Higher homotopy structures: then and now

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Dedicated to the memory of Masahiro Sugawara and John Coleman Moore

Program on Higher Structures in Geometry and Physics
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Abstract

Looking back over 55 years of higher homotopy structures, I will reminisce as I recall the early days and ponder how they developed and how I now see them. From the history of A_∞-structures and later of L_∞-structures, I will present selective highlights as they morphed into the topic of this Program on Higher Structures in Geometry and Physics.

Once upon a time: A_∞-spaces and algebras

The history of A_∞-structures begins, implicitly, in 1957 with the work of Masahiro Sugawara. He showed that, with a generalized notion of fibration, the Spanier-Whitehead condition for a space F to be an H-space:

A fibration with fibre F contractible in the total space

is necessary and sufficient.

He goes on to obtain similar criteria for F to be a homotopy-associative H-space or a loop space.

Problem

PROBLEM: When is a primitive cohomology class $u \in H^n(X, \pi)$ of a topological group or loop space X the suspension of a class in $H^{n+1}(BX, \pi)$.

In other words, when is an H-map $X \to K(\pi, n)$ induced as the loops on a map $BX \to K(\pi, n+1)$.

Projective spaces

Any H-space has a projective 'plane' $XP(2)$ and homotopy associativity implies the existence of $XP(3)$. Contrast this with classical projective geometry where the existence of a projective 3-space implies strict associativity.

Definition

An A_n space X consists of a space X together with a coherent set of maps

$$m_k : K_k \times X^k \to X \quad \text{for} \quad k \leq n$$

where K_k is the (by now) well known $k-2$-dimensional associahedron in one of its first realizations.

Note that is not the same as an A_∞-space with $m_k = *$ for $k > n$. For example, for $n = 3$ an associating homotopy need have no relation to the usual pentagon relation.

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Theorem

A 'nice' connected space X has the homotopy type of a based loop space ΩY for some Y if and only if X admits the structure of an A_∞-space.

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homotopy invariant algebraic structures

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Operads were crucial for studying an important issue in the ∞-version of commutative algebras:

whether to relax the commutativity up to homotopy or to keep the strict symmetry but relax the associativity or relax both.

As emphasized by Kontsevich, the triumvirate of A_∞ , L_∞ and C_∞-algebras play a dominant role.

By C_∞-algebra we mean what is also known as a balanced A_∞-algebra, that is, a strictly commutative A_∞-algebra defined in terms of a coherent set of n-ary products which vanish on shuffles.

C_∞-algebras and L_∞-algebras are in an adjoint relationship just as for strict associative and Lie algebras.

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L_∞-algebra

Definition

An L_∞*-algebra* is a differential graded vector space $(L = \{L_i\}, d = \ell_1)$ with a coherent set of n-ary brackets

$$\ell_n = [\, , \ldots, \,] : \Lambda^n X \to X.$$

Equivalently, an L_∞*-algebra* is a graded vector space $L = \{L_i\}$ with a coderivation differential of degree ± 1 on the graded symmetric coalgebra $C(L)$ on the shift sL.

For an ordinary Lie algebra, this is the classical Chevalley-Eilenberg *chain* complex.

Degree??

Remark

Usually L is either non positively or non-negatively graded.

Note the ambiguity as to the degree ± 1 *of* d *in defining an* L_∞*-algebra. The binary operation is always of degree 0; sometimes the 'manifest' grading in examples is not the right one; see examples below. The shift of the bracket now has the same degree as the shift of* ℓ_1.

Notice also this bracket extends to an action of the degree 0 piece on the piece of degree 1 (or -1 respectively), as for a module over an algebra. Notice d : *module* $\to$ *algebra* d : *algebra* $\to$ *module depending on the grading.*

L_∞ in deformation theory

L_∞-algebras arose by 1977 in my work with Mike Schlesinger on deformation theory of rational homotopy types.

The yoga of deformation theory:
any problem in deformation theory is “controlled" by a differential graded Lie algebra (unique up to homology equivalence of dg Lie algebras)
we extended to similar control by an L_∞-algebra.

Let $\mathcal{H}$ be a simply connected graded commutative algebra of finite type and $(\Lambda Z, d) \to \mathcal{H}$ a filtered model. Differential graded Lie algebras provide a natural setting in which to pursue the obstruction method for trying to integrate “infinitesimal deformations”, elements of $H^1(Der\Lambda Z)$, to full perturbations. In that regard, $H^*(Der\Lambda Z)$ appears not only as a graded Lie algebra (in the obvious way) but also as an L_∞-algebra.

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L_∞ in deformation theory

Our main result compares the set of augmented homotopy types of dgca's $(A, i : \mathcal{H} \approx H(A))$ with the path components of $C(L)$ where $L \subset \ Der \, \Lambda Z$ consists of the weight decreasing derivations.

L_∞ in deformation theory

Deformation theory as such began with deformations of complex structure. The algebraic version dates back to the work of Murray Gerstenhaber (1963 - it was a very good year!!). This led to an algebraic description of *deformation quantization*, a term derived from physics.

Given a Poisson algebra $(A, \{\ ,\ \})$, a deformation quantization is an associative unital $\star$ product on the algebra of formal power series $A[[\hbar]]$ subject to the following two axioms:

$$f \star g = fg + \mathcal{O}(\hbar) \tag{1}$$

$$f \star g - g \star f = \hbar\{f, g\} + \mathcal{O}(\hbar^2). \tag{2}$$

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L_∞-algebra in physics

L_∞-algebras are becoming increasingly useful in physics [1] in two ways:

- Solution of a physical problem leads to a structure which later is recognized as that of an L_∞-algebra.
- Solution of a physical problem is attacked using knowledge of L_∞-algebras.

There are some famous 'no go' theorems that rule out certain physical models, e.g. higher spin particles. What is ruled out is models in terms of Lie algebras and their representations. It has been evident for some time that L_∞-algebras bypass these obstacles.

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In 1987, the formulas of the BRST operator in the construction of Batalin-Fradkin-Vilkovisky constrained Hamiltonian systems could be recognized as corresponding to an L_∞-algebra as did the corresponding Lagrangian formulas of Batalin-Vilkovisky.

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L_∞-algebra in physics

A more exciting breakthrough into mathematical physics occurred when I recognized an L_∞-algebra structure in the closed field string theory of Zwiebach. This occurred in 1989 when he fortuitously gave a talk in Chapel Hill at the last GUT (Grand Unification Theory) conference.

Truncated L_∞-algebras

Warning!! Notice there is a real problem of nomenclature.

There are significant uses of *truncated* L_∞-algebras 'truncated' having trivial brackets for $k > n$. For example:

- Lie n-algebra means an L_∞-algebra concentrated in degrees 0 to n-1 (or -n+1 to 0) with d of degree -1 (or $+1$ respectively).
- n-Lie algebra means having a k-ary bracket only for $k = n$ and satisfying one of two *fundamental identities* generalizing Jacobi or that for the Nambu bracket.

Both of these types of algebras are important in geometry and in physics.

L_∞-algebras in geometry

Courant algebroids are structures which include as examples the doubles of Lie bialgebras and the bundles $TM \oplus T^*M$ with the bracket introduced by T. Courant.

Roytenberg showed in his PhD thesis the equivalence of this structure with a specific two-term L_∞ algebra, in which l_1 is determined by the de Rham differential, l_2 by the Courant bracket and the operation l_3 contains "flux" (e.g. a three-form known as H-flux).

In 1998, Roytenberg and Weinstein, building on Roytenberg's thesis, showed that Courant algebroids give rise to L_∞-algebras.

A Courant algebroid $E \to TM \to M$ comes equipped with an operator $\mathcal{D} : C^\infty(M) \to \Gamma(E)$.

Consider the total space of the following resolution X of $H = coker\mathcal{D}$:

$$X_2 = \ker\mathcal{D} \xrightarrow{d_2} X_1 = C^\infty(M) \xrightarrow{d_1} X_0 = \Gamma(E) \longrightarrow H \longrightarrow 0, \quad (3)$$

where with $d_1 = \mathcal{D}$ and d_2 is the inclusion $\iota : \ker\mathcal{D} \hookrightarrow C^\infty(M)$.

The Courant brackets on H come from Courant brackets on $\Gamma(E)$ for which the Jacobi identity is satisfied up to a $\mathcal{D}$ exact term.They extend the Courant bracket to an L_∞-structure on all of their resolution X, manifestly a Lie 3-algebra.

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In contrast, working from the physics side (in the context of Yang-Mills theory on a smooth Riemannian manifold), Zeitlin explored the *Maxwell complex* built from part of the de Rham complex using the Hodge star operator and $d \star d$:

$$0 \to \Omega^0(M) \to \Omega^1(M) \xrightarrow{d\star d} \Omega^{D-1}(M) \to \Omega^D(M) \to 0. \quad (4)$$

which he renames as $(\mathcal{F}^\cdot, \mathcal{Q})$:

$$0 \to \mathcal{F}^0 \xrightarrow{\mathcal{Q}} \mathcal{F}^1 \xrightarrow{\mathcal{Q}} \mathcal{F}^2 \xrightarrow{\mathcal{Q}} \mathcal{F}^3 \to 0. \quad (5)$$

He then tensors it with some reductive Lie algebra $\mathfrak{g}$.
He defines graded brackets $[\ ,\ ]$ based on the operator product expansion for open strings and $[\ ,\ ,\ ]$ by computation and shows they define an L_∞-algebra (with all higher brackets 0).

Notice this is not a Lie n-algebra since d is of degree 1 but the complex is non-negatively graded and d : *algebra* $\to$ *module* (cf. BBvD algebras).

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Lie n-algebras from multisymplectic manifolds

Chris Rogers:
A manifold is multisymplectic, or more specifically *n*-plectic, if it is equipped with a closed nondegenerate differential form of degree $n+1$. Just as a symplectic manifold gives rise to a Poisson algebra of functions, any n-plectic manifold gives rise to a Lie n-algebra of differential forms with multi-brackets specified via the *n*-plectic structure. The underlying graded vector space consists of a subspace of $(n-1)$-forms he calls *Hamiltonian* together with all *p*-forms for $0 \leq p \leq n-2$.

$$C^\infty(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{n-2}(M) \xrightarrow{d} \Omega^{n-1}_{Ham} \tag{6}$$

The bilinear bracket, as well as all higher *k*-ary brackets, are explicitly specified by the *n*-plectic structure.

This is a Lie *n*-algebra if we flip the indices.

Continuing in a more physical setting/language, Ritter and Saemann propose new physical field theory models by taking Lie n-algebras as ingredients. They consider 'zero-dimensional field theories' which means the fields are elements of an L_∞-algebra (think functions on a point!).

BBvD L_∞-algebras

In contrast to Lie n-algebras as above, it is possible to have an L_∞-algebra concentrated in degrees 0 to n-1 with d of degree $+1$ as for the higher spin algebras of Berends, Burgers and van Dam .

They start with a given space of ‘fields’ Φ which is a module over a Lie algebra Ξ of gauge symmetries.

BBvD L_∞-algebras

By a field dependent gauge transformation of Ξ on Φ, they mean a polynomial (or power series) map $\Xi \bigoplus \Phi \to \Phi$:

$$\delta_\xi(\phi) = \Sigma_{i \geq 0} T_i(\xi, \phi)$$

where T_i is linear in ξ and polynomial of homogeneous degree i in ϕ.

Note the operation $T_0 : \Xi \to \Phi$ from ‘algebra’ to ‘module’, in contrast to the above examples except for Zeitlin’s.

BBvD L_∞-algebras

They have a corresponding field dependent generalization of a Lie algebra structure on Ξ: a polynomial (or power series) map $\Xi \oplus \Xi \oplus \Phi \to \Xi$

$$[\ ,\ ] : (\xi, \eta)(\phi) = \Sigma_{i \geq 0} C_i(\xi, \eta, \phi)$$

where C_i is bilinear in ξ and η and of homogeneous degree i in ϕ.

These operations obey consistency relations which Fulp, Lada and I identified as structure relations of an L_∞ algebra.

Remark

Note the BBvD structure gives rise to an L_∞-algebra structure on the direct sum of the space of fields and the space of gauge parameters, **not** *of the form of an L_∞-algebra and its module. This is similar to what occurs in the BFV and BV formalisms.*

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Small L_∞-algebras

Zwiebach and/or other physicists asked about small examples of L_∞-agebras, (physicists' 'toy' models) leading to work of Tom Lada and his student Marilyn Daily. Daily classified all 3-dimensional L_∞-algebras: 2-graded with one 1-dimensional component and one 2-dimentional component; 3-graded where each component is 1-dimensional.

Daily and Lada showed that the 2-graded examples fit into the context of BBvD theory.

In the last two decades, what is called T-duality in string theory and supergravity required a formulation of differential geometry on a generalized tangent bundle (locally $TM \oplus T^*M$) as a *Courant algebroid*.

These have been further generalized to *higher Courant algebroids* are generalizations of $TM \oplus \Lambda^k(T^*M)$.

This in turn has led to Double Field Theory.

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A *generalized tangent bundle* E is an extension of T by T^*

$$0 \longrightarrow T^*M \longrightarrow E \xrightarrow{\pi} TM \longrightarrow 0. \tag{7}$$

Locally, the bundle E looks like $TM \oplus T^*M$. As above, there is an associated Courant bracket and hence an L_∞-algebra.

As a bundle over TM with a local trivialization with respect to a covering $\mathcal{U} = \{U_\alpha\}$, we have transition functions $a_{\alpha\beta} \in GL(2d, \mathbf{R})$ satisfying the usual cocycle condition, but, in terms of the local splitting $TM \oplus T^*M$, there is a higher order 'twist' τ depending on a 2-form ω. Now the cocycle condition fails, the failure depending on ω:

$$g_{\alpha\beta} g_{\beta\gamma} \neq g_{\alpha\gamma} \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma.$$

This is often described as a failure of associativity, but it is more accurately failure to correspond to a representation.

Since $g_{\alpha\beta} g_{\beta\gamma}$ can be expressed in terms of 1-forms, it could be that the difference is an exact form $d\lambda_{\alpha\beta\gamma}$ for some function $\lambda_{\alpha\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. One could say the transition functions form a representation up to homotopy (RUTH).

Representations up to homotopy/ RUTHs

An ordinary representation of a group or algebra is equivalent to a morphism to the endomorphisms of another object. An ‘up to homotopy’ analog appeared early on in in terms of the action of the loop space ΩB on the fiber F of a fibration $F \to E \to B$:

$$\Omega B \times F \to F \ \ or \ \ \Omega B \to H(F)$$

where $H(F)$ denotes the monoid of self homotopy equivalence $F \to F$.

Initially, this was referred to as a *homotopy action*, meaning only that $(f\lambda)\mu$ was homotopic to $f(\lambda\mu)$, with no higher order structure. Sugawara generalized this to *homotopy multiplicative maps* betwwen associative H-spaces.

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Notice the map $\Omega B \to H(F)$ above corresponds only to an H-map; the full equivalence between such fibrations and A_∞-maps $\Omega B \to H(F)$ had to wait until such maps were available.

The corresponding terminology is that of (strong or coherent or ∞) homotopy action, which has further variants under a variety of names.

In the case of a smooth fibre bundle $E \to B$,
the corresponding notion is parallel transport.
From a topological point of view, ∞-homotopy action is the more basic notion.

String theory and string field theory [have] inspired string topology, initiated by Chas and Sullivan and a variety of ∞-algebras.

The corresponding theory of ‘representation up to coherent homotopy’ should feed back into physics.

A_∞-functors and A_∞-categories

Since associativity is a key property of categories, it is not surprising that A_∞-categories were eventually defined. In 1993, Fukaya defined them to handle Morse theoretic homology.

Just as one considers A_∞-morphisms of A_∞-algebras, one can consider A_∞-functors (also known as *homotopy coherent functors*) between A_∞-categories. Such functors were first considered for ordinary strict but topological categories in the context of classification of fibre spaces.

For fibrations which are locally homotopy trivial with respect to a good open cover $\{U_\alpha\}$ of the base, one can define transition functions

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to H(F),$$

but instead of the cocycle condition for fibre bundles, one obtains only that $g_{\alpha\beta} g_{\beta\gamma}$ is homotopic to $g_{\alpha\gamma}$ as a map of $U_\alpha \cap U_\beta \cap U_\gamma$ into $H(F)$.

A_∞-functors and A_∞-categories

In 1965, Wirth showed how a set of coherent higher homotopies arise on multiple intersections. He calls that set a *homotopy transition cocycle.*

Regarding the disjoint union $\coprod U_\alpha$ as a topological category U, and $H(F)$ as a category with one object in the standard way, Wirth shows the transition cocycle web of higher homotopies is precisely equivalent to a *functor up to strong homotopy*

$$U \to H(F),$$

also known as a *homotopy coherent functor*.

For 'good' spaces, the usual classification of such fibrations is effected by the realization of this functor via a map

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A_∞ or L_∞-"geometry"

The coherent homotopy generalization of the definition of a dg-manifold is straightforward, but requires a coherent homotopy cocycle condition.

Definition

A dg ∞-manifold or sh-manifold is a locally ringed space $(M, \mathcal{O}_M)$ (in dg commutative algebras over $\mathbb{R}$), which is locally homotopy equivalent (as dcga's) to $(U, \mathcal{O}_U)$, where $\mathcal{O}_U = C^\infty(U) \otimes S(V^\bullet)$ with $\{U\}$ an open cover of M and $(S(V^\bullet), d)$ a dcga.

The analogs of classical transition functions with a cocycle condition are exactly Wirth's homotopy transition cocycles.

A_∞-algebra in physics

The most obvious presence of A_∞-structures in physics is in open and open-closed string field theory.

In OCSFT, there is an L_∞-algebra acting on an A_∞-algebra in a special way, well explicated in pictures.

Less well established are AA-structures involving what physicists call *flux*, especially H-flux but also R-flux and Q-flux.

Now and future

That brings us somewhat up to date; there is much work in progress even in this small part of the space of higher structures.

Now I find in Manin's *Mathematics, Art, Civilization*:

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Now and future

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