs dans la cohomologie de Deligne-Beilinson. En corollaire nous obtenons une nouvelle preuve du théorème de Burgos, que, pour un corps de nombres, le régulateur de Beilinson est deux fois le régulateur de Borel.

Contents

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Introduction

In a series of papers [**25, 26, 27, 29**] Karoubi introduced relative K-theory for Banach algebras A (the homotopy fibre of the map from algebraic to topological K-theory) and constructed the relative Chern character

$$\mathrm{Ch}_i^{\mathrm{rel}}\colon K_i^{\mathrm{rel}}(A) \to HC_{i-1}^{\mathrm{cont}}(A)$$

mapping relative K-theory to continuous cyclic homology. He also mentioned a geometric version of his relative Chern character and the possible connection with regulators.

The objective of this paper is to make these relations precise in the case of smooth affine varieties X over $\mathbf{C}$. In this situation the cyclic homology decomposes into a product of cohomology groups of the truncated de Rham complex and the relative Chern character becomes a morphism

$$\mathrm{Ch}_{n,i}^{\mathrm{rel}}\colon K_i^{\mathrm{rel}}(X) \to \mathbb{H}^{2n-i-1}(X, \Omega_X^{<n}).$$

We may formulate our main result as follows.

Theorem. — *Karoubi's relative Chern character factors naturally through a morphism*

$$K_i^{\mathrm{rel}}(X) \to H^{2n-i-1}(X, \mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X, \mathbf{C})$$

which we denote by the same symbol. The diagram

$$\begin{tikzcd}
K_i^{\mathrm{rel}}(X) \arrow[r] \arrow[d, "\mathrm{Ch}_{n,i}^{\mathrm{rel}}"] & K_i(X) \arrow[d, "\mathrm{Ch}_{n,i}^{\mathscr{D}}"] \\
H^{2n-i-1}(X, \mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X, \mathbf{C}) \arrow[r] & H_{\mathscr{D}}^{2n-i}(X, \mathbf{Q}(n)),
\end{tikzcd}$$

where $\mathrm{Ch}_{n,i}^{\mathscr{D}}$ *is Beilinson's Chern character with values in Deligne-Beilinson cohomology, commutes.*

As an application we give a new proof for the comparison of Borel's and Beilinson's regulators – the case $X = \mathrm{Spec}(\mathbf{C})$ –:

***Corollary* (Burgos' Theorem [6])**. — *Borel's regulator*

$$K_{2n-1}(\mathbf{C}) \to \mathbf{R}(n-1)$$

is twice Beilinson's regulator.

This result plays an important role in the study of special values of L-functions: Borel [**4**] established a precise relation between his regulator and special values of zeta functions of number fields. In [**1**] Beilinson formulated far reaching conjectures describing special values of L-functions of motives up to non-zero rational factors in terms of his regulator. He also proved that for a number field his regulator coincides with Borel's up to a non-zero rational factor (see also Rapoport's report [**32**]). This enabled him to view Borel's computations as a confirmation of his conjectures in the case of a number field.

However, in order to exploit Borel's result further and remove the $\mathbf{Q}^\times$-ambiguity it is important to have a precise comparison result for the regulators as provided by Burgos' Theorem.

In [**10**] Dupont, Hain, and Zucker proposed a strategy for the comparison of both regulators based on the comparison of Cheeger-Simons' and Beilinson's Chern character classes. While there remained some difficulties in carrying this out, they were led to the conjecture that the precise factor would be 2. This was then proven by Burgos using Beilinson's original argument.

So far Karoubi's relative Chern character has not much been studied in Arithmetics. One of its possible advantages is that it is defined in complete analogy in the classical, p-adic, and even non commutative situation and thus gives a unifying frame for the study of regulators in these different contexts. In the p-adic setting analogues of the results presented here have been obtained in [**37, 38**].

Karoubi's principal idea is to describe relative K-theory in terms of bundles with discrete structure group on certain simplicial sets together with a trivialization of the associated topological bundle. The relative Chern character is induced by secondary classes for these bundles constructed by means of Chern-Weil theory.

The first problem one encounters when trying to compare these classes with Beilinson's is that they live in the cohomology of the truncated de Rham complex which does not map naturally to Deligne-Beilinson cohomology. It is therefore necessary to construct refinements of these classes which are then to be compared with the corresponding classes in Deligne-Beilinson cohomology.

Our approach to this is to generalize Karoubi's formalism to simplicial manifolds and systematically use what we call *topological morphisms* and *bundles*. The Chern-Weil theoretic construction of secondary classes in this setup is described in Section 1. In the second Section we make essential use of the notion topological morphisms in order to construct the abovementioned refinements (Proposition 2.10) and compare them with Beilinson's classes (Theorem 2.11).

Our construction of the relative Chern character on K-theory is presented in Section 3. It differs slightly from Karoubi's original one. The comparison with Beilinson's regulator then follows formally from the results of the second Section.

The Corollary is finally proven in Section 4. By the Theorem it reduces to a comparison of Karoubi's relative Chern character for $X = \mathrm{Spec}(\mathbf{C})$ and Borel's regulator. A similar result has been obtained previously by Hamida [**22**]. She constructs an explicit map $K_{2n-1}(\mathbf{C}) \to K^{\mathrm{rel}}_{2n-1}(\mathbf{C})$ and composes it with the relative Chern character to obtain a map defined on the K-theory of $\mathbf{C}$ rather than the relative K-theory. This is then compared with Borel's regulator.

Related to ours is the work of Soulé [**35**] who showed that Beilinson's Chern character $\mathrm{Ch}^{\mathscr{D}}_{n,i}\colon K_i(X) \to H^{2n-i}_{\mathscr{D}}(X, \mathbf{Q}(n))$ factors through the *multiplicative K-theory* of X which may also be described in terms of bundles and connections on certain simplicial algebraic varieties. However, the relation with the relative Chern character or Borel's regulator is not treated in that paper.

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Notation. — For a complex manifold Y the sheaves of holomorphic functions, holomorphic n-forms, and smooth n-forms are denoted by $\mathscr{O}_Y, \Omega^n_Y$, and $\mathscr{A}^n_Y$, respectively. Global sections are denoted by $\Omega^n(Y)$, etc.

The ordered set $\{0, \dots, p\}$ is denoted by $[p]$. Simplicial objects are usually marked with a bullet like $X_\bullet$. The i^{th} face and degeneracy of a simplicial object are denoted by ∂_i and s_i, respectively. The i^{th} coface of a cosimplicial object is denoted by δ^i. The geometric realization of a simplicial set $S_\bullet$ is denoted by $|S_\bullet|$.

If $f\colon A^\bullet \to B^\bullet$ is a morphism of cochain complexes, we define $\mathrm{Cone}(f)$ to be the complex given in degree n by $A^{n+1} \oplus B^n$ with $d(a,b) = (-da, db - f(a))$. The complex $A[-1]$ is given in degree n by A^{n-1} with differential $-d$.

1. Karoubi's secondary classes

1.1. De Rham cohomology of simplicial complex manifolds. — Here we recall Dupont's computation of the de Rham cohomology of simplicial manifolds [**11**] in the setting of complex manifolds. This is fundamental for simplicial Chern-Weil theory in the following Sections.

Let $X_\bullet$ be a simplicial complex manifold and denote by $\Omega^{\geq r}_{X_\bullet}$ for $r \geq 0$ the naively truncated de Rham complex of sheaves of holomorphic forms, i.e. the r^{th} step of the bête filtration. Then we have

$$\mathbb{H}^*(X_\bullet, \Omega^{\geq r}_{X_\bullet}) = H^*(\operatorname{Tot}\operatorname{Fil}^r \mathscr{A}^*(X_\bullet)),$$

where $\operatorname{Tot}\operatorname{Fil}^r\mathscr{A}^*(X_\bullet)$ is the total complex associated with the cosimplicial complex $[p] \mapsto \operatorname{Fil}^r\mathscr{A}^*(X_p) = \bigoplus_{k+l=*, k\geq r} \mathscr{A}^*(X_p)$ (cf. [**9**, (5.2.7)]). For the purpose of simplicial Chern-Weil theory we need another version of the simplicial de Rham complex. Let

$$\Delta^p := \left\{(x_0, \dots, x_p) \in \mathbf{R}^{p+1} \;\middle|\; x_i \geq 0, \sum\nolimits_{i=0}^p x_i = 1\right\} \subseteq \mathbf{R}^{p+1}$$

denote the standard simplex. Then $[p] \mapsto \Delta^p$ is a cosimplicial space in a natural way. A function or form on Δ^p is called smooth, if it extends to a smooth function resp. form on a neighbourhood of Δ^p in $\{\sum x_i = 1\} \subseteq \mathbf{R}^{p+1}$. We recall from [**11**]:

Definition 1.1. — A smooth simplicial n-form on a simplicial complex manifold $X_\bullet$ is a family $\omega = (\omega_p)_{p\geq 0}$, where ω_p is a smooth n-form on $\Delta^p \times X_p$, and the compatibility condition

$$(\delta^i \times 1)^* \omega_p = (1 \times \partial_i)^* \omega_{p-1} \quad \text{ on } \quad \Delta^{p-1} \times X_p$$

$i = 0, \dots, p$, $p \geq 0$, is satisfied. The space of smooth simplicial n-forms on $X_\bullet$ is denoted by $A^n(X_\bullet)$.

The exterior derivative d and the usual wedge product applied component-wise make $A^*(X_\bullet)$ into a commutative differential graded $\mathbf{C}$-algebra.

Moreover, $A^*(X_\bullet)$ is naturally the total complex associated with the triple complex $(A^{k,l,m}(X_\bullet), d_\Delta, \partial_X, \bar{\partial}_X)$ where $A^{k,l,m}(X_\bullet)$ consists of the forms ω of type (k,l,m), that is, each ω_p is locally of the form $\sum_{I,J,K} f_{I,J,K} dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge d\zeta_{j_1} \wedge \dots \wedge d\zeta_{j_l} \wedge d\bar{\zeta}_{k_1} \wedge \dots \wedge d\bar{\zeta}_{k_m}$, where $x_0, \dots, x_p$ are the barycentric coordinates on Δ^p and the ζ_j are *holomorphic* coordinates on X_p. $d_\Delta, \partial_X, \bar{\partial}_X$ denote the exterior derivative in Δ- and the Dolbeault-derivations in X-direction, respectively. Write $\mathrm{Fil}^r A^*(X_\bullet) = \bigoplus_{k+l+m=*, l\geq r} A^{k,l,m}(X_\bullet)$.

On the other hand we have the triple complex $(\mathscr{A}^{k,l,m}(X_\bullet), \delta, \partial_X, \bar{\partial}_X)$, where $\mathscr{A}^{k,l,m}(X_\bullet) = \mathscr{A}^{l,m}(X_k)$ and $\delta = \sum_{i=0}^k (-1)^i \partial_i^* : \mathscr{A}^{k,l,m}(X_\bullet) \to \mathscr{A}^{k+1,l,m}(X_\bullet)$.

Theorem 1.2 **(Dupont)**. — *Let $X_\bullet$ be a simplicial complex manifold. For each $l, m \geq 0$ the two complexes $(A^{*,l,m}(X_\bullet), d_\Delta)$ and $(\mathscr{A}^{*,l,m}(X_\bullet), \delta)$ are naturally chain homotopy equivalent. The equivalence is given by integration over the standard simplex:*

$$I\colon A^{k,l,m}(X_\bullet) \to \mathscr{A}^{k,l,m}(X_\bullet), \quad \omega = (\omega_p)_{p\geq 0} \mapsto \int_{\Delta^k} \omega_k.$$

In particular, we get natural isomorphisms

$$\mathbb{H}^*(X_\bullet, \Omega^{\geq r}_{X_\bullet}) \cong H^*(\mathrm{Tot}\,\mathrm{Fil}^r \mathscr{A}^*(X_\bullet)) \cong H^*(\mathrm{Fil}^r A^*(X_\bullet)).$$

Proof. — This is essentially [**11**, Theorem 2.3]. One only has to check that the integration I, the homotopy inverse, and the homotopies constructed by Dupont in the proof of [**11**, Theorem 2.3] respect the (l,m)-type. This is left to the reader (or see [**36**, Theorem 1.3]). □

1.2. Bundles on simplicial manifolds. — This Section introduces the formalism of algebraic, holomorphic, and topological bundles on simplicial varieties. We follow Karoubi's approach [**27, 28**] describing bundles in terms of their transition functions. This is perfectly suited for computations and the construction of Chern character maps in K-theory as in Section 3.

Definition 1.3. — The *classifying simplicial manifold* for $\mathrm{GL}_r(\mathbf{C})$ is the simplicial complex manifold $B_\bullet\mathrm{GL}_r(\mathbf{C})$, where

$$B_p\mathrm{GL}_r(\mathbf{C}) = \mathrm{GL}_r(\mathbf{C}) \times \cdots \times \mathrm{GL}_r(\mathbf{C}) \quad (p \text{ factors}),$$

with faces and degeneracies

$$\partial_i(g_1,\dots,g_p) = \begin{cases} (g_2,\dots,g_p), & \text{if } i=0,\\ (g_1,\dots,g_ig_{i+1},\dots,g_p), & \text{if } 1\le i\le p-1,\\ (g_1,\dots,g_{p-1}), & \text{if } i=p,\end{cases}$$

$$s_i(g_1,\dots,g_p) = (g_1,\dots,g_i,1,g_{i+1},\dots,g_p), \qquad i=0,\dots,p.$$

The *universal principal* $\mathrm{GL}_r(\mathbf{C})$*-bundle* is the simplicial complex manifold $E_\bullet\mathrm{GL}_r(\mathbf{C})$, where

$$E_p\mathrm{GL}_r(\mathbf{C}) = \mathrm{GL}_r(\mathbf{C}) \times \cdots \times \mathrm{GL}_r(\mathbf{C}) \quad (p+1 \text{ factors}),$$

with faces and degeneracies

$$\partial_i(g_0,\dots,g_p) = (g_0,\dots,g_{i-1},g_{i+1},\dots,g_p), \qquad i=0,\dots,p, \tag{1.1}$$

$$s_i(g_0,\dots,g_p) = (g_0,\dots,g_i,g_i,\dots,g_p), \qquad i=0,\dots,p. \tag{1.2}$$

The canonical projection $p\colon E_\bullet\mathrm{GL}_r(\mathbf{C}) \to B_\bullet\mathrm{GL}_r(\mathbf{C})$ is given in degree p by

$$(g_0,\dots,g_p) \mapsto (g_0g_1^{-1},\dots,g_{p-1}g_p^{-1}).$$

Thus $B_\bullet\mathrm{GL}_r(\mathbf{C})$ is the quotient of $E_\bullet\mathrm{GL}_r(\mathbf{C})$ by the diagonal *right* action of $\mathrm{GL}_r(\mathbf{C})$. Obviously $E_\bullet\mathrm{GL}_r(\mathbf{C})$ is a simplicial group and it operates from the left on $B_\bullet\mathrm{GL}_r(\mathbf{C}) \cong E_\bullet\mathrm{GL}_r(\mathbf{C})/\mathrm{GL}_r(\mathbf{C})$. Explicitly, this action is given by

$$(g_0,\dots,g_p)\cdot(h_1,\dots,h_p) = (g_0h_1g_1^{-1},\dots,g_{p-1}h_pg_p^{-1}).$$

We define $B_\bullet G$ and $E_\bullet G$ in the same way if G is a discrete group, a group scheme, etc.

Definition 1.4. — Let $X_\bullet$ be a simplicial complex manifold. A *holomorphic* $\mathrm{GL}_r(\mathbf{C})$*-bundle* on $X_\bullet$ is a holomorphic morphism of simplicial complex manifolds

$$g\colon X_\bullet \to B_\bullet\mathrm{GL}_r(\mathbf{C}).$$

We also denote such a bundle by $E/X_\bullet$ and call g the *classifying map of* E. The *universal* $\mathrm{GL}_r(\mathbf{C})$*-bundle* E^{univ} is the bundle given by $\mathrm{id}\colon B_\bullet\mathrm{GL}_r(\mathbf{C}) \to B_\bullet\mathrm{GL}_r(\mathbf{C})$.

A *morphism* $\alpha\colon g \to h$ of $\mathrm{GL}_r(\mathbf{C})$-bundles on $X_\bullet$ is a morphism of simplicial complex manifolds $\alpha\colon X_\bullet \to E_\bullet\mathrm{GL}_r(\mathbf{C})$, such that $\alpha\cdot g = h$ with respect to the abovementioned action. Every morphism is an isomorphism.

Note that $B_\bullet\mathrm{GL}_r(\mathbf{C})$ may also be viewed as the $\mathbf{C}$-valued points of a simplicial $\mathbf{C}$-scheme which, by abuse of notation, will be denoted by the same symbol. We define an *algebraic* $\mathrm{GL}_r(\mathbf{C})$*-bundle* on a simplicial $\mathbf{C}$*-scheme* $X_\bullet$ to be a morphism $g\colon X_\bullet \to B_\bullet\mathrm{GL}_r(\mathbf{C})$ of simplicial $\mathbf{C}$-schemes.

Example 1.5. — Let Y be an arbitrary complex manifold and E a holomorphic vector bundle of rank r. Choose an open covering $\mathscr{U} = \{U_\alpha, \alpha \in A\}$ of Y such that $E\big|_{U_\alpha}$ is trivial for each $\alpha \in A$. A set of transition functions $g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbf{C})$ defininig E yields a holomorphic map $N_1\mathscr{U} = \coprod_{\alpha,\beta\in A} U_\alpha \cap U_\beta \to B_1\mathrm{GL}_r(\mathbf{C}) = \mathrm{GL}_r(\mathbf{C})$ and the cocycle condition ensures that this map extends uniquely to a holomorphic map $g\colon N_\bullet\mathscr{U} \to B_\bullet\mathrm{GL}_r(\mathbf{C})$, where $N_\bullet\mathscr{U}$ denotes the Čech nerve of $\mathscr{U}$, i.e. the simplicial manifold which in degree p is given by $N_p\mathscr{U} = \coprod_{\alpha_0,\dots,\alpha_p\in A} U_{\alpha_0} \cap \dots \cap U_{\alpha_p}$. Thus we get a $\mathrm{GL}_r(\mathbf{C})$-bundle on $N_\bullet\mathscr{U}$ in the above sense.

Example 1.6. — Again let Y be a complex manifold and in addition let S be a simplicial set. Let $\mathscr{O}(Y)$ denote the ring of holomorphic functions on Y and G the group $\mathrm{GL}_r(\mathscr{O}(Y))$. Then a G-fibre bundle ("G-fibré repéré") on S in the sense of Karoubi [**27**, 5.1] may be defined as a morphism of simplicial sets $S \to B_\bullet G$ (cf. the proof of [**27**, Théorème 5.4]). But $G = \mathrm{GL}_r(\mathscr{O}(Y))$ may be identified with the group of holomorphic maps $Y \to \mathrm{GL}_r(\mathbf{C})$ and thus a morphism of simplicial sets $S \to B_\bullet G$ is equivalent to a morphism of simplicial complex manifolds $Y \otimes S \to B_\bullet\mathrm{GL}_r(\mathbf{C})$, where $Y \otimes S$ is the simplicial manifold given in degree p by $\coprod_{\sigma\in S_p} Y$ with structure maps induced from those of S.

1.2.1. Topological morphisms and bundles. — The definition of a differential form on a simplicial complex manifold leads to the following notion of what we call *topological morphisms.*

Definition 1.7. — A *topological morphism* of simplicial manifolds $f\colon Y_\bullet \rightsquigarrow X_\bullet$ is a family of smooth maps

$$f_p\colon \Delta^p \times Y_p \to X_p, \qquad p \geq 0,$$

satisfying the following compatibility condition: For every increasing map $\phi\colon [p] \to [q]$ the diagram

$$\begin{tikzcd} & \Delta^q \times Y_q \arrow[r, "f_q"] & X_q \arrow[d, "\phi_X"] \\ \Delta^p \times Y_q \arrow[ur, "\phi_\Delta\times\mathrm{id}"] \arrow[dr, "\mathrm{id}\times\phi_Y"'] & & \\ & \Delta^p \times Y_p \arrow[r, "f_p"] & X_p \end{tikzcd}$$

commutes. Here $\phi_\Delta, \phi_Y, \phi_X$ denote the (co)simplicial structure maps induced by ϕ.

Every holomorphic or smooth morphism of simplicial complex manifolds $f\colon Y_\bullet \to X_\bullet$ induces a topological morphism $f\colon Y_\bullet \rightsquigarrow X_\bullet$ by composition with the natural projections $\Delta^p \times Y_p \to Y_p$.

Let $f\colon Z_\bullet \rightsquigarrow Y_\bullet$ be a topological morphism. For every increasing $\phi\colon [p] \to [q]$ we get a commutative diagram

$$
\begin{tikzcd}
 & \Delta^q \times Z_q \arrow[rr, "(\mathrm{pr}_{\Delta^q}, f_q)"] & & \Delta^q \times Y_q \\
\Delta^p \times Z_q \arrow[ur, "\phi_\Delta \times \mathrm{id}"] \arrow[rr, "(\mathrm{pr}_{\Delta^p}, f_q \circ (\phi_\Delta \times \mathrm{id}))"] \arrow[dr, "\mathrm{id} \times \phi_Z"] & & \Delta^p \times Y_q \arrow[ur, "\phi_\Delta \times \mathrm{id}"] \arrow[dr, "\mathrm{id} \times \phi_Y"] & \\
 & \Delta^p \times Z_p \arrow[rr, "(\mathrm{pr}_{\Delta^p}, f_p)"] & & \Delta^p \times Y_p.
\end{tikzcd}
\tag{1.3}
$$

Definition 1.8. — Let $f\colon Z_\bullet \rightsquigarrow Y_\bullet$ and $g\colon Y_\bullet \rightsquigarrow X_\bullet$ be topological morphisms. We define the *composition* $g \circ f\colon Z_\bullet \rightsquigarrow X_\bullet$ to be the topological morphism given in degree p by $g_p \circ (\mathrm{pr}_{\Delta^p}, f_p)\colon \Delta^p \times Z_p \to X_p$.

For a simplicial form $\omega = (\omega_p)_{p \geq 0} \in A^n(Y_\bullet)$ we define the *pullback* of ω by f to be the simplicial form $f^*\omega := ((\mathrm{pr}_{\Delta^p}, f_p)^*\omega_p)_{p \geq 0} \in A^n(Z_\bullet)$.

From diagram (1.3) one sees that these are well defined.

Definition 1.9. — Let $X_\bullet$ be a simplicial manifold. A *topological* $\mathrm{GL}_r(\mathbf{C})$*-bundle* on $X_\bullet$ is a topological morphism of simplicial manifolds

$$g\colon X_\bullet \rightsquigarrow B_\bullet \mathrm{GL}_r(\mathbf{C}).$$

A *morphism* $\alpha\colon g \to h$ of topological $\mathrm{GL}_r(\mathbf{C})$-bundles on $X_\bullet$ is a topological morphism of simplicial manifolds $\alpha\colon X_\bullet \rightsquigarrow E_\bullet \mathrm{GL}_r(\mathbf{C})$, such that $\alpha \cdot g = h$.

Example 1.10. — Let S be a simplicial set, A a complex Fréchet algebra and $A_\bullet$ the simplicial algebra $\mathscr{C}^\infty(\Delta^\bullet)\widehat{\otimes}_\pi A$, where $\mathscr{C}^\infty$ denotes smooth complex valued functions and $\widehat{\otimes}_\pi$ the projectively completed tensor product over $\mathbf{C}$. The simplicial classifying set $B_\bullet \mathrm{GL}_r(A_\bullet)$ for the simplicial group $\mathrm{GL}_r(A_\bullet)$ is by definition the diagonal of the bisimplicial set $([p],[q]) \mapsto B_p \mathrm{GL}_r(A_q)$. Karoubi defines a topological $\mathrm{GL}_r(A)$-bundle (= a "$\mathrm{GL}_r(A_\bullet)$-fibré repéré") on the simplicial set S to be a morphism $S \to B_\bullet \mathrm{GL}_r(A_\bullet)$ [**27**, 5.1, proof of 5.4 and 5.26].

In the special case, where A is the ring of smooth complex valued functions $\mathscr{C}^\infty(Y)$ on a complex manifold Y, this gives a topological bundle on the simplicial manifold $Y \otimes S$ (cf. Example 1.6) as follows:

First of all, there is a natural map $\mathscr{C}^\infty(\Delta^p)\widehat{\otimes}_\pi \mathscr{C}^\infty(Y) \to \mathscr{C}^\infty(\Delta^p \times Y)$. Next, $B_p \mathrm{GL}_r(\mathscr{C}^\infty(\Delta^p \times Y)) = \mathscr{C}^\infty(\Delta^p \times Y, B_p \mathrm{GL}_r(\mathbf{C}))$. Thus, a morphism of simplicial sets $f\colon S \to B_\bullet \mathrm{GL}_r(A_\bullet)$ gives rise to a family of smooth morphisms

$$\Delta^p \times Y \xrightarrow{f(\sigma)} B_p \mathrm{GL}_r(\mathbf{C}), \qquad \sigma \in S_p, p \geq 0.$$

That f is a morphism of simplicial sets is reflected in the fact that for every increasing $\phi\colon [p] \to [q]$ and $\sigma \in S_q$ the diagram

$$\begin{tikzcd} \Delta^q \times Y \arrow[r, "f(\sigma)"] & B_q\mathrm{GL}_r(\mathbf{C}) \arrow[d, "\phi_{B_\bullet G}"] \\ \Delta^p \times Y \arrow[u, "\phi_\Delta \times \mathrm{id}"] \arrow[r, "f(\phi_S^*\sigma)"] & B_p\mathrm{GL}_r(\mathbf{C}) \end{tikzcd}$$

commutes. Here $\phi_S^*\colon S_q \to S_p$ denotes the simplicial structure map induced by ϕ. Now the collection of maps $f(\sigma)$, $\sigma \in S_p$, defines a smooth morphism

$$\widetilde{f}_p\colon \Delta^p \times (Y \otimes S)_p = \coprod_{\sigma \in S_p} \Delta^p \times Y \xrightarrow{\coprod f(\sigma)} B_p\mathrm{GL}_r(\mathbf{C})$$

and the commutativity of the above diagrams is equivalent to the fact that the family of maps $\widetilde{f}_p$, $p \geq 0$, defines a topological morphism $Y \otimes S \rightsquigarrow B_\bullet\mathrm{GL}_r(\mathbf{C})$ in our sense.

1.3. Chern-Weil theory. — Chern-Weil theory on simplicial manifolds was developed by Dupont [**12**] and in the case of simplicial sets (but more general structure groups) by Karoubi [**27, 28**]. We recall Karoubi's formalism and adapt it to the setting of topological bundles.

In order to define the notion of a connection, we have to introduce some more notation. Any p-simplex x in the classifying space $B_\bullet\mathrm{GL}_r(\mathbf{C})$ may be written as $x = (g_{01}, g_{12}, \ldots, g_{p-1,p})$. Thus, if $(g_0, \ldots, g_p) \in E_p\mathrm{GL}_r(\mathbf{C})$ is a p-simplex lying over x, then $g_{01} = g_0 g_1^{-1}$ etc. and we define $g_{ji} := g_j g_i^{-1}$ for any $0 \leq i, j \leq p$. If $g\colon X_\bullet \rightsquigarrow B_\bullet\mathrm{GL}_r(\mathbf{C})$ is a topological $\mathrm{GL}_r(\mathbf{C})$-bundle, we write g_{ji} for the smooth maps $\Delta^p \times X_p \to \mathrm{GL}_r(\mathbf{C})$ obtained in the above way. If g is a holomorphic bundle then g_{ji} factors through a holomorphic map $X_p \to \mathrm{GL}_r(\mathbf{C})$ which, by abuse of notation, will also be denoted by g_{ji}.

Definition 1.11. — A *connection* in a topological $\mathrm{GL}_r(\mathbf{C})$-bundle $g\colon X_\bullet \rightsquigarrow B_\bullet\mathrm{GL}_r(\mathbf{C})$ is given by the following data: For any $p \geq 0$ and any $i \in [p] = \{0, \ldots, p\}$ a matrix valued 1-form $\Gamma_i = \Gamma_i^{(p)} \in \mathscr{A}^1(\Delta^p \times X_p; \mathrm{Mat}_r(\mathbf{C})) = \mathrm{Mat}_r(\mathscr{A}^1(\Delta^p \times X_p))$ subject to the conditions

(i) $(\phi_\Delta \times \mathrm{id})^*\Gamma_{\phi(i)}^{(q)} = (\mathrm{id} \times \phi_X)^*\Gamma_i^{(p)}$ for any increasing map $\phi\colon [p] \to [q]$ and

(ii) $\Gamma_i = g_{ji}^{-1} dg_{ji} + g_{ji}^{-1}\Gamma_j g_{ji}$.

Here Mat_r denotes $r \times r$-matrices. We view g_{ji} as a matrix of smooth functions on $\Delta^p \times X_p$. Thus dg_{ji} is a matrix valued 1-form on $\Delta^p \times X_p$.

If g is a holomorphic bundle, we call the connection *holomorphic*, if $\Gamma_i \in \mathscr{A}^{0,1,0}(\Delta^p \times X_p, \mathrm{Mat}_r(\mathbf{C})) \subseteq \mathscr{A}^1(\Delta^p \times X_p; \mathrm{Mat}_r(\mathbf{C}))$ (cf. the discussion before Theorem 1.2).

Example 1.12. — Every topological $\mathrm{GL}_r(\mathbf{C})$-bundle $g\colon X_\bullet \rightsquigarrow B_\bullet\mathrm{GL}_r(\mathbf{C})$ may be equipped with the *standard connection* given by

$$\Gamma_i = \sum_k x_k g_{ki}^{-1} dg_{ki},$$

where $x_0, \ldots, x_p$ denote the barycentric coordinates of Δ^p. If g is holomorphic, this connection is holomorphic.

Definition 1.13. — The *curvature* of the connection $\{\Gamma_i\}$ is defined as the family of matrix valued 2-forms

$$R_i := R_i^{(p)} := d\Gamma_i^{(p)} + \left(\Gamma_i^{(p)}\right)^2 \in \mathscr{A}^2(\Delta^p \times X_p; \mathrm{Mat}_r(\mathbf{C})),$$

$p \geq 0, i = 0, \ldots, p$.

Since in general $R_i \neq R_j$ for $i \neq j$ the entries of the $R_i^{(p)}, p \geq 0$, do *not* define a simplicial form (but see Definition 1.15 below).

Remarks 1.14. — (i) Let $g, h\colon X_\bullet \rightsquigarrow B_\bullet\mathrm{GL}_r(\mathbf{C})$ be two bundles, $\alpha\colon g \to h$ a morphism of bundles and $\Gamma = \{\Gamma_i\}$ a connection on h with curvature $\{R_i\}$. Then the *pullback* $\alpha^*\Gamma$ *of the connection* Γ is defined by the family of forms

$$(\alpha^*\Gamma)_i = \alpha_i^{-1} d\alpha_i + \alpha_i^{-1}\Gamma_i\alpha_i,$$

where $\alpha_i\colon \Delta^p \times X_p \to \mathrm{GL}_r(\mathbf{C})$ is the i-th component of the morphism α in simplicial degree p. The curvature of $\alpha^*\Gamma$ is given by the family of 2-forms $\alpha_i^{-1}R_i\alpha_i$.

(ii) If $E/X_\bullet$ is a topological bundle on $X_\bullet$ given by $g\colon X_\bullet \rightsquigarrow B_\bullet\mathrm{GL}_r(\mathbf{C})$, and $f\colon Y_\bullet \rightsquigarrow X_\bullet$ is a topological morphism, the pullback f^*E is given by $g \circ f$. If $\Gamma = \{\Gamma_i\}$ is a connection on E, the induced connection $f^*\Gamma$ on f^*E is given by

$$(f^*\Gamma)_i^{(p)} = (\mathrm{pr}_{\Delta^p}, f_p)^*\Gamma_i^{(p)}.$$

Consequently, its curvature is given by the family of forms $(\mathrm{pr}_{\Delta^p}, f_p)^*R_i^{(p)}$.

If Γ is the standard connection on E, then $f^*\Gamma$ is the standard connection on f^*E, as follows directly from the definitions.

Definition 1.15. — We define the *n-th Chern character form* $\mathrm{Ch}_n(\Gamma)$ of the connection $\Gamma = \{\Gamma_i\}$ to be the family of forms $\frac{(-1)^n}{n!}\mathrm{Tr}\left(\left(R_i^{(p)}\right)^n\right)$ on $\Delta^p \times X_p$, $p \geq 0$.

These forms do not depend on i since $R_i = g_{ji}^{-1}R_j g_{ji}$ by a straight forward computation. We summarize the results of Chern-Weil theory:

Theorem 1.16. — *Let $g : X_\bullet \rightsquigarrow B_\bullet\mathrm{GL}_r(\mathbf{C})$ be a topological bundle and Γ a connection on g.*

(i) $\mathrm{Ch}_n(\Gamma)$ *is a closed $2n$-form on $X_\bullet$, i.e. belongs to $A^{2n}(X_\bullet)$ and $d\mathrm{Ch}_n(\Gamma) = 0$.*
(ii) *The cohomology class of* $\mathrm{Ch}_n(\Gamma)$ *does not depend on the connection chosen.*

(iii) *If the bundle g and the connection are holomorphic,* $\mathrm{Ch}_n(\Gamma) \in \mathrm{Fil}^n A^{2n}(X_\bullet)$. *Moreover, the class of* $\mathrm{Ch}_n(\Gamma)$ *in* $H^{2n}(\mathrm{Fil}^n A^*(X_\bullet)) = \mathbb{H}^{2n}(X_\bullet, \Omega^{\geq n}_{X_\bullet})$ *does not depend on the holomorphic connection chosen.*

(iv) *If* $h\colon X_\bullet \rightsquigarrow B_\bullet \mathrm{GL}_r(\mathbf{C})$ *is a second bundle, and* $\alpha\colon h \to g$ *is a morphism, then* $\mathrm{Ch}_n(\alpha^*\Gamma) = \mathrm{Ch}_n(\Gamma)$.

(v) *If* $f\colon Y_\bullet \rightsquigarrow X_\bullet$ *is a topological morphism,* $\mathrm{Ch}_n(f^*\Gamma) = f^*\mathrm{Ch}_n(\Gamma)$.

Proof. — (i) Condition (i) in Definition 1.11 ensures that $(\phi_\Delta \times \mathrm{id}_{X_q})^* \mathrm{Tr}((R^{(q)}_{\phi(i)})^n) = (\mathrm{id}_{\Delta^p} \times \phi_X)^* \mathrm{Tr}((R^{(p)}_i)^n)$, hence the forms $\frac{(-1)^n}{n!} \mathrm{Tr}\left(\left(R^{(p)}_i\right)^n\right)$, $p \geq 0$, are indeed compatible and define $\mathrm{Ch}_n(\Gamma) \in A^{2n}(X_\bullet)$. For the closedness cf. the proof of [**27**, théorème 1.19].

(ii) This follows from a standard homotopy argument (cf. the construction of secondary forms in Section 1.4).

(iii) With the notations of Section 1.1 write

$$\mathrm{Fil}^i \mathscr{A}^*(\Delta^p \times X_p) = \bigoplus_{k+l+m=*, l \geq i} \mathscr{A}^{k,l,m}(\Delta^p \times X_p)$$

and similarly for matrix valued forms. These are subcomplexes and the product maps $\mathrm{Fil}^i \times \mathrm{Fil}^j$ to Fil^{i+j}. Now, if the connection is holomorphic, $\Gamma_i \in \mathrm{Fil}^1 \mathscr{A}^1(\Delta^p \times X_p, \mathrm{Mat}_r(\mathbf{C}))$, hence $R_i = d\Gamma_i + \Gamma_i^2 \in \mathrm{Fil}^1 \mathscr{A}^2(\Delta^p \times X_p; \mathrm{Mat}_r(\mathbf{C}))$ and then also $\mathrm{Ch}_n(\Gamma) \in \mathrm{Fil}^n A^{2n}(X_\bullet)$.

The independence of the associated cohomology class of the holomorphic connection chosen follows from a homotopy argument as before, where one has to take care about the filtration [**36**, Lemma 1.34].

(iv), (v) These follow directly from remarks 1.14 (i) and (ii) respectively. □

Definition 1.17. — If $E/X_\bullet$ is a topological bundle, we write $\mathrm{Ch}_n(E)$ for the cohomology class of $\mathrm{Ch}_n(\Gamma)$ in $H^{2n}(A^*(X_\bullet)) = H^{2n}(X_\bullet, \mathbf{C})$, where Γ is any connection on E. If E is *holomorphic*, we also denote by $\mathrm{Ch}_n(E)$ the class of $\mathrm{Ch}_n(\Gamma)$ in $\mathbb{H}^{2n}(X_\bullet, \Omega^{\geq n}_{X_\bullet})$, where Γ is any holomorphic connection.

Characteristic classes of holomorphic vector bundles. — In order to compare these Chern-Weil theoretic characteristic classes with other constructions we have to extend them to arbitrary holomorphic vector bundles on simplicial manifolds.

Recall from [**17**, Ex. 1.1] that a holomorphic vector bundle on the simplicial complex manifold $X_\bullet$ is a sheaf $\mathscr{E}_\bullet$ of $\mathscr{O}_{X_\bullet}$-modules such that each $\mathscr{E}_p$ is locally free and for every $\phi\colon [p] \to [q]$ the associated map $\phi_X^* \mathscr{E}_p \to \mathscr{E}_q$ is an isomorphism.

There exists a canonical holomorphic rank r vector bundle on $B_\bullet\mathrm{GL}_r(\mathbf{C})$ and it is well known that pulling back along classifying maps induces a bijection

$$\left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{holomorphic}\\ \mathrm{GL}_r(\mathbf{C})\text{-bundles on } X_\bullet\end{array}\right\} \xrightarrow{1-1} \left\{\begin{array}{c}\text{isomorphism classes of}\\ \text{degreewise trivial}\\ \text{holomorphic rank } r \text{ vector}\\ \text{bundles on } X_\bullet\end{array}\right\}$$

(for a proof see e.g. [**36**, Lemma 1.13]).

Now let $\mathscr{E}_\bullet$ be an arbitrary holomorphic vector bundle of rank r on $X_\bullet$. Let $(V_\alpha)_{\alpha\in A}$ be an open covering of X_0 that trivializes $\mathscr{E}_0$ and put $U_0 := \coprod_{\alpha\in A} V_\alpha$. Then $U_\bullet := \operatorname{cosk}_0^{X_\bullet} U_0$ (cf. [**9**, (5.1.1)]) is an open covering of $X_\bullet$ such that $\mathscr{E}_\bullet|_{U_\bullet}$ is degreewise trivial. Denote by $N_{X_\bullet}(U_\bullet)$ its Čech nerve (cf. [**14**, p. 20]) and by $\Delta N_{X_\bullet}(U_\bullet)$ its diagonal simplicial manifold. It follows from the Theorem of Eilenberg-Zilber [**9**, (6.4.2.2)] and [**9**, (5.3.7), (6.4.3)] that the natural maps

$$\mathbb{H}^*(X_\bullet, \Omega^{\geq n}_{X_\bullet}) \xrightarrow{\cong} \mathbb{H}^*(N_{X_\bullet}(U_\bullet), \Omega^{\geq n}_{N_{X_\bullet}(U_\bullet)}) \xrightarrow{\cong} \mathbb{H}^*(\Delta N_{X_\bullet}(U_\bullet), \Omega^{\geq n}_{\Delta N_{X_\bullet}(U_\bullet)})$$

are isomorphisms.

Now $\mathscr{E}_\bullet|_{\Delta N_{X_\bullet}(U_\bullet)}$ is degreewise trivial, hence corresponds to a $\mathrm{GL}_r(\mathbf{C})$-bundle E on $\Delta N_{X_\bullet}(U_\bullet)$, and we define

$$\mathrm{Ch}_n(\mathscr{E}_\bullet) \in \mathbb{H}^{2n}(X_\bullet, \Omega^{\geq n}_{X_\bullet})$$

to be the preimage of $\mathrm{Ch}_n(E)$ under the above isomorphisms. Using the fact that two open covers of X_0 admit a common refinement one checks that this is well defined.

In order to apply the splitting principle later on, we need the

Proposition 1.18 **(Whitney sum formula)**. — *Let* $0 \to \mathscr{E}'_\bullet \to \mathscr{E}_\bullet \to \mathscr{E}''_\bullet \to 0$ *be a short exact sequence of holomorphic vector bundles on* $X_\bullet$. *Then* $\mathrm{Ch}_n(\mathscr{E}_\bullet) = \mathrm{Ch}_n(\mathscr{E}'_\bullet) + \mathrm{Ch}_n(\mathscr{E}''_\bullet)$.

Proof. — Choosing suitable coverings we may assume without loss of generality that $0 \to \mathscr{E}'_0 \to \mathscr{E}_0 \to \mathscr{E}''_0 \to 0$ is a split short exact sequence of free $\mathscr{O}_{X_0}$-modules. If we denote the components of the classifying maps of $\mathscr{E}'_\bullet, \mathscr{E}_\bullet$, and $\mathscr{E}''_\bullet$ by g'_{ij}, g_{ij}, and g''_{ij}, respectively, it follows that g_{ij} is of the form

$$\begin{pmatrix} g'_{ij} & * \\ 0 & g''_{ij} \end{pmatrix}.$$

Using the standard connections for the computation of the Chern character classes the result follows easily from this. □

1.4. Secondary classes. — Here we give the construction of the secondary classes associated with a holomorphic bundle together with a trivialization of its underlying topological bundle. These are the classes we are primarily interested in since they are used in the construction of Karoubi's relative Chern character on relative K-theory.

Let $X_\bullet$ be a simplicial complex manifold and E a *holomorphic* $\mathrm{GL}_r(\mathbf{C})$-bundle given by the map $g\colon X_\bullet \to B_\bullet\mathrm{GL}_r(\mathbf{C})$. Assume that $\alpha\colon T \to E$ is a morphism from the trivial bundle T, given by the constant map $X_\bullet \to \{1\} \subseteq B_\bullet\mathrm{GL}_r(\mathbf{C})$, to E viewed as *topological* bundles. According to the definitions this means that we have a commutative diagram

$$\begin{tikzcd} & E_\bullet\mathrm{GL}_r(\mathbf{C}) \arrow[d, "p"] \\ X_\bullet \arrow[ur, squiggly, "\alpha"] \arrow[r, "g"'] & B_\bullet\mathrm{GL}_r(\mathbf{C}). \end{tikzcd}$$

Fix a holomorphic connection Γ^E on E. By Chern-Weil theory (Theorem 1.16) the form $\mathrm{Ch}_n(\Gamma^E) \in A^{2n}(X_\bullet)$ is exact. In fact, we can make a particular choice of a form $\mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^E, \alpha) \in A^{2n-1}(X_\bullet)$ which bounds $\mathrm{Ch}_n(\Gamma^E)$. It is constructed as follows: The standard homotopy operator from de Rham cohomology $K\colon A^{2n}(X_\bullet \times \mathbf{C}) \to A^{2n-1}(X_\bullet)$, $\omega \mapsto \int_0^1 (i_{\partial/\partial t}\omega)dt$, where t is the coordinate on $\mathbf{C}$ and $i_{\partial/\partial t}$ is inner multiplication with respect to the vector field $\partial/\partial t$, satisfies

$$dK + Kd = i_1^* - i_0^*$$

with the obvious inclusions $i_0, i_1\colon X_\bullet \hookrightarrow X_\bullet \times \mathbf{C}$. Let π denote the projection $X_\bullet \times \mathbf{C} \to X_\bullet$. On the trivial bundle T on $X_\bullet \times \mathbf{C}$ we have the trivial connection given by the zero matrix and the connection $\pi^*\alpha^*\Gamma^E$, and we may also consider the connection

$$\Gamma_t = t\pi^*\alpha^*\Gamma^E \tag{1.4}$$

which is an affine combination of both.

Definition 1.19. — $\mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^E, \alpha) := K(\mathrm{Ch}_n(\Gamma_t)) \in A^{2n-1}(X_\bullet)$.

We collect some properties.

Proposition 1.20. — (i) $d\mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^E, \alpha) = \mathrm{Ch}_n(\Gamma^E)$

(ii) *The class of* $\mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^E, \alpha)$ *in* $H^{2n-1}(A^*(X_\bullet)/\mathrm{Fil}^n A^*(X_\bullet)) = \mathbb{H}^{2n-1}(X_\bullet, \Omega_{X_\bullet}^{<n})$ *does not depend on the holomorphic connection chosen. We will denote it by* $\mathrm{Ch}_n^{\mathrm{rel}}(E, \alpha)$.

Proof. — (i) follows directly from the constructions and the properties of the homotopy operator. (ii) Let $\widetilde{\Gamma}^E$ be a second holomorphic connection on E. Denote by π the projection $X_\bullet \times \mathbf{C} \times \mathbf{C} \to X_\bullet$ and by s, t the variables on $\mathbf{C} \times \mathbf{C}$. On the trivial bundle on $X_\bullet \times \mathbf{C} \times \mathbf{C}$ consider the connection $\Gamma_{s,t} = (1-s)t\pi^*\alpha^*\Gamma^E + st\pi^*\alpha^*\widetilde{\Gamma}^E$. Denote by K_s and K_t the homotopy operators with respect to s and t, respectively. Then

$$\begin{aligned} & d(K_sK_t(\mathrm{Ch}_n(\Gamma_{s,t}))) + K_s(d(K_t(\mathrm{Ch}_n(\Gamma_{s,t})))) = \\ = \quad & K_t(\mathrm{Ch}_n(\Gamma_{s,t}))|_{s=1} - K_t(\mathrm{Ch}_n(\Gamma_{s,t}))|_{s=0} \\ = \quad & \mathrm{Ch}_n^{\mathrm{rel}}(\widetilde{\Gamma}^E) - \mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^E). \end{aligned}$$

But $dK_t(\mathrm{Ch}_n(\Gamma_{s,t})) = \mathrm{Ch}_n(\Gamma_{s,1}) - \mathrm{Ch}_n(\Gamma_{s,0}) = \mathrm{Ch}_n((1-s)\pi^*\alpha^*\Gamma^E + s\pi^*\alpha^*\widetilde{\Gamma}^E) = \mathrm{Ch}_n((1-s)\pi^*\Gamma^E + s\pi^*\widetilde{\Gamma}^E)$ as one easily checks using Remark 1.14 where we still denoted by π the projection $X_\bullet \times \mathbf{C} \to X_\bullet$. But $(1-s)\pi^*\Gamma^E + s\pi^*\widetilde{\Gamma}^E$ is a holomorphic connection on π^*E and it is not hard to see that $K_s(\mathrm{Ch}_n((1-s)\pi^*\Gamma^E + s\pi^*\widetilde{\Gamma}^E)) \in \mathrm{Fil}^n A^{2n-1}(X_\bullet)$ completing the proof. □

Remarks 1.21. — (i) The definition of the relative Chern character form involves the choice of a connection Γ_t on the trivial bundle on $X_\bullet \times \mathbf{C}$ satisfying $i_0^*\Gamma_t = 0, i_1^*\Gamma_t = \alpha^*\Gamma$. We stick to the choice (1.4) to have well defined forms which moreover satisfy a certain functoriality (see below). However, an argument similar to the proof of Proposition 1.20 (ii) shows that a different choice of Γ_t only alters $\mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^E, \alpha)$ by an exact form, hence leads to the same class in cohomology.

(ii) The definition of the relative Chern character *form* makes sense for arbitrary topological bundles $E/X_\bullet$ together with a trivialization $\alpha\colon X_\bullet \rightsquigarrow E_\bullet\mathrm{GL}_r(\mathbf{C})$ and a connection Γ^E. If $f\colon Y_\bullet \rightsquigarrow X_\bullet$ is a topological morphism the pullback f^*E admits the trivialization $\alpha\circ f$ and one can check that $\mathrm{Ch}_n^{\mathrm{rel}}(f^*\Gamma^E, \alpha\circ f) = f^*\mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^E, \alpha)$ (cf. [**36**, Lemma 1.35]). If the bundle E, the connection Γ^E, and the map f are *holomorphic* we have a pullback f^* on cohomology and $\mathrm{Ch}_n^{\mathrm{rel}}(f^*E, \alpha\circ f) = f^*\mathrm{Ch}_n^{\mathrm{rel}}(E, \alpha)$.

(iii) One may ask whether the relative Chern character class for holomorphic bundles with a topological trivialization is determined by functoriality and the fact that it transgresses the Chern character class. This is not clear since there seems to be no good classifying space for holomorphic bundles E together with a topological trivialization α in the sense that E and α are obtained by pullback of a universal pair $E^{\mathrm{univ}}_{\mathrm{top.\,triv.}}, \alpha^{\mathrm{univ}}_{\mathrm{top.\,triv.}}$ along a *holomorphic* map. This is in fact one of the main difficulties in the comparison of relative Chern character classes and Deligne-Beilinson Chern character classes.

However, it turns out that we can work around this problem using the functoriality of the relative Chern character form: The projection $p\colon E_\bullet\mathrm{GL}_r(\mathbf{C}) \to B_\bullet\mathrm{GL}_r(\mathbf{C})$ classifies the holomorphic bundle p^*E^{univ} which admits the tautological trivialization $\alpha^{\mathrm{univ}}\colon T \to p^*E^{\mathrm{univ}}$ given by $\alpha^{\mathrm{univ}} = \mathrm{id}\colon E_\bullet\mathrm{GL}_r(\mathbf{C}) \to E_\bullet\mathrm{GL}_r(\mathbf{C})$. We equip p^*E^{univ} with the standard connection and denote the corresponding relative form by $\mathrm{Ch}_n^{\mathrm{rel,univ}} := \mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^{p^*E^{\mathrm{univ}}}, \alpha^{\mathrm{univ}})$.

Proposition 1.22. — *If $E/X_\bullet$ is a holomorphic $\mathrm{GL}_r(\mathbf{C})$-bundle together with a topological trivialization $\alpha\colon T \to E$, then*

$$\mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^E, \alpha) = \alpha^*\mathrm{Ch}_n^{\mathrm{rel,univ}},$$

Γ^E denoting the standard connection on E.

Proof. — This follows from Remark 1.21 (ii) and the fact that $E = \alpha^*p^*E^{\mathrm{univ}}$ and that the pullback of the standard connection is the standard connection. □

Remark 1.23. — This description will be needed in Proposition 2.10 to compare the class $\mathrm{Ch}_n^{\mathrm{rel}}(E,\alpha)$ for an algebraic bundle E with a topological trivialization α with the class $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E,\alpha)$ constructed by a completely different strategy. It will be this latter class, that can be compared with the Deligne-Beilinson Chern character class $\mathrm{Ch}_n^{\mathscr{D}}(E)$. Note that this kind of "universal" description of relative Chern character forms would not be possible in Karoubi's setting of bundles on simplicial sets.

2. Secondary classes for algebraic bundles

The heart of this section is the comparison of relative and Deligne-Beilinson Chern character classes in the last subsection. To do this, we first construct a refinement of the secondary classes of Section 1.4 for an *algebraic* bundle on a simplicial variety $X_\bullet$ together with a topological trivialization. These classes live in $H^{2n}(X_\bullet,\mathbf{C})/\mathrm{Fil}^n H^{2n}(X_\bullet,\mathbf{C})$. Using the so called *refined Chern character classes* constructed in 2.3 the comparison will be reduced to the comparison of primary Chern character classes, which is done in 2.2. The first subsection recalls the definition of the Hodge filtration on the cohomology of a simplicial variety.

2.1. Preliminaries.

— Recall that a simplicial object in a category $\mathcal{C}$ is a functor $\Delta^{\mathrm{op}} \to \mathcal{C}$ where Δ denotes the category of finite ordered sets and increasing maps. Denote by Δ^{str} the subcategory of Δ with the same objects but only strictly increasing maps as morphisms. A *strict simplicial object* in $\mathcal{C}$ is a functor $(\Delta^{\mathrm{str}})^{\mathrm{op}} \to \mathcal{C}$.

In the following, a *variety* will be a *smooth, separated scheme of finite type* over $\mathbf{C}$ equipped with the classical topology and $\mathscr{O}_X, \Omega_X^*$ will denote the sheaves of holomorphic functions and differential forms, respectively. By abuse of notation we will denote the complex manifold associated with a variety X by the same letter. A simplicial or strict simplicial variety $X_\bullet$ is called *proper* if each X_p is proper over $\mathbf{C}$.

Let $X_\bullet$ be a simplicial variety. Using Nagata's compactification theorem and Hironaka's resolution of singularities one inductively constructs an open immersion $j\colon X_\bullet \hookrightarrow \overline{X}_\bullet$ into a proper strict simplicial variety $\overline{X}_\bullet$ such that the complement $D_p := \overline{X}_p - X_p$ is a divisor with normal crossings for each p [**35**, 1.2]. We call j a *good compactification*. The n^{th} step of the Hodge filtration is given as the image of the injective map $\mathbb{H}^*(\overline{X}_\bullet, \Omega^{\geq n}_{\overline{X}_\bullet}(\log D_\bullet)) \hookrightarrow \mathbb{H}^*(\overline{X}_\bullet, \Omega^{*}_{\overline{X}_\bullet}(\log D_\bullet)) = H^*(X_\bullet,\mathbf{C})$. It may be computed as follows [**8**, (3.2.3)], [**35**, 1.3]: For each p let $\mathscr{A}^{k,l}_{\overline{X}_p}(\log D_p)$ be the subsheaf $\Omega^k_{\overline{X}_p}(\log D_p) \otimes_{\mathscr{O}_{\overline{X}_p}} \mathscr{A}^{0,l}_{\overline{X}_p}$ of $j_{p*}\mathscr{A}^{k,l}_{\mathscr{O}_{X_p}}$ and denote its global sections by $\mathscr{A}^{k,l}(\overline{X}_p, \log D_p)$. Denote by $\mathrm{Fil}^n \mathscr{A}^*(\overline{X}_\bullet, \log D_\bullet)$ the complex which is given in degree $*$ by $\bigoplus_{k+l+p=*, k\geq n} \mathscr{A}^{k,l}(\overline{X}_p, \log D_p)$. There are natural isomorphisms $H^*(\mathrm{Fil}^n\mathscr{A}^*(\overline{X}_\bullet, \log D_\bullet)) \cong \mathrm{Fil}^n H^*(X_\bullet, \mathbf{C})$.

We have natural maps $\Omega^{\geq n}_{\overline{X}_\bullet}(\log D_\bullet) \to j_*\Omega^{\geq n}_{X_\bullet}$ which on cohomology induce

$$\mathbb{H}^*(\overline{X}_\bullet, \Omega^{\geq n}_{\overline{X}_\bullet}(\log D_\bullet)) = \mathrm{Fil}^n H^*(X_\bullet, \mathbf{C}) \to \mathbb{H}^*(\overline{X}_\bullet, j_*\Omega^{\geq n}_{X_\bullet}) \cong \mathbb{H}^*(X_\bullet, \Omega^{\geq n}_{X_\bullet}). \tag{2.1}$$

If we compose this further with the map induced by $\Omega^{\geq n}_{X_\bullet} \hookrightarrow \Omega^*_{X_\bullet}$ we obtain the natural inclusion $\mathrm{Fil}^n H^*(X_\bullet, \mathbf{C}) \hookrightarrow H^*(X_\bullet, \mathbf{C}) = \mathbb{H}^*(X_\bullet, \Omega^*_{X_\bullet})$. Hence (2.1) is injective, too, and we will view it as an inclusion.

Remark 2.1. — In the study of Chern character maps on higher K-theory, simplicial schemes of the form $X_\bullet = X \otimes S$, where X is a variety and S a simplicial set, occur naturally. These are in general not of finite type. Nevertheless, they admit a good compactification $\overline{X}_\bullet$ defined as $\overline{X} \otimes S$, where $X \hookrightarrow \overline{X}$ is a good compactification, and we can still consider the map $\mathbb{H}^*(\overline{X}_\bullet, \Omega^{\geq n}_{\overline{X}_\bullet}(\log D_\bullet)) \to \mathbb{H}^*(X_\bullet, \Omega^*_{X_\bullet}) = H^*(X_\bullet, \mathbf{C})$. It is not hard to see that this map is still injective. We will denote its image by $\mathrm{Fil}^n H^*(X_\bullet, \mathbf{C})$ also in this case.

In the following, all the results and constructions that are formulated for simplicial varieties are also valid for simplicial schemes of the form $X \otimes S$ and we will use them without mentioning them explicitly.

2.2. Chern classes of algebraic bundles. — Let E be an algebraic $\mathrm{GL}_r(\mathbf{C})$-bundle on the simplicial variety $X_\bullet$, i.e. a morphism of simplicial varieties $g\colon X_\bullet \to B_\bullet\mathrm{GL}_r(\mathbf{C})$. Since E may be viewed as a holomorphic bundle, we have the classes $\mathrm{Ch}_n(E) \in \mathbb{H}^{2n}(X_\bullet, \Omega^{\geq n}_{X_\bullet})$ constructed using Chern-Weil theory. On the other hand, one may also construct Chern character classes in $\mathrm{Fil}^n H^{2n}(X_\bullet, \mathbf{C})$ in the style of Grothendieck and Hirzebruch. We recall the construction, and show that these are mapped to the Chern-Weil theoretic classes under the natural map $\mathrm{Fil}^n H^{2n}(X_\bullet, \mathbf{C}) \to \mathbb{H}^{2n}(X_\bullet, \Omega^{\geq n}_{X_\bullet})$.

2.2.1. The first Chern class of a line bundle. — Let X be a complex manifold, or more generally a simplicial complex manifold. The group of isomorphism classes of *holomorphic* line bundles on X is $H^1(X, \mathscr{O}^*_X)$ (cf. [**17**, Ex. 1.1]).

Definition 2.2. — The first Chern class $c_1\colon H^1(X, \mathscr{O}^*_X) \to \mathbb{H}^2(X, \Omega^{\geq 1}_X)$ is the map on cohomology induced by the morphism of complexes $d\log\colon \mathscr{O}^*_X[-1] \to \Omega^{\geq 1}_X$.

Lemma 2.3. — *If $\mathscr{L}$ is an* algebraic *line bundle on the* variety *X, then $c_1(\mathscr{L}) \in \mathrm{Fil}^1 H^2(X, \mathbf{C}) \subseteq \mathbb{H}^2(X, \Omega^{\geq 1}_X)$.*

Proof. — We may assume that $X = X_\bullet$ is a simplicial variety and that $\mathscr{L}$ is classified by a morphism of simplicial varieties $g^{(\bullet)}\colon X_\bullet \to B_\bullet\mathbb{G}_m(\mathbf{C})$. Then $g^{(1)} \in \Gamma(X_1, \mathscr{O}^*_{X_1})$ represents a class in $H^1(\Gamma^*(X_\bullet, \mathscr{O}^*_{X_\bullet}))$ whose image in $H^1(X_\bullet, \mathscr{O}^*_{X_\bullet})$ is the class of $\mathscr{L}$ [**17**, Ex. 1.1]. Thus $c_1(\mathscr{L}) \in \mathbb{H}^2(X_\bullet, \Omega^{\geq 1}_{X_\bullet}) = H^2(\mathrm{TotFil}^1\mathscr{A}^*(X_\bullet))$ is the class represented by $(d\log(g^{(1)})) \oplus 0 \in \Gamma(X_1, \Omega^1_{X_1}) \oplus \Gamma(X_0, \Omega^2_{X_0}) \subseteq \mathrm{Fil}^1\mathscr{A}^1(X_1) \oplus \mathrm{Fil}^1\mathscr{A}^2(X_0)$.

Let $\overline{X}_1$ be any good compactification of X_1. Then $g^{(1)}$, being algebraic, is meromorphic along $\overline{X}_1 - X_1$, hence $d\log(g^{(1)}) \in \mathrm{Fil}^1\mathscr{A}^1(X_1, \log(\overline{X}_1 - X_1))$. □

Lemma 2.4. — *Let $X_\bullet$ be a simplicial complex manifold and $\mathscr{L}$ a holomorphic line bundle on $X_\bullet$. Then*

$$\mathrm{Ch}_1(\mathscr{L}_\bullet) = c_1(\mathscr{L}_\bullet)$$

in $\mathbb{H}^2(X_\bullet, \Omega^{\geq 1}_{X_\bullet})$.

Proof. — Again, we may assume that $\mathscr{L}_\bullet$ is classified by a holomorphic morphism of simplicial manifolds $g^{(\bullet)}\colon X_\bullet \to B_\bullet\mathbb{G}_m(\mathbf{C})$. Then $\mathrm{Ch}_1(\mathscr{L}_\bullet)$ can be computed explicitly: We equip the $\mathbb{G}_m(\mathbf{C})$-bundle L classified by $g^{(\bullet)}$ with the standard connection, given by the family of matrices $\Gamma_i^{(p)} = \sum_{k=0}^p x_k(g_{ki}^{(p)})^{-1}dg_{ki}^{(p)} = \sum_k x_k d\log(g_{ki}^{(p)})$ where the notations are as in Section 1.3. The curvature is then given by $R_i^{(p)} = \sum_k dx_k d\log(g_{ki}^{(p)}) + \sum_{k,l} x_k x_l d\log(g_{ki}^{(p)}) d\log(g_{li}^{(p)})$. This form does not depend on i, and the first Chern character form $\mathrm{Ch}_1(L)$ of L in $\mathrm{Fil}^1 A^2(X_\bullet)$ is given by the family $(\mathrm{Ch}_1(L)_p)_{p\geq 0} = (-R_i^{(p)})_{p\geq 0}$.

The isomorphism $H^2(\mathrm{Fil}^1 A^*(X_\bullet)) \to H^2(\mathrm{Fil}^1\mathscr{A}^*(X_\bullet)) = \mathbb{H}^2(X_\bullet, \Omega^{\geq 1}_{X_\bullet})$ is given by $\omega = (\omega_p)_{p\geq 0} \mapsto (\int_{\Delta^1}\omega_1, \int_{\Delta^0}\omega_0) \in \mathrm{Fil}^1\mathscr{A}^1(X_1) \oplus \mathrm{Fil}^1\mathscr{A}^2(X_0)$.

Since $g_{ii}^{(p)}$ is the constant map 1, $d\log(g_{ii}^{(p)}) = 0$ for all $p \geq 0, i = 0, \dots, p$ and in particular $\mathrm{Ch}_1(L)_0 = 0$. Next, $\mathrm{Ch}_1(L)_1 = -R_1^{(1)} = -dx_0 d\log(g_{01}^{(1)})$, and hence $\int_{\Delta^1}\mathrm{Ch}_1(L)_1 = d\log(g_{01}^{(1)}) = d\log(g^{(1)})$. Comparing with the computation in the proof of the last Lemma, this concludes the proof. □

2.2.2. Higher Chern classes. — These are constructed in the style of Grothendieck using the splitting principle.

Let $X_\bullet$ be a simplicial variety and $\mathscr{E}_\bullet$ an algebraic vector bundle on $X_\bullet$ of rank r. Denote by $\pi\colon \mathbb{P}(\mathscr{E}_\bullet) \to X_\bullet$ the associated projective bundle and by $\mathscr{O}(1)$ the tautological line bundle on $\mathbb{P}(\mathscr{E}_\bullet)$. Write $\xi := c_1(\mathscr{O}(1)) \in \mathrm{Fil}^1 H^2(\mathbb{P}(\mathscr{E}_\bullet), \mathbf{C}) \subseteq \mathbb{H}^2(\mathbb{P}(\mathscr{E}_\bullet), \Omega^{\geq 1}_{\mathbb{P}}(\mathscr{E}_\bullet))$.

Lemma 2.5. — *The maps*

$$\sum_{i=0}^{r-1}\pi^*(\_)\cup\xi^i\colon \bigoplus_{i=0}^{r-1}\mathbb{H}^{m-2i}(X_\bullet, \Omega^{\geq n-i}_{X_\bullet}) \to \mathbb{H}^m(\mathbb{P}(\mathscr{E}_\bullet), \Omega^{\geq n}_{\mathbb{P}(\mathscr{E}_\bullet)}) \quad \textit{and} \tag{2.2}$$

$$\sum_{i=0}^{r-1}\pi^*(\_)\cup\xi^i\colon \bigoplus_{i=0}^{r-1}\mathrm{Fil}^{n-i}H^{m-2i}(X_\bullet, \mathbf{C}) \to \mathrm{Fil}^n H^m(\mathbb{P}(\mathscr{E}_\bullet), \mathbf{C}) \tag{2.3}$$

are isomorphisms.

Proof. — By a spectral sequence argument as in [**16**, Lemma 2.4] the simplicial case follows from the classical case. There the second isomorphism follows by Hodge theory from the classical Leray-Hirsch Theorem, the first one is established in the proof of [**18**, Proposition 5.2]. □

The higher Chern classes $c_n(\mathscr{E}_\bullet) \in \mathrm{Fil}^n H^{2n}(X_\bullet, \mathbf{C})$ are now defined by the equation

$$\sum_{i=0}^{r} \pi^*(c_{r-i}(\mathscr{E}_\bullet)) \cup c_1(\mathscr{O}_{\mathbb{P}(\mathscr{E}_\bullet)}(1))^i = 0$$

and the conditions $c_n(\mathscr{E}_\bullet) = 0$ if $n > r$, $c_0(\mathscr{E}_\bullet) = 1$. Let $N_n \in \mathbf{Z}[X_1, \dots, X_n]$ be the n^{th} Newton polynomial. The n^{th} Chern character class is defined as

$$\widetilde{\mathrm{Ch}}_n(\mathscr{E}_\bullet) := \frac{1}{n!} N_n(c_1(\mathscr{E}_\bullet), \dots, c_n(\mathscr{E}_\bullet)) \in \mathrm{Fil}^n H^{2n}(X_\bullet, \mathbf{C}).$$

The theory of Chern character classes obtained in this way has the usual properties. In particular they are functorial and the Whitney sum formula holds [**21**].

Proposition 2.6. — *Let $\mathscr{E}_\bullet$ be an algebraic vector bundle on the simplicial variety $X_\bullet$. The natural morphism $\mathrm{Fil}^n H^{2n}(X_\bullet, \mathbf{C}) \to \mathbb{H}^{2n}(X_\bullet, \Omega^{\geq n}_{X_\bullet})$ maps $\widetilde{\mathrm{Ch}}_n(\mathscr{E}_\bullet)$ to $\mathrm{Ch}_n(\mathscr{E}_\bullet)$. In particular, $\mathrm{Ch}_n(\mathscr{E}_\bullet) \in \mathrm{Fil}^n H^{2n}(X_\bullet, \mathbf{C}) \subseteq \mathbb{H}^{2n}(X_\bullet, \Omega^{\geq n}_{X_\bullet})$.*

Proof. — Repeated use of the projective bundle construction gives a morphism of simplicial varieties $\pi\colon Q_\bullet \to X_\bullet$ such that $\pi^*\mathscr{E}_\bullet$ has a filtration whose subquotients are line bundles, and such that both maps $\pi^*\colon \mathrm{Fil}^n H^{2n}(X_\bullet, \mathbf{C}) \to \mathrm{Fil}^n H^{2n}(Q_\bullet, \mathbf{C})$ and $\pi^*\colon \mathbb{H}^{2n}(X_\bullet, \Omega^{\geq n}_{X_\bullet}) \to \mathbb{H}^{2n}(Q_\bullet, \Omega^{\geq n}_{Q_\bullet})$ are injective (Lemma 2.5).

By the Whitney sum formula it is thus enough to show, that for a line bundle $\mathscr{L}_\bullet$, $\widetilde{\mathrm{Ch}}_n(\mathscr{L}_\bullet)$ maps to $\mathrm{Ch}_n(\mathscr{L}_\bullet)$. But $\widetilde{\mathrm{Ch}}_n(\mathscr{L}_\bullet)$ is just $\frac{1}{n!} c_1(\mathscr{L}_\bullet)^n$ and similarly $\mathrm{Ch}_n(\mathscr{L}_\bullet) = \frac{1}{n!} (\mathrm{Ch}_1(\mathscr{L}_\bullet))^n$. Indeed, for the Chern character classes $\widetilde{\mathrm{Ch}}_n$ this follows from the explicit form of the Newton polynomials and the fact that $c_i(\mathscr{L}_\bullet) = 0$ if $i > 1$, while for the classes $\mathrm{Ch}_n(\mathscr{L}_\bullet)$ it follows directly from the construction. Hence the claim follows from Lemma 2.4. $\square$

2.3. Relative Chern character classes. — In this Section we construct refinements of the secondary classes of Definition 1.19 for algebraic bundles together with a trivialization of the associated topological bundle, which take the Hodge filtration into account.

First recall the following notion: If $A \xrightarrow{f} C \xleftarrow{g} B$ is a diagram of complexes in an abelian category, its *quasi-pullback* is the complex $\mathrm{Cone}(A \oplus B \xrightarrow{f-g} C)[-1]$. The natural projections give maps from the quasi-pullback to A and B and the diagram

$$\begin{tikzcd} \mathrm{Cone}(A \oplus B \xrightarrow{f-g} C)[-1] \arrow[r, "f'"] \arrow[d, "g'"'] & B \arrow[d, "g"] \\ A \arrow[r, "f"] & C \end{tikzcd}$$

commutes up to a canonical homotopy. Moreover, if g is a quasi-isomorphism so is g'. (For more details see e.g. [**36**, Lemma A.1].)

Now let E be an algebraic $\mathrm{GL}_r(\mathbf{C})$-bundle on the simplicial variety $X_\bullet$ classified by $g\colon X_\bullet \to B_\bullet\mathrm{GL}_r(\mathbf{C})$. Define the principal bundle $E_\bullet \xrightarrow{p} X_\bullet$ associated with E by the pullback diagram

$$\begin{tikzcd} E_\bullet \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner"] & E_\bullet\mathrm{GL}_r(\mathbf{C}) \arrow[d, "p"] \\ X_\bullet \arrow[r, "g"] & B_\bullet\mathrm{GL}_r(\mathbf{C}). \end{tikzcd}$$

Choose a good compactification $j\colon X_\bullet \hookrightarrow \overline{X}_\bullet$ and write $D_p = \overline{X}_p - X_p$. We define the complex $\mathrm{Fil}^n A^*(\overline{X}_\bullet, \log D_\bullet)$ as the quasi-pullback of the diagram

$$\begin{tikzcd} & A^*(X_\bullet) \arrow[d, "\mathrm{qis}"', "I"] \\ \mathrm{Fil}^n \mathscr{A}^*(\overline{X}_\bullet, \log D_\bullet) \arrow[r, "\iota_{\mathscr{A}}"] & \mathscr{A}^*(X_\bullet). \end{tikzcd}$$

Then the natural projection $\mathrm{Fil}^n A^*(\overline{X}_\bullet, \log D_\bullet) \to \mathrm{Fil}^n \mathscr{A}^*(\overline{X}_\bullet, \log D_\bullet)$ is a quasi-isomorphism and the diagram

$$\begin{tikzcd} \mathrm{Fil}^n A^*(\overline{X}_\bullet, \log D_\bullet) \arrow[r, "\iota_A"] \arrow[d, "\mathrm{qis}"'] & A^*(X_\bullet) \arrow[d, "\mathrm{qis}"', "I"] \\ \mathrm{Fil}^n \mathscr{A}^*(\overline{X}_\bullet, \log D_\bullet) \arrow[r, "\iota_{\mathscr{A}}"] & \mathscr{A}^*(X_\bullet), \end{tikzcd}$$

is commutative up to canonical homotopy.

Definition 2.7. — For a given good compactification $X_\bullet \overset{j}{\hookrightarrow} \overline{X}_\bullet$ write

$$H^{E,*}_{\mathrm{rel}}(X_\bullet, n)_{\overline{X}_\bullet} := H^*\left(\mathrm{Cone}(\mathrm{Fil}^n A^*(\overline{X}_\bullet, \log D_\bullet) \xrightarrow{p^*\circ\iota_A} A^*(E_\bullet))\right),$$
$$H^{*}_{\mathrm{rel}}(X_\bullet, n)_{\overline{X}_\bullet} := H^*\left(\mathrm{Cone}(\mathrm{Fil}^n A^*(\overline{X}_\bullet, \log D_\bullet) \xrightarrow{\iota_A} A^*(X_\bullet))\right),$$

and define $H^{E,*}_{\mathrm{rel}}(X, n) := \varinjlim_{\overline{X}_\bullet} H^{E,*}_{\mathrm{rel}}(X_\bullet, n)_{\overline{X}_\bullet}$, $H^{*}_{\mathrm{rel}}(X, n) := \varinjlim_{\overline{X}_\bullet} H^{*}_{\mathrm{rel}}(X_\bullet, n)_{\overline{X}_\bullet}$ where the limit runs over the direct system of good compactifications of $X_\bullet$.

All the transition maps in the above direct systems are isomorphisms. In particular, for any good compactification $\overline{X}_\bullet$ the groups $H^{E,*}_{\mathrm{rel}}(X_\bullet, n)_{\overline{X}_\bullet}$ and $H^{*}_{\mathrm{rel}}(X_\bullet, n)_{\overline{X}_\bullet}$ are isomorphic to $H^{E,*}_{\mathrm{rel}}(X_\bullet, n)$ and $H^{*}_{\mathrm{rel}}(X_\bullet, n)$, respectively. Moreover, $H^{*}_{\mathrm{rel}}(X_\bullet, n) \cong H^*(X_\bullet, \mathbf{C})/\mathrm{Fil}^n H^*(X_\bullet, \mathbf{C})$.

Obviously there is a morphism $p^*\colon H^*_{\mathrm{rel}}(X_\bullet, n) \to H^{E,*}_{\mathrm{rel}}(X_\bullet, n)$ which yields a morphism of long exact sequences

$$\begin{tikzcd} \dots \arrow[r] & H^{E,i-1}_{\mathrm{rel}}(X_\bullet,n) \arrow[r] & \mathrm{Fil}^n H^i(X_\bullet,\mathbf{C}) \arrow[r] & H^i(E_\bullet,\mathbf{C}) \arrow[r] & H^{E,i}_{\mathrm{rel}}(X_\bullet,n) \arrow[r] & \dots \\ \dots \arrow[r] & H^{i-1}_{\mathrm{rel}}(X_\bullet,n) \arrow[r] \arrow[u, "p^*"] & \mathrm{Fil}^n H^i(X_\bullet,\mathbf{C}) \arrow[r] \arrow[u, equal] & H^i(X_\bullet,\mathbf{C}) \arrow[r] \arrow[u, "p^*"] & H^{i}_{\mathrm{rel}}(X_\bullet,n) \arrow[r] \arrow[u, "p^*"] & \dots . \end{tikzcd} \tag{2.4}$$

Let $f\colon Y_\bullet \to X_\bullet$ be a morphism of simplicial varieties and $E/X_\bullet$ as before. Given good compactifications $X_\bullet \hookrightarrow \overline{X}_\bullet$ and $Y_\bullet \hookrightarrow \overline{Y}_\bullet$, we may construct inductively (similarly as in [**35**, 1.2]) a good compactification $\widetilde{Y}_\bullet$ together with a morphism of compactifications $\widetilde{Y}_\bullet \to \overline{Y}_\bullet$, such that f extends to a morphism $\widetilde{Y}_\bullet \to \overline{X}_\bullet$. Hence we can define pullback maps $f^*\colon H^*_{\mathrm{rel}}(X_\bullet, n) \to H^*_{\mathrm{rel}}(Y_\bullet, n)$ and $f^*\colon H^{E,*}_{\mathrm{rel}}(X_\bullet, n) \to H^{f^*E,*}_{\mathrm{rel}}(Y_\bullet, n)$.

Proposition 2.8. — *There is a unique way to assign to every algebraic* $\mathrm{GL}_r(\mathbf{C})$-*bundle E on a simplicial variety $X_\bullet$ a class* $\widetilde{\mathrm{Ch}}^{\mathrm{rel}}_n(E) \in H^{2n-1,E}_{\mathrm{rel}}(X_\bullet, n)$ *which maps to the n-th Chern character class* $\mathrm{Ch}_n(E)$ *in* $\mathrm{Fil}^n H^{2n}(X_\bullet, \mathbf{C})$ *and which is functorial in $X_\bullet$ in the sense that for every morphism of simplicial varieties $f\colon Y_\bullet \to X_\bullet$ and every algebraic* $\mathrm{GL}_r(\mathbf{C})$*-bundle E on $X_\bullet$ we have* $f^*(\widetilde{\mathrm{Ch}}^{\mathrm{rel}}_n(E)) = \widetilde{\mathrm{Ch}}^{\mathrm{rel}}_n(f^*E)$.

Proof. — Consider the universal situation: Since the geometric realization of $E_\bullet\mathrm{GL}_r(\mathbf{C})$ is contractible $H^i(E_\bullet\mathrm{GL}_r(\mathbf{C}), \mathbf{C})$ vanishes for all $i > 0$. Hence the natural map $H^{E^{\mathrm{univ}},2n-1}_{\mathrm{rel}}(B_\bullet\mathrm{GL}_r(\mathbf{C}), n) \to \mathrm{Fil}^n H^{2n}(B_\bullet\mathrm{GL}_r(\mathbf{C}), \mathbf{C})$ is an isomorphism by the exactness of the top line in (2.4), and the proposition follows. ☐

Now assume that the algebraic bundle $E/X_\bullet$, classified by $g\colon X_\bullet \to B_\bullet\mathrm{GL}_r(\mathbf{C})$, admits a topological trivialization $\alpha\colon T \to E$, i.e. a topological morphism $\alpha\colon X_\bullet \rightsquigarrow E_\bullet\mathrm{GL}_r(\mathbf{C})$ such that $p \circ \alpha = g$. Since $E_\bullet$ is the pullback of $E_\bullet\mathrm{GL}_r(\mathbf{C})$ along g, α induces a topological morphism $\alpha\colon X_\bullet \rightsquigarrow E_\bullet$ such that $p \circ \alpha = \mathrm{id}_{X_\bullet}$. Hence we can define a map $\alpha^*\colon H^{E,*}_{\mathrm{rel}}(X_\bullet, n) \to H^*_{\mathrm{rel}}(X_\bullet, n)$ left inverse to p^*.

Definition 2.9. — $\widetilde{\mathrm{Ch}}^{\mathrm{rel}}_n(E, \alpha) := \alpha^*\widetilde{\mathrm{Ch}}^{\mathrm{rel}}_n(E) \in H^{2n-1}_{\mathrm{rel}}(X_\bullet, n) \cong H^{2n-1}(X_\bullet, \mathbf{C})/\mathrm{Fil}^n$.

Proposition 2.10. — *The class* $\widetilde{\mathrm{Ch}}^{\mathrm{rel}}_n(E, \alpha)$ *maps to the class* $\mathrm{Ch}^{\mathrm{rel}}_n(E, \alpha)$ *by the natural map* $H^{2n-1}(X_\bullet, \mathbf{C})/\mathrm{Fil}^n H^{2n-1}(X_\bullet, \mathbf{C}) \to \mathbb{H}^{2n-1}(X_\bullet, \Omega^{<n}_{X_\bullet})$.

Proof. — Abbreviate $\mathrm{GL}_r(\mathbf{C})$ to G. Let $g\colon X_\bullet \to B_\bullet G$ be the classifying map of E and choose compatible good compactifications $B_\bullet G \hookrightarrow \overline{B_\bullet G}$ and $X_\bullet \hookrightarrow \overline{X}_\bullet$.

Choose any representative c of $\mathrm{Ch}_n(E^{\mathrm{univ}})$ in $\mathrm{Fil}^n \mathscr{A}^{2n}(\overline{B_\bullet G}, \log D_\bullet)$. Then $\iota_{\mathscr{A}}(c) \in \mathscr{A}^{2n}(B_\bullet G)$ lies in $\mathrm{Fil}^n \mathscr{A}^{2n}(B_\bullet G)$ and represents $\mathrm{Ch}_n(E^{\mathrm{univ}})$ considered as a class in $\mathbb{H}^{2n}(B_\bullet G, \Omega^{\geq n}_{B_\bullet G})$. But this class is also represented by the form $I(\mathrm{Ch}_n(\Gamma^{\mathrm{univ}}))$, where Γ^{univ} denotes the standard connection on the universal bundle. Hence there exists $\eta \in \mathrm{Fil}^n \mathscr{A}^{2n-1}(B_\bullet G)$ such that $d\eta = \iota_{\mathscr{A}}(c) - I(\mathrm{Ch}_n(\Gamma^{\mathrm{univ}}))$ and $ch_n := (c, \mathrm{Ch}_n(\Gamma^{\mathrm{univ}}), \eta)$ is a representative for $\mathrm{Ch}_n(E^{\mathrm{univ}})$ in $\mathrm{Fil}^n A^{2n}(\overline{B_\bullet G}, \log D_\bullet)$. With this choice we have

$p^*(\iota_A(ch_n)) = p^*\mathrm{Ch}_n(\Gamma^{\mathrm{univ}}) = d\mathrm{Ch}_n^{\mathrm{rel,univ}}$, where the form $\mathrm{Ch}_n^{\mathrm{rel,univ}}$ was defined before Proposition 1.22. Hence the universal class $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E^{\mathrm{univ}})$ is represented by the cycle $(ch_n, \mathrm{Ch}_n^{\mathrm{rel,univ}})$.

Let $g'\colon E_\bullet \to E_\bullet G$ be the map induced by g on the principal bundles. Then $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E)$ is represented by $(g^*ch_n, g'^*\mathrm{Ch}_n^{\mathrm{rel,univ}})$ and $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E,\alpha)$ is represented by $(g^*ch_n, \alpha^*g'^*\mathrm{Ch}_n^{\mathrm{rel,univ}}) = (g^*ch_n, \alpha^*\mathrm{Ch}_n^{\mathrm{rel,univ}}) = (g^*ch_n, \mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^E,\alpha))$, where on the left we view α as a morphism $X_\bullet \rightsquigarrow E_\bullet$, in the middle as a morphism $X_\bullet \rightsquigarrow E_\bullet G$, Γ^E denotes the standard connection, and we used Proposition 1.22.

Now the natural map

$$\begin{aligned} H^*_{\mathrm{rel}}(X_\bullet, n)_{\overline{X}_\bullet} \cong H^*\left(\mathrm{Cone}(\mathrm{Fil}^n A^*(\overline{X}_\bullet, \log D_\bullet) \xrightarrow{\iota_A} A^*(X_\bullet))\right) \\ \to \mathbb{H}^*(X_\bullet, \Omega^{<n}_{X_\bullet}) = H^*(A^*(X_\bullet)/\mathrm{Fil}^n A^*(X_\bullet)) \end{aligned}$$

is induced by the morphism of complexes $\mathrm{Cone}(\mathrm{Fil}^n A^*(\overline{X}_\bullet, \log D_\bullet) \xrightarrow{\iota_A} A^*(X_\bullet)) \to A^*(X_\bullet)/\mathrm{Fil}^n A^*(X_\bullet)$, $(\omega,\eta) \mapsto \eta$. In particular $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E,\alpha)$ maps to the class represented by $\mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^E,\alpha)$, that is to $\mathrm{Ch}_n^{\mathrm{rel}}(E,\alpha)$. □

2.4. Comparison with Deligne-Beilinson Chern character classes. — Let us first recall the definition and relevant facts about Deligne-Beilinson cohomology [**1, 13**].

Let A be a subring of $\mathbf{R}$ and write $A(n) := (2\pi i)^n A \subseteq \mathbf{C}$. Let $X_\bullet$ be a simplicial algebraic variety and choose a good compactification $j\colon X_\bullet \hookrightarrow \overline{X}_\bullet$.

The *Deligne-Beilinson cohomology* $H^*_{\mathscr{D}}(X_\bullet, A(n))$ of $X_\bullet$ is by definition

$$\mathbb{H}^*\left(\overline{X}_\bullet, \mathrm{Cone}\left(\mathbb{R}j_*A(n) \oplus \mathrm{Fil}^n\Omega^*_{\overline{X}_\bullet}(\log D_\bullet) \xrightarrow{\varepsilon-\iota} \mathbb{R}j_*\Omega^*_{X_\bullet}\right)[-1]\right).$$

By construction we have long exact sequences

$$\begin{aligned} \cdots \to H^k_{\mathscr{D}}(X_\bullet, A(n)) \to H^k(X_\bullet, A(n)) \oplus \mathrm{Fil}^n H^k(X_\bullet, \mathbf{C}) \xrightarrow{\varepsilon-\iota} H^k(X_\bullet, \mathbf{C}) \\ \to H^{k+1}_{\mathscr{D}}(X_\bullet, A(n)) \to \dots \qquad \text{and} \end{aligned} \tag{2.5}$$

$$\cdots \to H^{k-1}(X_\bullet, \mathbf{C})/\mathrm{Fil}^n \to H^k_{\mathscr{D}}(X_\bullet, A(n)) \to H^k(X_\bullet, A(n)) \to \dots \tag{2.6}$$

An *algebraic* vector bundle $\mathscr{E}$ on $X_\bullet$ has Chern character classes $\mathrm{Ch}_n^{\mathscr{D}}(\mathscr{E}) \in H^{2n}_{\mathscr{D}}(X_\bullet, \mathbf{Q}(n))$. These are functorial and mapped to the usual Chern character classes in singular cohomology (to be recalled in 2.4.2 below) by the natural map $H^{2n}_{\mathscr{D}}(X_\bullet, \mathbf{Q}(n)) \to H^{2n}(X_\bullet, \mathbf{Q}(n))$. In fact, these two properties determine them uniquely [**1**, 1.7], [**13**, Prop. 8.2]. Since any algebraic $\mathrm{GL}_r(\mathbf{C})$-bundle E may also be viewed as an algebraic vector bundle, we may also consider the classes $\mathrm{Ch}_n^{\mathscr{D}}(E)$.

Theorem 2.11. — *Let E be an algebraic $\mathrm{GL}_r(\mathbf{C})$-bundle on the simplicial variety $X_\bullet$ and α a trivialization of the associated topological bundle. The relative Chern character class $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E,\alpha)$ maps to $\mathrm{Ch}_n^{\mathscr{D}}(E)$ under the natural map $H^{2n-1}(X_\bullet, \mathbf{C})/\mathrm{Fil}^n H^{2n-1}(X_\bullet, \mathbf{C}) \to H^{2n}_{\mathscr{D}}(X_\bullet, \mathbf{Q}(n))$ from sequence* (2.6).

Before we enter the proof we provide concrete complexes computing Deligne-Beilinson cohomology that are adapted to the setting of topological morphisms (Section 2.4.1) and fix the normalization of Chern classes in singular and hence Deligne-Beilinson cohomology (Section 2.4.2).

2.4.1. Complexes. — First some notation. For an arbitrary manifold Y we denote by $\mathscr{C}^*(Y,A)$ the complex of smooth singular cochains with coefficients in A. We define the complex of *modified differential forms* $\widetilde{\mathscr{A}^*}(Y,A(n))$ to be the quasi-pullback of the diagram

$$\begin{tikzcd} & \mathscr{A}^*(Y) \arrow[d, "\mathrm{qis}"', "\mathscr{I}"] \\ \mathscr{C}^*(Y,A(n)) \arrow[r, hook, "\mathrm{incl}"] & \mathscr{C}^*(Y,\mathbf{C}), \end{tikzcd}$$

where $\mathscr{I}$ denotes the de Rham quasi-isomorphism given by integration over simplices.

Now let $X_\bullet$ be a simplicial manifold. Let $\mathscr{C}^*(X_\bullet,A)$ be the total complex associated with the cosimplicial complex $[p]\mapsto \mathscr{C}^*(X_p,A)$. Then we have a natural isomorphism $H^*(X_\bullet,A)=H^*(\mathscr{C}^*(X_\bullet,A))$.

As in the case of de Rham cohomology, $H^*(X_\bullet,A)$ may also be computed using *compatible singular cochains*: We define the complex of compatible singular cochains $C^*(X_\bullet,A)$ in analogy with that of simplicial differential forms:

$$\begin{aligned} C^n(X_\bullet,A) := \big\{(\sigma_p)_{p\geq 0} \mid \sigma_p\in \mathscr{C}^n(\Delta^p\times X_p,A),\\ (\delta^i\times \mathrm{id})^*\sigma_p = (\mathrm{id}\times\partial_i)^*\sigma_{p-1}, i=0,\dots,p, p\geq 1\big\} \end{aligned}$$

There is a natural quasi-isomorphism $\Phi\colon C^*(X_\bullet,A)\to \mathscr{C}^*(X_\bullet,A)$ given as follows (cf. [**35**, 2.1.3]): For a compatible n-cochain $\sigma=(\sigma_p)_{p\geq 0}$, define $\Phi(\sigma)_{p,n-p}\in\mathscr{C}^{n-p}(X_p,A)$ to be the cochain that sends a singular $(n-p)$-simplex $f\colon \Delta^{n-p}\to X_p$ to $\sigma_p(\mathrm{id}_{\Delta^p}\times f)\in A$. Here $\times$ denotes the cross product of singular chains and $\mathrm{id}_{\Delta^p}\colon\Delta^p\to\Delta^p$ is the canonical singular p-chain. More precisely, to every $(p,n-p)$-shuffle μ corresponds an n-simplex $\mu_*\colon \Delta^n\to\Delta^p\times\Delta^{n-p}$, and the singular chain $\mathrm{id}_{\Delta^p}\times f$ is given by $\sum_\mu \mathrm{sgn}(\mu)(\mathrm{id}_{\Delta^p}\times f)\circ\mu_*$ where the sum runs over all $(p,n-p)$-shuffles μ and the last $\times$ is the usual product of maps (see [**23**, Section 3.B, p. 278–279]).

Integration over simplices induces an integration map $\mathscr{I}\colon A^*(X_\bullet)\to C^*(X_\bullet,\mathbf{C})$ and one checks that the diagram

$$\begin{tikzcd} A^*(X_\bullet) \arrow[r, "\mathscr{I}"] \arrow[d, "I"] & C^*(X_\bullet,\mathbf{C}) \arrow[d, "\Phi"] \\ \mathscr{A}^*(X_\bullet) \arrow[r, "\mathscr{I}"] & \mathscr{C}^*(X_\bullet,\mathbf{C}) \end{tikzcd}$$

commutes (cf. [**36**, Lemma 2.15]).

As before we define the modified complex $\widetilde{A}^*(X_\bullet,A(n))$ as the quasi-pullback of the diagram $C^*(X_\bullet,A(n))\to C^*(X_\bullet,\mathbf{C})\xleftarrow{\mathscr{I}} A^*(X_\bullet)$.

The following follows quite directly from the definitions.

Lemma 2.12. — *Let $X_\bullet$ be a simplicial variety and $X_\bullet \overset{j}{\hookrightarrow} \overline{X}_\bullet$ a good compactification. The Deligne-Beilinson cohomology $H^*_{\mathscr{D}}(X_\bullet, A(n))$ is naturally isomorphic to the cohomology of the complex*

$$\mathrm{Cone}\left(\widetilde{A}^*(X_\bullet, A(n)) \oplus \mathrm{Fil}^n A^*(\overline{X}_\bullet, \log D_\bullet) \xrightarrow{\varepsilon-\iota} A^*(X_\bullet)\right)[-1].$$

The advantage of this description of the Deligne-Beilinson cohomology of simplicial varieties is that we may define a pullback map $\alpha^*\colon \widetilde{A}^*(X_\bullet, A(n)) \to \widetilde{A}^*(Y_\bullet, A(n))$, whenever $\alpha\colon Y_\bullet \rightsquigarrow X_\bullet$ is a topological morphism:

Lemma 2.13. — *Let $\alpha\colon Y_\bullet \rightsquigarrow X_\bullet$ be a topological morphism of simplicial manifolds. Then there is a well defined pullback map $\alpha^*\colon \widetilde{A}^*(X_\bullet, A(n)) \to \widetilde{A}^*(Y_\bullet, A(n))$. It is compatible with the natural maps $\widetilde{A}^* \to A^*$.*

Proof. — By definition $\widetilde{A}^*(X_\bullet, A(n))$ is the quasi-pullback of the diagram $C^*(X_\bullet, A(n)) \to C^*(X_\bullet, \mathbf{C}) \xleftarrow{\mathscr{I}} A^*(X_\bullet)$. Obviously, α^* is well defined on each of the three complexes (cf. (1.3) and Definition 1.8) and we only have to check, that it is compatible with the maps between them. This is is clear for the left hand map. For $\mathscr{I}$ this follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathscr{A}^n(\Delta^p \times X_p) \arrow[r, "(\mathrm{id}_{\Delta^p}, \alpha_p)^*"] \arrow[d, "\mathscr{I}"] & \mathscr{A}^n(\Delta^p \times Y_p) \arrow[d, "\mathscr{I}"] \\
\mathscr{C}^n(\Delta^p \times X_p, \mathbf{C}) \arrow[r, "(\mathrm{id}_{\Delta^p}, \alpha_p)^*"] & \mathscr{C}^n(\Delta^p \times Y_p, \mathbf{C})
\end{tikzcd}$$

which is established as follows: Let $\omega \in \mathscr{A}^n(\Delta^p \times X_p)$ and $\tau : \Delta^n \to \Delta^p \times Y_p$ be a smooth simplex. Then $(\mathrm{id}_{\Delta^p}, \alpha_p)^* \mathscr{I}(\omega)(\tau) = \int_{\Delta^n} ((\mathrm{id}_{\Delta^p}, \alpha_p) \circ \tau)^* \omega = \int_{\Delta^n} \tau^*((\mathrm{id}_{\Delta^p}, \alpha_p)^* \omega) = \mathscr{I}((\mathrm{id}_{\Delta^p}, \alpha_p)^* \omega)(\tau)$. □

2.4.2. Chern character classes. — We recall the defintion of Chern classes in singular cohomology.

Definition 2.14. — Let X be a simplicial complex manifold. The first Chern class c_1^{top} in singular cohomology for holomorphic line bundles is the connecting homomorphism

$$c_1^{\mathrm{top}}\colon H^1(X, \mathscr{O}_X^*) \to H^2(X, \mathbf{Z}(1))$$

associated with the short exact sequence of sheaves on X

$$0 \to \mathbf{Z}(1) \to \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^* \to 0.$$

Remark 2.15. — If $c_1^{\text{Milnor-Stasheff}}$ denotes the classical integer valued first Chern class as constructed in [**30**] then $c_1^{\mathrm{top}} = -2\pi i c_1^{\text{Milnor-Stasheff}}$. This follows e.g. from [**30**, Appendix C, Theorem (p. 306)] together with [**20**, Ch. I §1, Proposition (p. 141)].

For later reference we note that Burgos [**6**] uses Milnor-Stasheff's normalization for his integer valued Chern classes b_i and defines the "twisted Chern classes" $c_i^{\mathrm{Burgos}} := (2\pi i)^i b_i$. In fact, the construction in [**6**, Section 4.2] is exactly the same as that in [**30**, §14] (alternatively, one may look at the Chern-Weil theoretic approach in [**6**, Proposition 5.27]). In particular, $c_1^{\mathrm{top}} = -c_1^{\mathrm{Burgos}}$ and we have corresponding signs for the higher Chern and Chern character classes.

The splitting principle also holds for singular cohomology and one constructs higher Chern and Chern character classes $c_n^{\mathrm{top}}(\mathscr{E}) \in H^{2n}(X_\bullet, \mathbf{Z}(n))$, $\mathrm{Ch}_n^{\mathrm{top}}(\mathscr{E}) \in H^{2n}(X_\bullet, \mathbf{Q}(n))$ for holomorphic vector bundles $\mathscr{E}$ as in Section 2.2.2.

Remark 2.16. — It is easy to see that for a holomorphic line bundle $\mathscr{L}$ the classes $c_1(\mathscr{L})$ and $c_1^{\mathrm{top}}(\mathscr{L})$ have the same image in $H^2(X, \mathbf{C})$. In particular, if $\mathscr{E}$ is a holomorphic vector bundle, the higher Chern and Chern character classes $c_n^{\mathrm{top}}(\mathscr{E})$, $\mathrm{Ch}_n^{\mathrm{top}}(\mathscr{E})$ map to $c_n(\mathscr{E})$, $\widetilde{\mathrm{Ch}}_n(\mathscr{E}) \in \mathrm{Fil}^n H^{2n}(X, \mathbf{C})$ under the natural map $H^{2n}(X, \mathbf{Z}(n)) \to H^{2n}(X, \mathbf{C})$.

Proof of Theorem 2.11. — Let $X_\bullet, E$, and α be as in the statement of the Theorem and $\overline{X}_\bullet$ some good compactification of $X_\bullet$. The natural morphism

$$H^{*-1}_{\mathrm{rel}}(X_\bullet, n) = H^{*-1}(X_\bullet, \mathbf{C})/\mathrm{Fil}^n H^{*-1}(X_\bullet, \mathbf{C}) \to H^*_{\mathscr{D}}(X_\bullet, \mathbf{Q}(n))$$

is induced on the defining cones by the maps in the commutative diagram

$$\begin{tikzcd} \mathrm{Fil}^n A^*(\overline{X}_\bullet, \log D_\bullet) \arrow[r, "\iota"] \arrow[d, "\mathrm{incl.}"'] & A^*(X_\bullet) \arrow[d, "-\mathrm{id}"] \\ \widetilde{A}^*(X_\bullet, \mathbf{Q}(n)) \oplus \mathrm{Fil}^n A^*(\overline{X}_\bullet, \log D_\bullet) \arrow[r, "\varepsilon - \iota"] & A^*(X_\bullet). \end{tikzcd}$$

Denote by $E_\bullet \xrightarrow{p} X_\bullet$ the principal bundle associated with E and define

$$H^{E,*}_{\mathscr{D}}(X_\bullet, \mathbf{Q}(n)) := \varinjlim_{\overline{X}_\bullet} H^* \left(\mathrm{Cone} \left(\widetilde{A}^*(E_\bullet, \mathbf{Q}(n)) \oplus \mathrm{Fil}^n A^*(\overline{X}_\bullet, \log D_\bullet) \xrightarrow{\varepsilon - p^* \circ \iota} A^*(E_\bullet) \right) [-1] \right),$$

the limit running over the good compactifications of $X_\bullet$. As in the case of relative cohomology groups, we have a natural map $p^* \colon H^*_{\mathscr{D}}(X_\bullet, \mathbf{Q}(n)) \to H^{E,*}_{\mathscr{D}}(X_\bullet, \mathbf{Q}(n))$ and a left inverse α^* of p^* for a topological trivialization α of E. Moreover, there is a natural map $H^{E,*-1}_{\mathrm{rel}}(X_\bullet, n) \to H^{E,*}_{\mathscr{D}}(X_\bullet, \mathbf{Q}(n))$ fitting in a commutative diagram (in the obvious sense)

$$\begin{tikzcd} H^{E,*-1}_{\mathrm{rel}}(X_\bullet, n) \arrow[r] \arrow[d, bend left, "\alpha^*"] & H^{E,*}_{\mathscr{D}}(X_\bullet, \mathbf{Q}(n)) \arrow[d, bend left, "\alpha^*"] \\ H^{*-1}_{\mathrm{rel}}(X_\bullet, n) \arrow[r] \arrow[u, "p^*"] & H^*_{\mathscr{D}}(X_\bullet, \mathbf{Q}(n)). \arrow[u, "p^*"] \end{tikzcd}$$

We claim that the refined class $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E)$ maps to $p^*\mathrm{Ch}_n^{\mathscr{D}}(E)$ by the upper horizontal map. Since both classes are functorial it suffices to treat the case of the universal bundle $E^{\mathrm{univ}}/B_\bullet\mathrm{GL}_r(\mathbf{C})$. Write $G := \mathrm{GL}_r(\mathbf{C})$. Since the cohomology of $E_\bullet G$ vanishes in positive degrees and the cohomology of $B_\bullet G$ vanishes in odd degrees we have the following commutative diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & H_{\mathscr{D}}^{E^{\mathrm{univ}},2n}(B_\bullet G, \mathbf{Q}(n)) \arrow[r] & \mathrm{Fil}^n H^{2n}(B_\bullet G, \mathbf{C}) \arrow[r] & 0 \\ 0 \arrow[r] & H_{\mathscr{D}}^{2n}(B_\bullet G, \mathbf{Q}(n)) \arrow[r] \arrow[u, "p^*"] & H^{2n}(B_\bullet G, \mathbf{Q}(n)) \oplus \mathrm{Fil}^n H^{2n}(B_\bullet G, \mathbf{C}) \arrow[r, "\varepsilon-\iota"] \arrow[u, "\mathrm{pr}_2"] & H^{2n}(B_\bullet G, \mathbf{C}). \end{tikzcd}$$

By definition, $\mathrm{Ch}_n^{\mathscr{D}}(E^{\mathrm{univ}})$ maps to $\mathrm{Ch}_n^{\mathrm{top}}(E^{\mathrm{univ}})$ in $H^{2n}(B_\bullet G, \mathbf{Q}(n))$. Since $\varepsilon(\mathrm{Ch}_n^{\mathrm{top}}(E^{\mathrm{univ}})) = \iota(\mathrm{Ch}_n(E^{\mathrm{univ}}))$ (cf. Proposition 2.6 and Remark 2.16), it follows from the above diagram that $p^*\mathrm{Ch}_n^{\mathscr{D}}(E^{\mathrm{univ}})$ maps to $\mathrm{Ch}_n(E^{\mathrm{univ}})$ in $\mathrm{Fil}^n H^{2n}(B_\bullet G, \mathbf{C})$. The defining property of $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E^{\mathrm{univ}})$ and the commutativity of the diagram

$$\begin{tikzcd} H_{\mathrm{rel}}^{E^{\mathrm{univ}},2n-1}(B_\bullet G, n) \arrow[d] \arrow[dr, "\cong"] & \\ H_{\mathscr{D}}^{E^{\mathrm{univ}},2n}(B_\bullet G, \mathbf{Q}(n)) \arrow[r, "\cong"] & \mathrm{Fil}^n H^{2n}(B_\bullet G, \mathbf{C}) \end{tikzcd}$$

imply that $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E^{\mathrm{univ}})$ maps to $p^*\mathrm{Ch}_n^{\mathscr{D}}(E^{\mathrm{univ}})$, whence our claim.

But then $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E, \alpha) = \alpha^*\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E)$ maps to $\alpha^*p^*\mathrm{Ch}_n^{\mathscr{D}}(E) = \mathrm{Ch}_n^{\mathscr{D}}(E)$. $\square$

3. Relative K-theory and regulators

Let $X = \mathrm{Spec}(A)$ be a smooth affine scheme of finite type over $\mathbf{C}$. Then the algebraic and topological K-theory of X resp. its underlying complex manifold are given (for $i > 0$) by

$$K_i(X) = \pi_i(B\mathrm{GL}(A)^+) \quad \text{resp.} \quad K_{\mathrm{top}}^{-i}(X) = \pi_i(BU^X)$$

and there is a natural morphism $B\mathrm{GL}(A)^+ \to BU^X$ in the homotopy category of spaces. We define the *relative K-group* $K_i^{\mathrm{rel}}(X)$ as the i-th homotopy group of the homotopy fibre of this map. The goal of this Section is to construct relative Chern character maps $\mathrm{Ch}_{n,i}^{\mathrm{rel}}\colon K_i^{\mathrm{rel}}(X) \to H^{2n-i-1}(X, \mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X, \mathbf{C})$ and to compare these with the Chern character in Deligne-Beilinson cohomology.

3.1. Topological K-theory. — Our first task is to give an adequate simplicial model for the topological K-groups of a manifold X in terms of smooth maps $\Delta^p \times X \to \mathrm{GL}_r(\mathbf{C})$ in order to be able to apply our theory of topological bundles.

Let X be a smooth manifold having the structure of a finite dimensional CW complex. We call a singular p-simplex in the mapping space $\mathrm{GL}_r(\mathbf{C})^X$ *smooth* if the associated map $\Delta^p \times X \to \mathrm{GL}_r(\mathbf{C})$ is smooth. Denote by $S^\infty_\bullet(\mathrm{GL}_r(\mathbf{C})^X)$ the simplicial set of smooth singular simplices in $\mathrm{GL}_r(\mathbf{C})^X$. This is a simplicial group in a natural way. Write $G_\bullet = \varinjlim_r S^\infty_\bullet(\mathrm{GL}_r(\mathbf{C})^X)$ and let $B_\bullet G_\bullet$ be its classifying simplicial set, i.e. the diagonal of the bisimplicial set $[p],[q] \mapsto B_p G_q$.

Proposition 3.1. — *There are natural isomorphisms $K^{-i}_{\mathrm{top}}(X) = \pi_i(B_\bullet G_\bullet)$.*

Proof. — It is well known that for a manifold Y the inclusion $S^\infty_\bullet(Y) \hookrightarrow S_\bullet(Y)$ of smooth singular simplices in the full simplicial set of singular simplices is a homotopy equivalence. Similarly one shows that the inclusion $S^\infty_\bullet(\mathrm{GL}_r(\mathbf{C})^X) \hookrightarrow S_\bullet(\mathrm{GL}_r(\mathbf{C})^X)$ is a homotopy equivalence (see [**36**, Prop. 3.2] for details). Hence we have isomorphisms

$$\begin{aligned}\pi_i(B_\bullet G_\bullet) &\cong \pi_{i-1}(G_\bullet) \cong \varinjlim_r \pi_{i-1}(S^\infty_\bullet(\mathrm{GL}_r(\mathbf{C})^X)) \cong \varinjlim_r \pi_{i-1}(S_\bullet(\mathrm{GL}_r(\mathbf{C})^X)) \\ &\cong \varinjlim_r \pi_{i-1}(\mathrm{GL}_r(\mathbf{C})^X) \cong \varinjlim_r \pi_{i-1}(U(r)^X) \cong \varinjlim_r \pi_i(BU(r)^X),\end{aligned}$$

where we used the fact, that $BU(r)^X$ is a classifying space for $U(r)^X$ (cf. the argument in the proof of [**19**, Lemma in Section 6.1]). Since X is a finite dimensional CW complex it follows by cellular approximation that $\varinjlim_r \pi_i(BU(r)^X) = \pi_i(BU^X) = K^{-i}_{\mathrm{top}}(X)$ finishing the proof of the proposition. □

3.2. Relative K-theory. — Now let $X = \mathrm{Spec}(A)$ be a smooth affine scheme of finite type over $\mathbf{C}$. By abuse of notation we denote the associated complex manifold by the same letter. Note that X has the structure of a finite dimensional CW complex, so our above description of the topological K-theory of X applies.

The natural map from A to the ring of smooth complex valued functions $\mathscr{C}^\infty(X)$ on X induces a map from the constant simplicial group $\mathrm{GL}_r(A)$ to $S^\infty_\bullet(\mathrm{GL}_r(\mathbf{C})^X)$ and hence, taking the limit over r and classifying simplicial sets, $B_\bullet\mathrm{GL}(A) \to B_\bullet G_\bullet$.

The *algebraic K-groups* of X are by definition

$$K_i(X) = \pi_i(|B_\bullet \mathrm{GL}(A)|^+), \qquad i > 0,$$

where $|B_\bullet\mathrm{GL}(A)|^+$ denotes Quillen's plus-construction with respect to the commutator subgroup $\mathrm{GL}(A)'$. A functorial version is given by the integral completion functor $\mathbf{Z}_\infty$ of Bousfield and Kan [**5**] (see [**15**, Theorem 2.16]): $K_i(X) = \pi_i(\mathbf{Z}_\infty B_\bullet\mathrm{GL}(A))$. Since $B_\bullet G_\bullet$ has the homotopy type of an H-space the natural map $B_\bullet G_\bullet \to \mathbf{Z}_\infty B_\bullet G_\bullet$ is a weak homotopy equivalence [**15**, 2.15]. The desired map from algebraic to topological K-theory $K_i(X) \to K^{-i}_{\mathrm{top}}(X)$ is the map induced by $\mathbf{Z}_\infty B_\bullet\mathrm{GL}(A) \to \mathbf{Z}_\infty B_\bullet G_\bullet$ on homotopy groups.

Define F and $\widetilde{F}$ by the pull-back diagrams

$$\begin{tikzcd} F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \widetilde{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbf{Z}_\infty E_\bullet G_\bullet \arrow[d, "\mathbf{Z}_\infty p"] \\ B_\bullet\mathrm{GL}(A) \arrow[r] & \mathbf{Z}_\infty B_\bullet\mathrm{GL}(A) \arrow[r] & \mathbf{Z}_\infty B_\bullet G_\bullet. \end{tikzcd} \tag{3.1}$$

According to [**5**, I 4.2] $\mathbf{Z}_\infty p$ is a fibration and so are the other two vertical arrows. Then, since $B_\bullet\mathrm{GL}(A) \to \mathbf{Z}_\infty B_\bullet\mathrm{GL}(A)$ is acyclic, so is $F \to \widetilde{F}$ [**2**, (4.1)]. Since $E_\bullet G_\bullet$ is contractible, so is $\mathbf{Z}_\infty E_\bullet G_\bullet$ and hence $\widetilde{F}$ is weakly homotopy equivalent to the homotopy fibre of the map $\mathbf{Z}_\infty B_\bullet\mathrm{GL}(A) \to \mathbf{Z}_\infty B_\bullet G_\bullet$ and we define the *relative K-groups*

$$K_i^{\mathrm{rel}}(X) := \pi_i(\widetilde{F}), \qquad i > 0.$$

By construction we have a long exact sequence

$$\cdots \to K_{\mathrm{top}}^{-i-1}(X) \to K_i^{\mathrm{rel}}(X) \to K_i(X) \to K_{\mathrm{top}}^{-i}(X) \to \dots. \tag{3.2}$$

Example 3.2. — Consider the case of a point: $X = \mathrm{Spec}(\mathbf{C})$. The topological K-groups of X equal $\mathbf{Z}$ in even degrees and vanish in odd degrees. Hence we get exact sequences

$$0 \to K_{2n}^{\mathrm{rel}}(X) \to K_{2n}(X) \to \mathbf{Z} \to K_{2n-1}^{\mathrm{rel}}(X) \to K_{2n-1}(X) \to 0, \quad n > 0.$$

We can say even more: One knows that for any smooth, projective $\mathbf{C}$-scheme Y and $i > 0$ the image of $K_i(Y)$ in $K_{\mathrm{top}}^{-i}(Y)$ is torsion [**19**, 6.3] . For $X = \mathrm{Spec}(\mathbf{C})$ this and the above sequence imply that we have isomorphisms

$$K_{2n}^{\mathrm{rel}}(X) \cong K_{2n}(X), \quad n > 0,$$

in even degrees and short exact sequences

$$0 \to \mathbf{Z} \to K_{2n-1}^{\mathrm{rel}}(X) \to K_{2n-1}(X) \to 0, \quad n > 0,$$

in odd degrees.

We resume the discussion before Example 3.2. We need the following description of the homology of $\widetilde{F}$. Define $\mathscr{F}$ by the pull-back diagram of simplicial sets:

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E_\bullet G_\bullet \arrow[d, "p"] \\ B_\bullet\mathrm{GL}(A) \arrow[r] & B_\bullet G_\bullet \end{tikzcd}$$

Then the natural map $\mathscr{F} \to F$ is a weak homotopy equivalence, too, and since $F \to \widetilde{F}$ is acyclic, we have isomorphisms in homology

$$H_*(\mathscr{F}, \mathbf{Z}) \xrightarrow{\cong} H_*(F, \mathbf{Z}) \xrightarrow{\cong} H_*(\widetilde{F}, \mathbf{Z}).$$

3.3. The relative Chern character. — Let $X = \mathrm{Spec}(A)$ be as before. We define relative Chern character maps

$$\mathrm{Ch}_{n,i}^{\mathrm{rel}} \colon K_i^{\mathrm{rel}}(X) \to H^{2n-i-1}(X, \mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X, \mathbf{C})$$

as follows: By definition, $K_i^{\mathrm{rel}}(X) = \pi_i(\widetilde{F})$, and we have the Hurewicz map $K_i^{\mathrm{rel}}(X) \to H_i(\widetilde{F}, \mathbf{Z}) \cong H_i(\mathscr{F}, \mathbf{Z})$. It is thus enough to construct a homomorphism $H_i(\mathscr{F}, \mathbf{Z}) \to H_{\mathrm{rel}}^{2n-i-1}(X, n) = H^{2n-i-1}(X, \mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X, \mathbf{C})$. We will use the following

Lemma 3.3. — *Let S be a simplicial set and X an algebraic variety. Form the simplicial variety $X_\bullet := X \otimes S$ as in Example 1.6. Then we have natural isomorphisms*

$$H^k_{\mathrm{rel}}(X_\bullet, n) \cong \bigoplus_{p+q=k} \mathrm{Hom}(H_p(S,\mathbf{Z}), H^q(X,\mathbf{C})/\mathrm{Fil}^n H^q(X,\mathbf{C})),$$
$$H^k_{\mathscr{D}}(X_\bullet, \mathbf{Q}(n)) \cong \bigoplus_{p+q=k} \mathrm{Hom}(H_p(S,\mathbf{Z}), H^q_{\mathscr{D}}(X,\mathbf{Q}(n))).$$

Proof. — This is standard and follows easily from the explicit form of the complexes in question. □

Remark 3.4. — A similar statement also holds for the group $\mathbb{H}^k(X_\bullet, \Omega^{<n}_{X_\bullet})$, which is computed by the complex $A^*(X_\bullet)/\mathrm{Fil}^n A^*(X_\bullet)$. We have a commutative diagram

$$\begin{tikzcd}
H^k(X_\bullet,\mathbf{C})/\mathrm{Fil}^n H^k(X_\bullet,\mathbf{C}) \arrow[r] \arrow[d] & \mathbb{H}^k(X_\bullet, \Omega^{<n}_{X_\bullet}) \arrow[d] \\
\mathrm{Hom}(H_p(S,\mathbf{Z}), H^{k-p}(X,\mathbf{C})/\mathrm{Fil}^n) \arrow[r] & \mathrm{Hom}(H_p(S,\mathbf{Z}), \mathbb{H}^{k-p}(X, \Omega^{<n}_X))
\end{tikzcd}$$

and the right vertical arrow is given explicitly as follows: A class in $\mathbb{H}^k(X_\bullet, \Omega^{<n}_{X_\bullet})$ may be represented by a form $\omega \in A^k(X_\bullet)$, closed modulo $\mathrm{Fil}^n A^{k+1}(X_\bullet)$. The simplicial form ω is given by a family of k-forms on $\Delta^q \times (X\otimes S)_q$, $q \geq 0$, and in particular we can consider the restriction $\sigma^*\omega$ of ω_p to the copy of $\Delta^p \times X$ corresponding to $\sigma \in S_p$. Integration along Δ^p gives the $(k-p)$-form $\int_\sigma \omega = \int_{\Delta^p} \sigma^*\omega \in \mathscr{A}^{k-p}(X)$. By linearity this extends to a map $\mathbf{Z}S_p \to \mathscr{A}^{k-p}(X)$, $\sigma \mapsto \int_{\Delta^p} \sigma^*\omega$, which induces a well defined homomorphism $H_p(S,\mathbf{Z}) \to H^{k-p}(\mathscr{A}^*(X)/\mathrm{Fil}^n \mathscr{A}^*(X)) = \mathbb{H}^{k-p}(X, \Omega^{<n}_X)$.

To construct the relative Chern character map on K-theory we thus have to construct classes in $H^{2n-1}(X\otimes\mathscr{F}, \mathbf{C})/\mathrm{Fil}^n H^{2n-1}(X\otimes\mathscr{F},\mathbf{C})$. This is achieved as follows. First write $G_{r,\bullet} := S^\infty_\bullet(\mathrm{GL}_r(\mathbf{C})^X)$, so that $G_\bullet = \varinjlim_r G_{r,\bullet}$, and define $\mathscr{F}_r$ by the cartesian diagram of simplicial sets

$$\begin{tikzcd}
\mathscr{F}_r \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E_\bullet G_{r,\bullet} \arrow[d, "p"] \\
B_\bullet \mathrm{GL}_r(A) \arrow[r] & B_\bullet G_{r,\bullet}.
\end{tikzcd} \tag{3.3}$$

Then $\mathscr{F} = \varinjlim_r \mathscr{F}_r$, $H_*(\mathscr{F},\mathbf{Z}) = \varinjlim_r H_*(\mathscr{F}_r,\mathbf{Z})$ and by the Lemma

$$H^*(X\otimes\mathscr{F},\mathbf{C})/\mathrm{Fil}^n = \varprojlim_r H^*(X\otimes\mathscr{F}_r,\mathbf{C})/\mathrm{Fil}^n.$$

By construction, a p-simplex in the simplicial group $G_{r,\bullet}$ is a smooth map $\Delta^p \times X \to \mathrm{GL}_r(\mathbf{C})$, and a p-simplex in $E_\bullet G_{r,\bullet}$ may be viewed as a smooth map $\Delta^p \times X \to E_p\mathrm{GL}_r(\mathbf{C})$. On the other hand, every p-simplex in $B_\bullet \mathrm{GL}_r(A)$ may be seen as a morphism of varieties $X \to B_p\mathrm{GL}_r(\mathbf{C})$. As in Example 1.10 diagram (3.3) then gives rise to a commutative diagram

$$\begin{tikzcd} & E_\bullet \mathrm{GL}_r(\mathbf{C}) \arrow[d, "p"] \\ X \otimes \mathscr{F}_r \arrow[ur, rightsquigarrow, "\alpha_r"] \arrow[r, "g_r"] & B_\bullet \mathrm{GL}_r(\mathbf{C}), \end{tikzcd}$$

where g_r is a morphism of simplicial varieties.

Phrased differently, if we denote by E_r the algebraic bundle classified by $g_r \colon X \otimes \mathscr{F}_r \to B_\bullet\mathrm{GL}_r(\mathbf{C})$ and by T_r the trivial $\mathrm{GL}_r(\mathbf{C})$-bundle, we have the trivialization $\alpha_r \colon T_r \to E_r$ of the underlying topological bundles and corresponding relative Chern character classes $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E_r, \alpha_r) \in H^{2n-1}_{\mathrm{rel}}(X \otimes \mathscr{F}_r, n) = H^{2n-1}(X \otimes \mathscr{F}_r, \mathbf{C})/\mathrm{Fil}^n H^{2n-1}(X \otimes \mathscr{F}_r, \mathbf{C})$. It is easy to see that these classes are compatible for different r [**36**, Lemma 3.7]. Hence the family $(\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E_r, \alpha_r))_{r\geq 0}$ defines a class in $H^{2n-1}(X \otimes \mathscr{F}, \mathbf{C})/\mathrm{Fil}^n H^{2n-1}(X \otimes \mathscr{F}, \mathbf{C})$. By Lemma 3.3 this class gives morphisms $H_i(\mathscr{F}, \mathbf{Z}) \to H^{2n-i-1}(X, \mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X, \mathbf{C})$, $i = 0, \dots, 2n-1$.

Definition 3.5. — Let X be a smooth, affine $\mathbf{C}$-scheme of finite type as before. We define the relative Chern character $\mathrm{Ch}^{\mathrm{rel}}_{n,i}$ on $K_i^{\mathrm{rel}}(X)$ to be the composition

$$\begin{aligned}\mathrm{Ch}^{\mathrm{rel}}_{n,i} \colon K_i^{\mathrm{rel}}(X) = \pi_i(\widetilde{F}) \xrightarrow{\mathrm{Hur.}} H_i(\widetilde{F}, \mathbf{Z}) \cong H_i(\mathscr{F}, \mathbf{Z}) \to \\ \to H^{2n-i-1}(X, \mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X, \mathbf{C}).\end{aligned}$$

Remarks 3.6. — (i) If in the above construction one replaces $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E_r, \alpha_r)$ with the Chern-Weil theoretic classes $\mathrm{Ch}_n^{\mathrm{rel}}(E_r, \alpha_r)$ one gets relative Chern character maps $K_i^{\mathrm{rel}}(X) \to \mathbb{H}^{2n-i-1}(X, \Omega_X^{<n})$, which are essentially Karoubi's original ones. Obviously, these are just the composition of $\mathrm{Ch}^{\mathrm{rel}}_{n,i}$ with the natural map $H^{2n-i-1}(X, \mathbf{C})/\mathrm{Fil}^n \to \mathbb{H}^{2n-i-1}(X, \Omega_X^{<n})$.

(ii) Let $X = \mathrm{Spec}(A)$ be an affine variety as above. Karoubi [**27, 26**] developed a theory of bundles, connections, and characteristic classes for $\mathrm{GL}_r(A)$-fibre bundles on simplicial sets S which enabled him to construct the relative Chern character on relative K-theory. In our setting, these bundles correspond to $\mathrm{GL}_r(\mathbf{C})$-bundles on the simplicial variety $X \otimes S$ (cf. Example 1.6). To compare the relative Chern character with the Chern character in Deligne-Beilinson cohomology however, it is necessary to extend the theory to general simplicial varieties.

3.4. Comparison with the Chern character in Deligne-Beilinson cohomology. — The Chern character in Deligne-Beilinson cohomology is constructed in exactly the same way as the relative Chern character above (cf. e.g. [**34**, 2.3]):

Let $X = \mathrm{Spec}(A)$ be a smooth affine $\mathbf{C}$-scheme of finite type as in the previous Section. Again we have the natural morphisms of simplicial varieties $X \otimes B_\bullet \mathrm{GL}_r(A) \to$

$B_\bullet\mathrm{GL}_r(\mathbf{C})$. Call the corresponding algebraic bundle G_r. As in the relative case the Chern character classes $\mathrm{Ch}_n^{\mathscr{D}}(G_r) \in H_{\mathscr{D}}^{2n}(X \otimes B_\bullet\mathrm{GL}_r(A), \mathbf{Q}(n))$ are compatible for different r and thus yield a well defined class in $H_{\mathscr{D}}^{2n}(X \otimes B_\bullet\mathrm{GL}(A), \mathbf{Q}(n))$. This class in turn yields maps $H_i(B_\bullet\mathrm{GL}(A), \mathbf{Z}) \to H_{\mathscr{D}}^{2n-i}(X, \mathbf{Q}(n))$ and, for $i > 0$, we define the Chern character maps $\mathrm{Ch}_{n,i}^{\mathscr{D}}$ on K-theory to be the composition

$$\begin{aligned}\mathrm{Ch}_{n,i}^{\mathscr{D}} \colon K_i(X) = \pi_i(\mathbf{Z}_\infty B_\bullet\mathrm{GL}(A)) \xrightarrow{\text{Hur.}} H_i(\mathbf{Z}_\infty B_\bullet\mathrm{GL}(A), \mathbf{Z}) \cong \\ \cong H_i(B_\bullet\mathrm{GL}(A), \mathbf{Z}) \to H_{\mathscr{D}}^{2n-i}(X, \mathbf{Q}(n)).\end{aligned}$$

Theorem 3.7. — *Let X be a smooth affine $\mathbf{C}$-scheme of finite type. The diagram*

$$\begin{tikzcd} K_i^{\mathrm{rel}}(X) \arrow[r] \arrow[d, "\mathrm{Ch}_{n,i}^{\mathrm{rel}}"] & K_i(X) \arrow[d, "\mathrm{Ch}_{n,i}^{\mathscr{D}}"] \\ H^{2n-i-1}(X, \mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X, \mathbf{C}) \arrow[r] & H_{\mathscr{D}}^{2n-i}(X, \mathbf{Q}(n)) \end{tikzcd}$$

commutes.

Proof. — This is now an easy consequence of Theorem 2.11 and the constructions.

We use the notations of the last two Sections. Then $E_r/X \otimes \mathscr{F}_r$ is just the pullback of $G_r/X \otimes B_\bullet\mathrm{GL}_r(A)$ by the morphism $X \otimes \mathscr{F}_r \to X \otimes B_\bullet\mathrm{GL}_r(A)$. It follows from Theorem 2.11 and functoriality that $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E_r, \alpha_r) \in H^{2n-1}(X \otimes \mathscr{F}_r, \mathbf{C})/\mathrm{Fil}^n$ and $\mathrm{Ch}_n^{\mathscr{D}}(G_r) \in H_{\mathscr{D}}^{2n}(X \otimes B_\bullet\mathrm{GL}(A), \mathbf{Q}(n))$ map to the same class in $H_{\mathscr{D}}^{2n}(X \otimes \mathscr{F}_r, \mathbf{Q}(n))$, namely to $\mathrm{Ch}_n^{\mathscr{D}}(E_r)$. Hence we have commutative diagrams

$$\begin{tikzcd} H_i(\mathscr{F}_r, \mathbf{Z}) \arrow[r, "\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E_r,\alpha_r)"] \arrow[d] \arrow[dr, "\mathrm{Ch}_n^{\mathscr{D}}(E_r)"] & H^{2n-i-1}(X, \mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X, \mathbf{C}) \arrow[d] \\ H_i(B_\bullet\mathrm{GL}_r(A)), \mathbf{Z}) \arrow[r, "\mathrm{Ch}_n^{\mathscr{D}}(G_r)"'] & H_{\mathscr{D}}^{2n-i}(X, \mathbf{Q}(n)), \end{tikzcd}$$

where the arrows are induced by the specified classes. Going to the limit $r \to \infty$ and using the commutativity of diagram (3.1) the claim follows. □

3.5. Extension to non-affine schemes. — Using Jouanolou's trick we extend the construction of the relative Chern character to all smooth, separated schemes of finite type over $\mathbf{C}$ (cf. [**40**, §4], [**19**, §6]).

By a *Jouanolou torsor* over a scheme X we mean an affine scheme W together with an affine map $W \to X$ which is a torsor for some vector bundle on X. According to Jouanolou and Thomason every smooth, separated scheme of finite type over a field admits a Jouanolou torsor [**40**, Proposition 4.4].

Let X be a smooth variety over $\mathbf{C}$ and fix a Jouanolou torsor $\pi \colon W \to X$. By the homotopy invariance of Quillen's K-theory for regular schemes [**31**, §7 Proposition 4.1] and the homotopy invariance of topological K-theory π induces isomorphisms

$\pi^*\colon K_*(X) \xrightarrow{\cong} K_*(W)$ and $\pi^*\colon K^*_{\mathrm{top}}(X) \xrightarrow{\simeq} K^*_{\mathrm{top}}(W)$. It follows, that if we define $K^{\mathrm{rel}}_*(X)_W := K^{\mathrm{rel}}_*(W)$, we get exact sequences

$$\cdots \to K^{-i-1}_{\mathrm{top}}(X) \to K^{\mathrm{rel}}_i(X)_W \to K_i(X) \to K^{-i}_{\mathrm{top}}(X) \to \dots$$

as in (3.2). To get a definition of relative K-theory which does not depend on the particular choice of W we proceed as follows: If $W' \to X$ is a second Jouanolou torsor, so is $W'' := W \times_X W'$ and both maps $W \leftarrow W'' \to W'$ induce isomorphisms on algebraic, topological and hence also relative K-groups. To avoid set theoretic problems we replace the category of Jouanolou torsors over X by a small skeletal subcategory. Then we can consider the set $\mathcal{J}$ of all finite sets of Jouanolou torsors over X. This is partially ordered by inclusion. For any $A \in \mathcal{J}$ write $K^{\mathrm{rel}}_i(X)_A := K^{\mathrm{rel}}_i(\prod_{W\in A} W)$ where $\prod$ denotes the fibered product over X. Any inclusion $A \subseteq B$ induces an isomorphism $K^{\mathrm{rel}}_i(X)_A \xrightarrow{\cong} K^{\mathrm{rel}}_i(X)_B$.

Definition 3.8. — We define the relative K-groups of X as

$$K^{\mathrm{rel}}_i(X) := \varinjlim_{A\in\mathcal{J}} K^{\mathrm{rel}}_i(X)_A.$$

For every $A \in \mathcal{J}$ the projection $\pi_A\colon \prod_{W\in A} W \to X$ induces an isomorphism $\pi_A^*\colon K_i(X) \xrightarrow{\cong} K_i(\prod_{W\in A} W)$ and for varying A the compositions

$$K^{\mathrm{rel}}_i(X)_A \to K_i(\prod_{W\in A} W) \xrightarrow{(\pi_A^*)^{-1}} K_i(X)$$

assemble to give a map $K^{\mathrm{rel}}_i(X) \to K_i(X)$.

The relative Chern character is now constructed as follows: Let $\pi\colon W \to X$ be any Jouanolou torsor. By the homotopy invariance of singular cohomology π induces an isomorphism $\pi^*\colon H^*(X,\mathbf{C}) \xrightarrow{\cong} H^*(W,\mathbf{C})$. Since π^* is in fact a morphism of mixed Hodge structures, the induced maps $\mathrm{Fil}^n H^*(X,\mathbf{C}) \to \mathrm{Fil}^n H^*(W,\mathbf{C})$ and $H^*_{\mathscr{D}}(X,\mathbf{Q}(n)) \to H^*_{\mathscr{D}}(W,\mathbf{Q}(n))$ are isomorphisms, too.

Hence, for $A \in \mathcal{J}$ as above, we can consider the composition

$$\begin{aligned} K^{\mathrm{rel}}_i(X)_A = K^{\mathrm{rel}}_i(\prod_{W\in A} W) &\xrightarrow{\mathrm{Ch}^{\mathrm{rel}}_{n,i}} H^{2n-i-1}(\prod_{W\in A} W,\mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(\prod_{W\in A} W,\mathbf{C}) \\ &\xrightarrow{(\pi_A^*)^{-1}} H^{2n-i-1}(X,\mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X,\mathbf{C}). \end{aligned}$$

For varying $A \in \mathcal{J}$ these maps induce the *relative Chern character*

$$\mathrm{Ch}^{\mathrm{rel}}_{n,i}\colon K^{\mathrm{rel}}_i(X) \to H^{2n-i-1}(X,\mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X,\mathbf{C}).$$

Similarly, the Chern character in Deligne-Beilinson cohomology is the composition

$$\mathrm{Ch}^{\mathscr{D}}_{n,i}\colon K_i(X) \xrightarrow{\pi^*} K_i(W) \xrightarrow{\mathrm{Ch}^{\mathscr{D}}_{n,i}} H^{2n-i}_{\mathscr{D}}(W,\mathbf{Q}(n)) \xrightarrow{(\pi^*)^{-1}} H^{2n-i}_{\mathscr{D}}(X,\mathbf{Q}(n))$$

where $\pi\colon W \to X$ is any Jouanolou torsor for X.

Theorem 3.9. — *Let X be a smooth, separated scheme of finite type over $\mathbf{C}$. The diagram*

$$\begin{tikzcd} K_i^{\mathrm{rel}}(X) \arrow[r] \arrow[d, "\mathrm{Ch}_{n,i}^{\mathrm{rel}}"] & K_i(X) \arrow[d, "\mathrm{Ch}_{n,i}^{\mathscr{D}}"] \\ H^{2n-i-1}(X,\mathbf{C})/\mathrm{Fil}^n H^{2n-i-1}(X,\mathbf{C}) \arrow[r] & H_{\mathscr{D}}^{2n-i}(X,\mathbf{Q}(n)) \end{tikzcd}$$

commutes.

Proof. — This follows directly from Theorem 3.7 and the constructions. □

Example 3.10. — Let X be a smooth, projective $\mathbf{C}$-scheme. Then the image of $K_i(X)$ in $K_{\mathrm{top}}^{-i}(X)$ is torsion [**19**, 6.3]. Hence, upon tensoring with $\mathbf{Q}$, the long exact sequence (3.2) breaks up into short exact sequences

$$0 \to K_{\mathrm{top}}^{-i-1}(X)_{\mathbf{Q}} \to K_i^{\mathrm{rel}}(X)_{\mathbf{Q}} \to K_i(X)_{\mathbf{Q}} \to 0, \quad i > 0.$$

On the other hand, the sequence of cohomology groups (2.6) breaks up into short exact sequences for weight reasons [**33**, Lemma p. 8] so that for $i > 0$ we have the following picture:

$$\begin{tikzcd} 0 \arrow[r] & K_{\mathrm{top}}^{-i-1}(X)_{\mathbf{Q}} \arrow[r] \arrow[d] & K_i^{\mathrm{rel}}(X)_{\mathbf{Q}} \arrow[r] \arrow[d, "\mathrm{Ch}_{n,i}^{\mathrm{rel}}"] & K_i(X)_{\mathbf{Q}} \arrow[r] \arrow[d, "\mathrm{Ch}_{n,i}^{\mathscr{D}}"] & 0 \\ 0 \arrow[r] & H^{2n-i-1}(X,\mathbf{Q}(n)) \arrow[r] & H^{2n-i-1}(X,\mathbf{C})/\mathrm{Fil}^n \arrow[r] & H_{\mathscr{D}}^{2n-i}(X,\mathbf{Q}(n)) \arrow[r] & 0. \end{tikzcd}$$

The unlabeled vertical arrow turns out to be the usual Chern character from topological K-theory to singular cohomology. This follows similarly as the analogous assertion in [**27**, Théorème 6.23]. We skip the details.

4. Application: the regulators of Beilinson and Borel

The goal of this Section is to use the above results to give a new proof of Burgos' Theorem that Borel's regulator is twice Beilinson's regulator.

4.1. Definition of the regulators. —

Definition 4.1. — The *Beilinson regulator* is by definition the Chern character with values in *real* Deligne-Beilinson cohomology:

$$r_{\mathrm{Be}} \colon K_{2n-1}(\mathbf{C}) \xrightarrow{\mathrm{Ch}_{n,2n-1}^{\mathscr{D}}} H_{\mathscr{D}}^1(\mathrm{Spec}(\mathbf{C}),\mathbf{Q}(n)) \to H_{\mathscr{D}}^1(\mathrm{Spec}(\mathbf{C}),\mathbf{R}(n)).$$

Here $H_{\mathscr{D}}^1(\mathrm{Spec}(\mathbf{C}),\mathbf{R}(n))$ is the cohomology in degree 1 of the complex $\mathbf{R}(n) \to \mathbf{C}$, hence canonically isomorphic to $\mathbf{C}/\mathbf{R}(n)$ which in turn is isomorphic to $\mathbf{R}(n-1)$ via the projection $\pi_{n-1} \colon \mathbf{C} \to \mathbf{R}(n-1)$, $z \mapsto \frac{1}{2}(z + (-1)^{n-1}\bar{z})$, and we will view r_{Be} as a map with values in $\mathbf{R}(n-1)$.

The definition of Borel's regulator ([**3**], see also [**6**, Ch. 9], [**32**]) needs some preparation. Consider $\mathrm{GL}_r(\mathbf{C})$ as a real Lie group with maximal compact subgroup $U(r)$. Denote the corresponding Lie algebras by $\mathfrak{gl}_r$ and $\mathfrak{u}_r$, respectively. We have the van Est isomorphism

$$H^*_{\mathrm{cts}}(\mathrm{GL}_r(\mathbf{C}), \mathbf{R}) \cong H^*(\mathfrak{gl}_r, \mathfrak{u}_r; \mathbf{R}) \tag{4.1}$$

between continuous group cohomology and relative Lie algebra cohomology. The right hand side of (4.1) is computed as follows (cf. e.g. [**32**]): The compact real form of $\mathfrak{gl}_r \otimes \mathbf{C}$ is $\mathfrak{u}_r \oplus \mathfrak{u}_r$, so we have isomorphisms $H^{2n-1}(\mathfrak{gl}_r, \mathfrak{u}_r; \mathbf{C}) \cong H^{2n-1}(\mathfrak{u}_r \oplus \mathfrak{u}_r, \mathfrak{u}_r; \mathbf{C}) \cong H^{2n-1}(\mathfrak{u}_r, \mathbf{C}) \cong H^{2n-1}(U(r), \mathbf{C}) \cong H^{2n-1}(\mathrm{GL}_r(\mathbf{C}), \mathbf{C})$ carrying the $\mathbf{R}$-cohomology to $i^{2n-1}H^{2n-1}(\mathrm{GL}_r(\mathbf{C}), \mathbf{R})$. Combining this with the van Est isomorphism and the natural map from continuous to discrete group cohomology yields natural maps

$$\begin{aligned} H^{2n}(B_\bullet \mathrm{GL}_r(\mathbf{C}), \mathbf{R}(n)) &\xrightarrow{\text{suspension}} H^{2n-1}(\mathrm{GL}_r(\mathbf{C}), \mathbf{R}(n)) \cong \\ &\cong H^{2n-1}(\mathfrak{gl}_r, \mathfrak{u}_r; \mathbf{R}(n-1)) \cong H^{2n-1}_{\mathrm{cts}}(\mathrm{GL}_r(\mathbf{C}), \mathbf{R}(n-1)) \to \\ &\to H^{2n-1}_{\mathrm{grp}}(\mathrm{GL}_r(\mathbf{C}), \mathbf{R}(n-1)) = H^{2n-1}(B_\bullet \mathrm{GL}_r(\mathbf{C})^\delta, \mathbf{R}(n-1)). \end{aligned}$$

Here and in the following $\mathrm{GL}_r(\mathbf{C})^\delta$ denotes the group $\mathrm{GL}_r(\mathbf{C})^\delta$ equipped with the discrete topology. Denote by Bo_n the image of the n-th universal Chern character class $\mathrm{Ch}^{\mathrm{top}}_n(E^{\mathrm{univ}}) \in H^{2n}(B_\bullet \mathrm{GL}_r(\mathbf{C}), \mathbf{R}(n))$ under the above composition.

Definition 4.2. — The Borel regulator is the composition

$$r_{\mathrm{Bo}} \colon K_{2n-1}(\mathbf{C}) \xrightarrow{\text{Hur.}} H_{2n-1}(B_\bullet \mathrm{GL}(\mathbf{C})^\delta, \mathbf{Z}) \cong H_{2n-1}(B_\bullet \mathrm{GL}_r(\mathbf{C})^\delta, \mathbf{Z}) \xrightarrow{\mathrm{Bo}_n} \mathbf{R}(n-1)$$

(r large enough).

***Theorem 4.3* (Burgos)**. —

$$r_{\mathrm{Bo}} = 2 r_{\mathrm{Be}}.$$

Remark 4.4. — Beilinson [**1**] proved that both regulators coincide up to a non zero rational factor. Many details of Beilinson's proof were provided by Rapoport [**32**]. Dupont, Hain, and Zucker [**10**] conjectured that the factor should be 2. This was finally proven by Burgos [**6**] using Beilinson's original argument.

Using the comparison of Karoubi's relative Chern character and Beilinson's regulator (Theorem 3.7) in the case $X = \mathrm{Spec}(\mathbf{C})$ our proof of Burgos' Theorem will be reduced to a comparison of Borel's regulator and the relative Chern character. This in turn will be done comparing explicit cocycles.

4.2. An explicit cocycle for Karoubi's relative Chern character. — In the notations of Section 3 we fix $A = \mathbf{C}$, $X = \mathrm{Spec}(\mathbf{C})$. In particular, we have the simplicial groups $G_{r,\bullet} = S^\infty_\bullet(\mathrm{GL}_r(\mathbf{C}))$, whose realization is equivalent to $\mathrm{GL}_r(\mathbf{C})$ with the usual topology, and the simplicial set $\mathscr{F}_r$, defined by diagram (3.3) and homotopy equivalent to the homotopy fibre of $B_\bullet \mathrm{GL}_r(\mathbf{C}) \to B_\bullet G_{r,\bullet}$. Recall that by construction the relative Chern character factors through the homology of the simplicial set $\mathscr{F} = \varinjlim_r \mathscr{F}_r$.

In the present situation the model for $\mathscr{F}_r$ used by Karoubi in [**27**] is more convenient: We have a commutative diagram of simplicial sets

$$\begin{tikzcd}
 & \mathscr{F}_r \arrow[r, "\alpha_r"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E_\bullet G_{r,\bullet} \arrow[d, "p"] \\
\mathrm{GL}_r(\mathbf{C})\backslash G_{r,\bullet} \arrow[r, "\rho_r"] \arrow[ur, dotted, "\eta_r"] \arrow[urr, bend left, "\beta_r"] & B_\bullet \mathrm{GL}_r(\mathbf{C})^\delta \arrow[r] & B_\bullet G_{r,\bullet},
\end{tikzcd}$$

$\beta_r(\sigma) = (\sigma(e_0)^{-1}\sigma, \ldots, \sigma(e_p)^{-1}\sigma)$, $\rho_r(\sigma) = (\sigma(e_0)^{-1}\sigma(e_1), \ldots, \sigma(e_{p-1})^{-1}\sigma(e_p))$ for $\sigma \in G_{r,p}$, and the map η_r, induced by β_r and ρ_r, is a weak homotopy equivalence (cf. [**27**, Proposition 6.16], [**36**, Lemma A.6]). Here e_i denotes the i-th standard basis vector $(0, \ldots, 1, \ldots, 0)$. This translates into a commutative diagram of topological morphisms of simplicial manifolds

$$\begin{tikzcd}
 & & E_\bullet \mathrm{GL}_r(\mathbf{C}) \arrow[d, "p"] \\
X \otimes \mathrm{GL}_r(\mathbf{C})\backslash G_{r,\bullet} \arrow[r, "\eta_r"] \arrow[urr, squiggly, "\beta_r"] & X \otimes \mathscr{F}_r \arrow[r, "g_r"] \arrow[ur, squiggly, "\alpha_r"] & B_\bullet \mathrm{GL}_r(\mathbf{C}).
\end{tikzcd}$$

Proposition 4.5. — *The composition*

$$H_{2n-1}(\mathrm{GL}_r(\mathbf{C})\backslash G_{r,\bullet}, \mathbf{Z}) \xrightarrow{\cong} H_{2n-1}(\mathscr{F}_r, \mathbf{Z}) \xrightarrow{\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E_r, \alpha_r)} H^0(X, \mathbf{C})/\mathrm{Fil}^n = \mathbf{C}$$

is given by the cocycle

$$\sigma \mapsto -\frac{(n-1)!}{(2n-1)!} \mathrm{Tr} \int_{\Delta^{2n-1}} (\sigma^{-1} d\sigma)^{2n-1}.$$

Remark 4.6. — Hamida obtained a similar result [**22**].

Proof. — Since r is fixed, we drop the subscript r in the following. Since X is proper, it makes no difference if we work with $\widetilde{\mathrm{Ch}}_n^{\mathrm{rel}}(E, \alpha)$ or with $\mathrm{Ch}_n^{\mathrm{rel}}(E, \alpha)$. It is clear from the commutativity of the above diagram that the composition in the statement of the proposition is induced by $\mathrm{Ch}_n^{\mathrm{rel}}(\eta^* E, \beta)$. This class can be computed explicitly: Since X is a point, the standard connection on the bundle $\eta^* E$ is given by the zero matrix (cf. the formula in Example 1.12). Then the pullback to the trivial bundle via β is given by $\beta_i^{-1} d\beta_i$ (see Remark 1.14 (i)). On the p-simplex $\sigma \in \mathrm{GL}_r(\mathbf{C})\backslash G_{r,p}$ the function β_i is given by the matrix $\sigma(e_i)^{-1}\sigma \in G_{r,p} = \mathscr{C}^\infty(\Delta^p, \mathrm{GL}_r(\mathbf{C}))$, hence $\beta_i^{-1} d\beta_i = \sigma^{-1} d\sigma$ on the simplex σ. We denote the corresponding simplicial form simply by $\sigma^{-1} d\sigma$.

By construction $\mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^{\eta^* E}, \beta)$ is given by $\int_0^1 (i_{\partial/\partial t}\mathrm{Ch}_n(\Gamma))dt$, where Γ is the connection given by $\Gamma_i = t\beta_i^{-1} d\beta_i = t\sigma^{-1}d\sigma$ on the trivial $\mathrm{GL}_r(\mathbf{C})$-bundle on $(X \otimes (\mathrm{GL}_r(\mathbf{C})\backslash G_{r,\bullet})) \times \mathbf{C}$, t denoting the coordinate on $\mathbf{C}$.

The curvature of Γ is given by

$$\begin{aligned} R_i = d\Gamma_i + \Gamma_i^2 = dt(\sigma^{-1}d\sigma) - t(\sigma^{-1}d\sigma)^2 + t^2(\sigma^{-1}d\sigma)^2 \\ = dt(\sigma^{-1}d\sigma) + (t^2 - t)(\sigma^{-1}d\sigma)^2. \end{aligned}$$

Hence $R_i^n = (t^2-t)^n(\sigma^{-1}d\sigma)^{2n} + ndt(t^2-t)^{n-1}(\sigma^{-1}d\sigma)^{2n-1}$ and

$$\begin{aligned} \mathrm{Ch}_n^{\mathrm{rel}}(\Gamma^{\rho^* E}, \beta) &= \frac{(-1)^n}{n!}\int_0^1 i_{\partial/\partial t}\mathrm{Tr}(R_i^n)dt \\ &= (-1)^n \frac{n}{n!}\mathrm{Tr}\int_0^1 (t^2-t)^{n-1}(\sigma^{-1}d\sigma)^{2n-1}dt \\ &= \frac{(-1)^n}{(n-1)!}\Big(\int_0^1 (t^2-t)^{n-1}dt\Big)\mathrm{Tr}\left((\sigma^{-1}d\sigma)^{2n-1}\right) \\ &= -\frac{(n-1)!}{(2n-1)!}\mathrm{Tr}\left((\sigma^{-1}d\sigma)^{2n-1}\right). \end{aligned}$$

Here we used that $\int_0^1 (t^2-t)^{n-1}dt = (-1)^{n-1}\int_0^1 t^{n-1}(1-t)^{n-1}dt = (-1)^{n-1}\mathrm{B}(n,n) = (-1)^{n-1}\frac{\Gamma(n)\cdot\Gamma(n)}{\Gamma(n+n)} = (-1)^{n-1}\frac{((n-1)!)^2}{(2n-1)!}$, where B is Euler's Beta function [**7**, Section 4.2]. Now the claim follows from Remark 3.4. □

4.3. The van Est isomorphism. — Recall that the relative Lie algebra cohomology $H^*(\mathfrak{gl}_r, \mathfrak{u}_r; \mathbf{R})$ is the cohomology of the complex $\mathscr{A}^*(\mathrm{GL}_r(\mathbf{C})/U(r); \mathbf{R})^{\mathrm{GL}_r(\mathbf{C})}$ of invariant real valued differential forms on the homogeneous space $\mathrm{GL}_r(\mathbf{C})/U(r)$.

To compare Borel's regulator with the relative Chern character we need the following description of the composition of the van Est isomorphism with the natural map $H^*_{\mathrm{cts}}(\mathrm{GL}_r(\mathbf{C}), \mathbf{R}) \to H^*_{\mathrm{grp}}(\mathrm{GL}_r(\mathbf{C}), \mathbf{R}) = H^*(B_\bullet \mathrm{GL}_r(\mathbf{C})^\delta, \mathbf{R})$ from continuous to discrete group cohomology.

Proposition 4.7. — *We have a commutative diagram*

$$\begin{tikzcd} H^*_{\mathrm{cts}}(\mathrm{GL}_r(\mathbf{C}), \mathbf{R}) \arrow[r] & H^*(B_\bullet \mathrm{GL}_r(\mathbf{C})^\delta, \mathbf{R}) \arrow[r, "\rho_r^*"] & H^*(\mathrm{GL}_r(\mathbf{C})\backslash G_{r,\bullet}, \mathbf{R}), \\ H^*(\mathfrak{gl}_r, \mathfrak{u}_r; \mathbf{R}) \arrow[u, "\text{van Est}", "\cong"'] \arrow[r] & H^*(\mathfrak{gl}_r; \mathbf{R}) \arrow[ur, "\phi"'] & \end{tikzcd}$$

where ϕ is induced by the chain map ϕ sending a left invariant form ω to the simplicial cocycle

$$\mathrm{GL}_r(\mathbf{C})\backslash S_p^\infty(\mathrm{GL}_r(\mathbf{C})) \ni \sigma \mapsto \int_{\Delta^p} \sigma^*\omega. \tag{4.2}$$

Proof. — This is a modification of Tillmann's argument in [**39**, Theorem 4.3].

Write $G = \mathrm{GL}_r(\mathbf{C})$, $U = U(r)$, and π for the projection $G \to G/U$. We have a commutative diagram of complexes of continuous G-modules

$$\begin{tikzcd}
\mathbf{R} \arrow[r, "\simeq"] \arrow[dr, "\simeq"] & \mathscr{C}(E_\bullet G, \mathbf{R}) \arrow[r] & \mathscr{C}(E_\bullet G, \mathscr{A}^*(G, \mathbf{R})) \\
& \mathscr{A}^*(G/U, \mathbf{R}) \arrow[r, "\pi^*"] & \mathscr{A}^*(G, \mathbf{R}), \arrow[u, "\simeq"']
\end{tikzcd}$$

all modules apart from $\mathbf{R}$ being injective and $\simeq$ denoting quasi-isomorphisms (cf. [**24**, p. 370, p. 385] and [**39**, proof of Theorem 4.3]). The right vertical arrow is the inclusion as constant "functions" and $\mathscr{C}(E_\bullet G, \_)$ denotes continuous functions. Taking continuous group cohomology the quasi-isomorphisms on the left induce the van Est isomorphism and we get the commutativity of the lower left square of the diagram

$$\begin{tikzcd}
H^*(\mathscr{C}(B_\bullet G, \mathbf{R})) \arrow[r] \arrow[d, equal] & H^*(\mathscr{C}(B_\bullet G^\delta, \mathbf{R})) \arrow[r, equal] & H^*(B_\bullet G^\delta, \mathbf{R}) \arrow[d] \arrow[dd, bend left=60, "\rho_r^*"] \\
H^*_{\mathrm{cts}}(G, \mathbf{R}) \arrow[r] & H^*(\mathscr{C}(B_\bullet G, \mathscr{A}^*(G, \mathbf{R}))) \arrow[r] & H^*(\mathscr{C}(B_\bullet G^\delta, \mathscr{C}(G_\bullet, \mathbf{R}))) \\
H^*(\mathfrak{gl}_r, \mathfrak{u}_r; \mathbf{R}) \arrow[r] \arrow[u, "\text{van Est}", "\cong"'] & H^*(\mathfrak{gl}_r; \mathbf{R}) \arrow[r, "\phi"] \arrow[u, "\cong"'] & H^*(G \backslash G_\bullet, \mathbf{R}). \arrow[u, "\cong"']
\end{tikzcd}$$

Here $\mathscr{C}(G_\bullet, \mathbf{R})$ denotes the complex of simplicial cochains on $G_\bullet$ (= the complex of singular cochains on G) and the unlabeled arrows are induced by the natural map $G^\delta \to G$ and the de Rham map (integration of differential forms) respectively.

The commutativity of the remaining parts is established in [**39**, Theorem 4.3, (4.4)] (ρ_r^* and ϕ are called eval and *de R* there) finishing the proof of the proposition. $\square$

4.4. Proof of Theorem 4.3. — Since the odd topological K-theory of $\mathrm{Spec}(\mathbf{C})$ vanishes, the map $K^{\mathrm{rel}}_{2n-1}(\mathbf{C}) \to K_{2n-1}(\mathbf{C})$ is surjective. By construction of the regulators resp. the relative Chern character and the comparison result of Theorem 3.7 it then suffices to show that the diagram

$$\begin{tikzcd}
H_{2n-1}(\mathrm{GL}_r(\mathbf{C}) \backslash G_{r,\bullet}, \mathbf{Z}) \arrow[r, "\rho_r^*"] \arrow[d, "\mathrm{Ch}^{\mathrm{rel}}_{n,2n-1}"'] & H_{2n-1}(B_\bullet \mathrm{GL}_r(\mathbf{C})^\delta, \mathbf{Z}) \arrow[ddr, bend left, "\frac{1}{2}\mathrm{Bo}_n"] & \\
H^0(\mathrm{Spec}(\mathbf{C}), \mathbf{C})/\mathrm{Fil}^n \arrow[r] \arrow[d, equal] & H^1_{\mathscr{D}}(\mathrm{Spec}(\mathbf{C}), \mathbf{R}(n)) \arrow[d, equal] & \\
\mathbf{C} \arrow[r] & \mathbf{C}/\mathbf{R}(n) \arrow[r, "\pi_{n-1}", "\cong"'] & \mathbf{R}(n-1)
\end{tikzcd}$$

commutes.

Recall the definition of Bo_n in Section 4.1. By abuse of notation we also denote by Bo_n the image of $\mathrm{Ch}_n^{\mathrm{top}}(E^{\mathrm{univ}})$ in the relative Lie algebra cohomology $H^{2n-1}(\mathfrak{gl}_r, \mathfrak{u}_r; \mathbf{R}(n-1))$. We need Burgos' description of its image in absolute Lie algebra cohomology:

Lemma 4.8. — *The image of* Bo_n *in* $H^{2n-1}(\mathfrak{gl}_r, \mathbf{R}(n-1))$ *is represented by the left invariant differential form*

$$-2\frac{(n-1)!}{(2n-1)!}\pi_{n-1} \circ \mathrm{Tr}((g^{-1}dg)^{2n-1}),$$

$g^{-1}dg$ *denoting the Maurer-Cartan form on* $\mathrm{GL}_r(\mathbf{C})$ *and* π_{n-1} *the projection* $\mathbf{C} \to \mathbf{R}(n-1)$.

Using Proposition 4.7 we conclude that the composition $\frac{1}{2}\mathrm{Bo}_n \circ \rho_r^*$ is induced by the cocycle

$$\mathrm{GL}_r(\mathbf{C})\backslash G_{r,\bullet} \ni \sigma \mapsto -\frac{(n-1)!}{(2n-1)!}\pi_{n-1}\mathrm{Tr}\int_{\Delta^{2n-1}}(\sigma^{-1}d\sigma)^{2n-1}$$

and finish the proof of the Theorem using Proposition 4.5.

Proof of the Lemma. — Obviously, the form in the statement is left invariant. At the unit element the Maurer-Cartan form is just the identity $\mathfrak{gl}_r \to \mathfrak{gl}_r$. Hence the above form corresponds to the alternating form on $\mathfrak{gl}_r$ that is given by

$$x_1 \wedge \cdots \wedge x_{2n-1} \mapsto -2\frac{(n-1)!}{(2n-1)!}\pi_{n-1}\left(\sum_{\tau \in \mathfrak{S}_{2n-1}} \mathrm{sgn}(\tau)\mathrm{Tr}(x_{\tau(1)} \cdots x_{\tau(2n-1)})\right),$$

where $\mathfrak{S}_{2n-1}$ denotes the symmetric group on $2n-1$ elements.

It follows from [**6**, Proposition 9.26] that this represents the image of Bo_n in $H^{2n-1}(\mathfrak{gl}_r, \mathbf{R}(n-1))$. Note that Burgos' cocycle differs from ours by the factor $(-1)^n$. This is explained by the fact that Burgos uses another normalization of the Chern classes. His "twisted Chern character class" ch_n is $(-1)^n\mathrm{Ch}_n^{\mathrm{top}}$, cf. Remark 2.15. □

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