THE STABLE HOMOTOPY OF COMPLEX PROJECTIVE SPACE

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1. Introduction

THE object of this note is to prove that the space BU is a direct factor of the space $Q(\mathbf{CP}^\infty) = \Omega^\infty S^\infty(\mathbf{CP}^\infty) = \varinjlim_n \Omega^n S^n(\mathbf{CP}^\infty)$. This is not very surprising, as Toda [cf. (6) (2.1)] has shown that the homotopy groups of $Q(\mathbf{CP}^\infty)$, i.e. the stable homotopy groups of $\mathbf{CP}^\infty$, split in the appropriate way. But the method, which is Quillen's technique (7) of reducing to a problem about finite groups and then using the Brauer induction theorem, may be interesting.

If X and Y are spaces, I shall write $\{X; Y\}^k$ for $\left[X; \varinjlim_n \Omega^n S^{n+k}(Y_+)\right]$, where Y_+ means Y together with a disjoint base-point, and [;] means homotopy classes of maps with no conditions about base-points. For fixed Y, $X \mapsto \{X; Y\}^*$ is a representable cohomology theory. If Y is a topological abelian group the composition $Y \times Y \to Y$ induces

$$S^p(Y_+) \wedge S^q(Y_+) \to S^{p+q}(Y_+),$$

and makes $\{ ; Y\}^*$ into a multiplicative cohomology theory. In fact it is easy to see that $\{X; Y\}^0$ is then even a λ-ring.

Let $P = \mathbf{CP}^\infty$, and embed it in the space $\mathbf{Z} \times BU$ which represents the functor K by $P = 1 \times BU_1 \subset \mathbf{Z} \times BU$. This corresponds to the natural inclusion {line bundles} $\subset$ {virtual vector bundles}. There is an induced map from the suspension-spectrum of P to the spectrum representing K-theory, inducing a transformation of multiplicative cohomology theories $T: \{ ; P\}^* \to K^*$.

PROPOSITION 1. *For any space X the ring-homomorphism*

$$T: \{X; P\}^0 \to K^0(X)$$

is surjective.

COROLLARY. *The space QP is (up to homotopy) the product of BU and a space with finite homotopy groups.*

The functor $\{ ; P\}$ is the 'minimal' representable functor containing the group of formal sums of line bundles. Thus Proposition 1 is trivial for a space X such that $K(X)$ is generated by line bundles. But because $\{\ ; P\}$ is a cohomology theory it has a 'transfer' or 'Gysin homomorphism' [see (**3**), (**4**)] for finite covering maps, and so 'contains' vector bundles obtained as direct images of line bundles on finite covering spaces of X. By Quillen's results in (**7**) one finds it is enough to prove Proposition 1 when X is the classifying space of a finite group G. Then elements of $K(X)$ correspond (roughly) to representations of G, and Brauer's theorem [see (**8**) 11–29] tells one that they are obtained by the transfer from one-dimensional representations of subgroups H of G, i.e. from line bundles on finite coverings BH of BG.

It is interesting to compare Proposition 1 with the following theorem of Kahn and Priddy [see (**3**)], where π_s^* denotes stable cohomotopy.

PROPOSITION 2. *There is a transformation of cohomology theories* $T\colon \{\ ;\mathbf{RP}^\infty\}^* \to \pi_s^*$, *and, for any space* X,

$$T\colon \{X;\mathbf{RP}^\infty\}^0 \to \pi_s^0(X)$$

is surjective.

2. Proof of Proposition 1

We shall use Sullivan's technique of completion at a prime p, described in (**9**). Let us recall that if h^* is a representable cohomology theory such that $h^k(\text{point})$ is finitely generated for all k one can define a theory h_p^* which takes its values in the category of compact abelian groups and which has the properties:

(*a*) if X is a finite CW-complex, $h_p^k(X)$ is the p-adic completion of $h^k(X)$,

(*b*) if X is the direct limit of closed subspaces X_α, then

$$h_p^k(X) \cong \varprojlim_\alpha h_p^k(X_\alpha)$$

as topological group.

In fact (*a*) and (*b*) define h_p^*, and Brown's theorem [see (**2**)] shows it is representable. We shall also need theories $h_{\mathbf{Q}}^*$ and $h_{\mathbf{Q}_p}^*$ defined by the conditions

(*a′*) $\quad h_{\mathbf{Q}}^*(X) = h^*(X) \otimes_{\mathbf{Z}} \mathbf{Q} \quad$ and $\quad h_{\mathbf{Q}_p}^*(X) = h_p^*(X) \otimes_{\mathbf{Z}} \mathbf{Q}_p$

when X is a finite CW-complex, and

(*b′*) the functors $h_{\mathbf{Q}}^*$ and $h_{\mathbf{Q}_p}^*$ are representable (in fact by products of Eilenberg–Maclane spaces).

There are natural transformations

$$\begin{array}{ccc} h^* & \to & \prod_p h_p^* \\ \downarrow & & \downarrow \\ h_{\mathbf{Q}}^* & \to & \prod_p h_{\mathbf{Q}_p}^*, \end{array}$$

and the associated diagram of classifying spaces is a fibre-square, as one sees by inspecting the homotopy groups.

Write P^* for the cohomology theory $\{\ ; P\}^*$. To show that

$$P^0(X) \to K^0(X)$$

is surjective for all X it is enough to show that the canonical element η in $K^0(BU)$ comes from $P^0(BU)$. Let η_p, $\eta_{\mathbf{Q}}$ be the images of η in $K^0_p(BU)$ and $K^0_{\mathbf{Q}}(BU)$. Suppose one has found $\tilde{\eta}_p$ and $\tilde{\eta}_{\mathbf{Q}}$ in $P^0_p(BU)$ and $P^0_{\mathbf{Q}}(BU)$ which agree in $\prod_p P^0_{\mathbf{Q}_p}$ and are such that $\tilde{\eta}_p \mapsto \eta_p$, $\tilde{\eta}_{\mathbf{Q}} \mapsto \eta_{\mathbf{Q}}$. Then there is an element $\tilde{\eta} \in P^0(BU)$ inducing $\tilde{\eta}_p$ and $\tilde{\eta}_{\mathbf{Q}}$, and it maps to η, as desired, because $K^0(BU) \to K^0_{\mathbf{Q}}(BU)$ is injective. It remains to find $\tilde{\eta}_p$ and $\tilde{\eta}_{\mathbf{Q}}$; but their existence follows at once from the following two lemmas.

LEMMA 1. *For each prime p, $P^0_p(BU) \to K^0_p(BU)$ is surjective.*

LEMMA 2. *The transformations $P^*_{\mathbf{Q}} \to K^*_{\mathbf{Q}}$ and $P^*_{\mathbf{Q}_p} \to K^*_{\mathbf{Q}_p}$ are isomorphisms.*

Proof of Lemma 1. Let q be a prime distinct from p, and let $\overline{\mathbf{F}}_q$ be the algebraic closure of the field $\mathbf{F}_q$ with q elements. Quillen has shown [in (7) 1.6] that there is a map $BGL(\infty, \overline{\mathbf{F}}_q) \to BU$ which induces an isomorphism of cohomology with coefficients $\mathbf{Z}/p^n$ for all n. It therefore induces an isomorphism of cohomology with coefficients in any finitely generated $\mathbf{Z}_p$-module, where $\mathbf{Z}_p$ denotes the p-adic numbers. So, by obstruction theory,

$$P^*_p(BU) \xrightarrow{\cong} P^*_p(BGL(\infty, \overline{\mathbf{F}}_q)) \quad \text{and} \quad K^*_p(BU) \xrightarrow{\cong} K^*_p(BGL(\infty, \overline{\mathbf{F}}_q)),$$

for both functors are represented by spaces whose homotopy groups are finitely generated $\mathbf{Z}_p$-modules. So it suffices to show that

$$P^*_p(BGL(\infty, \overline{\mathbf{F}}_q)) \to K^*_p(BGL(\infty, \overline{\mathbf{F}}_q))$$

is surjective; and hence (using (*a*) above, and observing that $\varprojlim$ is exact for compact groups) that $P^*_p(BGL(m, \mathbf{F}_{q^n})) \to K^*_p(BGL(m, \mathbf{F}_{q^n}))$ is surjective. Thus Lemma 1 is reduced to:

LEMMA 3. *If G is a finite group, $P^0_p(BG) \to K^0_p(BG)$ is surjective.*

Proof of Lemma 3. If H is a subgroup of G, let $\mathrm{Pic}(H)$ denote the set of one-dimensional representations of H. This can be identified with $[BH; P] \subset \{BH; P\} = P^0(BH)$. Construction of the induced representation gives a map ind: $\mathrm{Pic}(H) \to R(G)$, where $R(G)$ is the representation ring; and the following diagram commutes:

$$\begin{array}{ccccccc} & & \prod_{H<G} P^0(BH) & \xrightarrow{\text{transfer}} & P^0(BG) & \to & P^0_p(BG) \\ & \nearrow & & & \big\downarrow T & & \big\downarrow T \\ \prod_{H<G} \mathrm{Pic}(H) & & & & & & \\ & \searrow & & & & & \\ & & R(G) & \longrightarrow & K^0(BG) & \to & K^0_p(BG). \end{array}$$

(The pentagon commutes because (i) the transfer $P^0(BH) \to P^0(BG)$ corresponds via T to the transfer $K^0(BH) \to K^0(BG)$, as any transformation of cohomology theories commutes with transfers; and (ii) the transfer $K^0(BH) \to K^0(BG)$ corresponds to the homomorphism

$$\mathrm{ind}\colon R(H) \to R(G),$$

as proved in (4) 540.)

Brauer's theorem [(8) 11–29] asserts that the image of ind generates $R(G)$ additively. By the main theorem of (1), the image of $R(G)$ is dense in $K^0_p(BG)$—in fact $K^0_p(BG)$ is the $(pR(G)+I_G)$-adic completion of $R(G)$. So by the commutativity of the diagram the image of $P^0_p(BG)$ is dense in $K^0_p(BG)$; but $P^0_p(BG)$ is compact, so its image must be all of $K^0_p(BG)$, as desired.

Proof of Lemma 2. It is sufficient to show that $P^*_{\mathbf{Q}}(\text{point}) \xrightarrow{\cong} K^*_{\mathbf{Q}}(\text{point})$, i.e. that $\pi^s_*(P) \otimes \mathbf{Q} \xrightarrow{\cong} \pi_*(\mathbf{Z} \times BU) \otimes \mathbf{Q}$, i.e. that

$$H_*(P; \mathbf{Q}) \xrightarrow{\cong} \operatorname{prim} H_*(\mathbf{Z} \times BU; \mathbf{Q}),$$

as $\pi^s_*(X) \otimes \mathbf{Q} \cong H_*(X; \mathbf{Q})$ for any space X, and

$$\pi_*(Y) \otimes \mathbf{Q} \cong \operatorname{prim} H_*(Y; \mathbf{Q})$$

for any H-space Y by (5) 263. Lemma 2 now follows by a very well-known and simple calculation.

3. Proof of the corollary

The functor P^0 is represented by $Q(P_+) = QS^0 \times QP$. By Proposition 1 there is a map $f\colon \mathbf{Z} \times BU \to QS^0 \times QP$ such that $T \circ f\colon \mathbf{Z} \times BU \to \mathbf{Z} \times BU$ is homotopic to the identity. But if one considers just the component $f'\colon BU \to QP$ of f then $T \circ f'\colon BU \to BU$ must be a homotopy equivalence because it must have the same effect on homotopy groups as $T \circ f$, the homotopy groups of QS^0 being finite, and those of BU torsion-free. Thus QP is decomposed as $BU \times (\text{fibre: } QP \to BU)$.

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