Group structure on spheres and the Hopf fibration

Spheres of spheres over spheres

Saifuddin Syed

UBC Grad Student Seminar

Outline

Spheres

Definition

We define the *n*-**sphere** S^n to be the set of points in $\mathbb{R}^{n+1}$ of unit distance from the origin. ie,

$$S^n = \{x \in \mathbb{R}^{n+1} | |x| = 1\}$$

Example

$$S^0 = \{-1, 1\}$$

$$S^1 = \{e^{i\theta} \in \mathbb{C} \cong \mathbb{R}^2 | \theta \in [0, \pi)\}$$

S^2 is the standard sphere in $\mathbb{R}^3$

Spheres

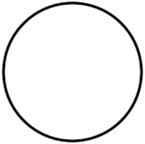

Figure: 1-sphere

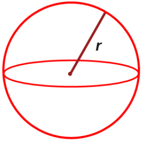

Figure: 2-sphere

Groups

Definition

A **group** is a set G with a multiplication defined such that

1. $\exists e \in G$ such that $\forall g \in G, eg = ge = g$
2. $\forall g \in G, \exists g^{-1}$ such that $gg^{-1} = g^{-1}g = e$
3. The multiplication is associative, as in $\forall g, h, k \in G, (gh)k = g(hk)$

Example

1. S^0 is finite group $\mathbb{Z}_2$
2. S^1 is $\mathrm{U}(1)$, the set of 1-dimensional unitary matrices

It is natural to ask, is S^n always a group? If so why, and if not which ones are?

Why are spheres groups

What makes S^0 a group is that we can multiply the unit normed elements of $\mathbb{R}$, and the elements of S^0 are closed under real multiplication.

Similarly what makes S^1 a group is that the elements can be viewed as unit normed elements in $\mathbb{C} \cong \mathbb{R}^2$. The set of unit normed elements are closed under complex multiplication.

Basically $\mathbb{R}$ and $\mathbb{C}$ are "nice".

Normed real division algebras

It turns out the common thread between $\mathbb{R}, \mathbb{C}$ is that they are both normed real division algebras.

Definition

An n-dimensional **normed real division algebra** $\mathbb{A}$ satisfies the following

1. $\mathbb{A}$ is a normed real vector space
2. $\mathbb{A}$ is a division ring, that may or may not be associative.
3. The norm respects multiplication, as in $\forall a, b \in \mathbb{A}$ we have $|ab| = |a||b|$

Construction of group from $\mathbb{A}$

In general, if one has an associative *n*-dimensional normed real division algebra $\mathbb{A}$ then we have a group structure on $S^{n-1} = \{x \in \mathbb{A} | |x| = 1\}$ given by the multiplication of $\mathbb{A}$.

Construction of group from $\mathbb{A}$

Conversely, if one has a group structure on S^{n-1}, one can construct an associative n-dimensional normed real division algebra $\mathbb{A}$, via $a, b \in \mathbb{R}^n$ then

$$ab \equiv |a||b| \left(\frac{a}{|a|} * \frac{b}{|b|} \right).$$

Classification of $\mathbb{A}$

So we have translated this problem of finding all the spheres with a group structure to finding all normed real division algebras.

It turns out there are a very limited class of normed real division algebras.

Classification of $\mathbb{A}$

So we have translated this problem of finding all the spheres with a group structure to finding all normed real division algebras.

It turns out there are a very limited class of normed real division algebras.

Theorem (Hurwitz, 1898)

There are only 4 normed real division algebras upto isomorphism. They are denoted by $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ *and are of dimension* $1, 2, 4, 8$ *respectively. Where* $\mathbb{H}$ *are the quaternions and* $\mathbb{O}$ *are the octonions.*

Classification of $\mathbb{A}$

The intuitive reason as to why there are only 4 is that you lose structure every time dimension increases:

- $\mathbb{R}$ to $\mathbb{C}$ one loses ordering
- $\mathbb{C}$ to $\mathbb{H}$ one loses commutativity
- $\mathbb{H}$ to $\mathbb{O}$ one loses associativity

For dimension greater than 8, you lose too much structure.

Summary

So we have that the only spheres that are groups are

$$
\begin{aligned}
S^0 &\cong \mathbb{Z}_2 \cong \mathrm{O}(1), \\
S^1 &\cong \mathrm{U}(1), \\
S^3 &\cong \mathrm{Sp}(1) \cong \mathrm{SU}(2) \cong \mathrm{SO}(3).
\end{aligned}
$$

S^7 is almost a group, because it lacks associativity.

They will be crucial to the construction of the Hopf fibrations.

Outline

Fibrations

Definition

Let E, B, F be topological spaces. A **fibre bundle** is denoted by

$$F \hookrightarrow E \xrightarrow{p} B$$

where $p : E \to B$ satisfies,

1. $p^{-1}(b) \cong F$
2. $\forall b \in B$ there is a neighbourhood U of b such that $p^{-1}(U)$ is homeomorphic to $U \times F$ via some homeomorphism $\psi : U \times F \to p^{-1}(U)$.
3. We have $p \circ \psi = \pi$ where $\pi : U \times F \to U$ is the projection from $U \times F$ to U.

We say that E is the **total space**, B is the **base**, F is the **fibre** and E is the fibre bundle (or fibration) over B with fibre F.

Fibrations

In other words. . .

$$F \hookrightarrow E \xrightarrow{p} B$$

Is a fancy way of saying E **locally** looks like "$B \times F$" (with some mild technical conditions).

Fibrations

Given $F \hookrightarrow E \xrightarrow{p} B$, you can think of E as a family of F parametrized by B.

In general for all topologogical spaces A, B, the trivial fibration is

$$B \hookrightarrow A \times B \xrightarrow{p} A$$

where $p((a, b)) = a$

NOTE: A fibration is NOT a cartesian product!

Examples: Cylinder

Let I be a closed interval and $p : I \times S^1, p(t, e^{i\theta}) = e^{i\theta}$

$$I \hookrightarrow I \times S^1 \xrightarrow{p} S^1$$

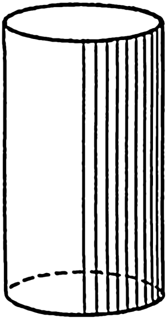

Figure: $I \times S^1$ or a cylinder, Source: Wikipedia

Examples: Möbius strip

Let I be a closed interval, M the Möbius strip, and p projects to the central circle S^1.

$$I \hookrightarrow M \xrightarrow{p} S^1$$

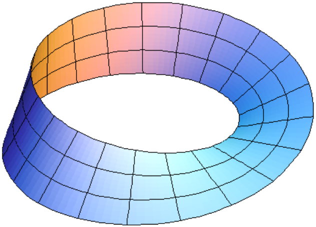

Figure: The Möbius strip, Source: virtualmathmuseum.org

Examples

Both $I \times S^1$ and M are fibrations over S^1 with fibres I, but $I \times S^1 \not\cong M$.

Real projective space

Definition

The real projective space $\mathbb{RP}^n$ is the of set 1 dimensional real subspaces in $\mathbb{R}^{n+1}$. It is a compact, n-dimensional smooth manifold.

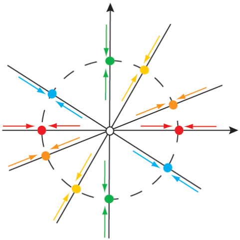

Figure: $\mathbb{RP}^1$, Source: Wikipedia

Real projective space

Points in $\mathbb{RP}^n$ are the set of equivalence classes in $\mathbb{R}^{n+1}$ such that

$$x, y \in \mathbb{R}^{n+1}, [x] = [y] \iff x = \lambda y \quad \text{for some} \quad 0 \neq \lambda \in \mathbb{R}.$$

We can restrict our relation to lines intersecting S^n (by picking the representatives of the equivalents classes of unit norm). So we have the set of points in $\mathbb{RP}^n$ are the set of equivalence classes in S^n such that

$$x, y \in S^n, [x] = [y] \iff x = \lambda y \quad \text{for some} \quad 1 = |\lambda|, \lambda \in \mathbb{R}.$$

Real projective space

Note that $\mathbb{RP}^n$ can be though of as the set of orbits of the group action of S^0 on S^n by left multiplication. The action is free because $\lambda x = x \Rightarrow \lambda = 1$. So each orbit (ie. fibre) is isomorphic to S^0.

Let $\pi : S^n \to \mathbb{RP}^n, \pi(x) = [x]$ be the quotient map. Then we have S^n is a fibration over $\mathbb{RP}^n$ with fibre

$$\pi^{-1}(x) = \{\lambda x | |\lambda| = 1, \lambda \in \mathbb{R}\} = \{x, -x\} \cong S^0.$$

So we have constructed:

$$S^0 \hookrightarrow S^n \xrightarrow{\pi} \mathbb{RP}^n$$

Complex projective spaces

Definition

The complex projective space $\mathbb{CP}^n$ is the set of 1 dimensional complex subspaces in $\mathbb{C}^{n+1}$. It is a compact, $2n$-dimensional smooth manifold.

Points in $\mathbb{CP}^n$ are the set of equivalence classes in $\mathbb{C}^{n+1}$ such that

$$x, y \in \mathbb{C}^{n+1}, [x] = [y] \iff x = \lambda y \quad \text{for some} \quad 0 \neq \lambda \in \mathbb{C}.$$

Since $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$ we pick restrict our relation to lines intersecting S^{2n+1}, as before. So we have the set of points in $\mathbb{CP}^n$ are the set of equivalence classes in S^{2n+1} such that

$$x, y \in S^{2n+1}, [x] = [y] \iff x = \lambda y \quad \text{for some} \quad 1 = |\lambda|, \lambda \in \mathbb{C}.$$

Complex projective space

Note that $\mathbb{CP}^n$ can be though of as the set of orbits of the group action of S^1 on S^{2n+1} by left multiplication. The action is free because $\lambda x = x \Rightarrow \lambda = 1$. So each orbit (ie. fibre) is isomorphic to S^1.

Let $\pi : S^{2n+1} \to \mathbb{CP}^n, \pi(x) = [x]$ be the quotient map. Then we have S^{2n+1} is a fibration over $\mathbb{CP}^n$ with fibre

$$\pi^{-1}(x) = \{\lambda x | |\lambda| = 1, \lambda \in \mathbb{C}\} \cong S^1.$$

So we have constructed:

$$S^1 \hookrightarrow S^{2n+1} \xrightarrow{\pi} \mathbb{CP}^n$$

Definition

The quaternionic projective space $\mathbb{HP}^n$ is the set of 1 dimensional quaternionic subspaces in $\mathbb{H}^{n+1}$. It is a compact, $4n$-dimensional smooth manifold. One has to be a bit careful with multiplication since $\mathbb{H}$ is not commutative.

After repeating the identical process for $\mathbb{RP}^n$, and $\mathbb{CP}^n$, we have S^{4n+3} is a fibration over $\mathbb{HP}^n$ with fibre

$$\pi^{-1}(x) = \{\lambda x || \lambda| = 1, \lambda \in \mathbb{H})\} \cong S^3.$$

So we have constructed:

$$S^3 \hookrightarrow S^{4n+3} \xrightarrow{\pi} \mathbb{HP}^n$$

Octionic projective spaces

It seems natural to repeat the process with $\mathbb{O}$, however the non-associativity of the octonions makes this difficult. It turns out that you cannot define $\mathbb{OP}^n$ for $n > 2$ and can only form a fibration for $\mathbb{OP}^1$, but not over $\mathbb{OP}^2$.

Repeating the previous process we get the following fibtations.

$$S^7 \hookrightarrow S^{8n+7} \xrightarrow{\pi} \mathbb{OP}^1$$

The Hopf Fibrations

To summarize we have constructed the following fibrations. These are known as the Hopf fibrations.

$$S^0 \longrightarrow S^n \xrightarrow{\pi} \mathbb{RP}^n$$
$$S^1 \longrightarrow S^{2n+1} \xrightarrow{\pi} \mathbb{CP}^n$$
$$S^3 \longrightarrow S^{4n+3} \xrightarrow{\pi} \mathbb{HP}^n$$
$$S^7 \longrightarrow S^{8+7} \xrightarrow{\pi} \mathbb{OP}^1$$

They are usually stated in the case where $n = 1$ to get

$$S^0 \longrightarrow S^1 \xrightarrow{\pi} \mathbb{RP}^1 \cong S^1$$
$$S^1 \longrightarrow S^3 \xrightarrow{\pi} \mathbb{CP}^1 \cong S^2$$
$$S^3 \longrightarrow S^7 \xrightarrow{\pi} \mathbb{HP}^1 \cong S^4$$
$$S^7 \longrightarrow S^{15} \xrightarrow{\pi} \mathbb{OP}^1 \cong S^8$$

Classical Hopf Fibration

Lets now look at $S^1 \hookrightarrow S^3 \xrightarrow{\pi} S^2$. This allows us to visualize S^3.

If we apply stereographic projection from S^3 to $\mathbb{R}^3 \cup \{\infty\}$, have R^3 is completely filled by disjoint circles and a line (circle through ∞). Not only that, but all these circles are pairwise "linked".

Visualization of the $3 - sphere$

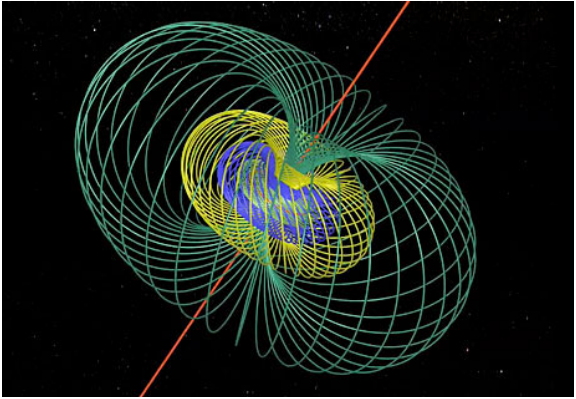

Figure: Stereographic projection of S^3. Each circle is a fibre of S^3.
Source: sciencenews.org

Outline

Qubit system

In quantum mechanics, we study systems corresponding to separable Hilbert spaces, which are complete inner product spaces, with a countable dense set.

The simplest non-trivial system is $\mathcal{H} = \mathbb{C}^2$ corresponds is the qubit system (or spin $\frac{1}{2}$-system).

Besides being an easy system to introduce to an undergrad quantum class, the qubit system is of great importance in quantum cryptography and quantum computing.

Setup

Definition

- We define $\{|0\rangle, |1\rangle\}$to be an orthonormal basis of $\mathbb{C}^2$ to be , and $\{\langle 0|, \langle 1|\}$ to be an orthonormal basis for the dual of $\mathbb{C}^2$.
- So for a general $|\psi\rangle \in \mathbb{C}^2$ there are some $a, b \in \mathbb{C}$ such that $|\psi\rangle = a|0\rangle + b|1\rangle$. We also have that $\langle\psi| = \overline{a}\langle 0| + \overline{b}\langle 1|$ is the **dual vector** of $|\psi\rangle$.
- Given $|\psi\rangle = a|0\rangle + b|1\rangle$ and $|\varphi\rangle = c|0\rangle + d|1\rangle$, we define the inner product on $\mathbb{C}^2$ by

$$\langle\psi|\varphi\rangle := \overline{a}c + \overline{b}d$$

- We define the norm on $\mathbb{C}^2$ to be $\| |\psi\rangle \| := \sqrt{\langle\psi|\psi\rangle}$

States

A quantum **state** is defined to be a vector $|\psi\rangle = a|0\rangle + b|1\rangle$ such that $\| \, |\psi\rangle \, \| = |a|^2 + |b|^2 = 1$.

The set of quantum states can be identified with $(u + iv, x + iy)$ in $\mathbb{C}^2$ such that

$$u^2 + v^2 + x^2 + y^2 = 1.$$

Therefore set of quantum states is precisely S^3, viewed as a subset of $\mathbb{C}^2$.

States

In quantum mechanics we don't particularly care about states, but rather what can be observed by them.

If 2 states, always output the same outcomes when "observed", then we want to say these states are equivalent. So we need a way to determine how to measure states, and distinguish them.

Observables

Definition

If $\mathcal{H} = \mathbb{C}^2$ is a separable Hilbert space, then an **observable** is an Hermitian operator $A : \mathcal{H} \to \mathcal{H}$, such that $A^* = A$.

Since A is Hermitian, it has a real eigenvalues, and a can be decomposed as

$$A = \sum_{\lambda \in Spec(A)} \lambda P_\lambda$$

Where P_λ is the projection onto the eigenspace for λ.

Outcomes

Definition

- Given an observable $A = \sum_{\lambda \in Spec(A)} \lambda P_\lambda$, the **outcomes** of A are defined to be the eigenvalues of A.
- Given a state $|\psi\rangle$ the **probability** of observing an outcome λ with $|\psi\rangle$ is

$$\mathrm{Pr}_\lambda(|\psi\rangle) = \langle\psi| P_\lambda |\psi\rangle$$

i.e. the "percentage" of $|\psi\rangle$ that lies in the λ eigenspace.

Bloch Sphere

It is natural to define two states to be equal if they they always produce the same probabilities.

It is clear from the definition that for all $|\psi\rangle$

$$\mathrm{Pr}_\lambda(|\psi\rangle) = \mathrm{Pr}_\lambda(e^{i\theta}|\psi\rangle).$$

Therefore we define states to be equal if they differ by some $e^{i\theta}$, which is precisely how we defined $\mathbb{CP}^1$.

Bloch Sphere

Thus in the qubit system quantum states can be viewed as elements of $\mathbb{CP}^1$.

This allows us to use the Hopf fibration to view the set of states in S^3 as fibres of S^1 parametrized by S^2.

In quantum mechanics this is parametrization is called the **Bloch sphere**. It allows us visualize this non trivial space. Fairly complicated actions can be shown to be rotations on S^2.