THE SPHERE SPECTRUM

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Following J.H.C. Whitehead [14] and Lima [7], a *sequential spectrum* E is a sequence of based topological spaces (or simplicial sets) E_n and structure maps $\sigma_n \colon \Sigma E_n \to E_{n+1}$, for all $n \geq 0$. Here the suspension $\Sigma E_n = S^1 \wedge E_n$ equals the smash product with the based topological (or simplicial) circle S^1. The *sphere spectrum* $\mathbb{S}$ is the most basic example: its n-th space is the n-sphere $S^n = S^1 \wedge \dots \wedge S^1$ and its structure maps are the resulting homeomorphisms $\sigma_n \colon \Sigma S^n \to S^{n+1}$.

The *k-th homotopy group* $\pi_k(E)$ is the colimit over n of the (unstable) homotopy groups $\pi_{k+n}(E_n)$. As k varies, the $\pi_k(E)$ assemble to a graded abelian group $\pi_*(E)$. In the case of the sphere spectrum, $\pi_k(\mathbb{S})$ is the colimit of the homotopy groups of spheres $\pi_{k+n}(S^n)$, i.e., of the homotopy classes of based maps $S^{k+n} \to S^n$. By the Freudenthal suspension theorem [5] the homomorphisms in this colimit are isomorphisms for $n \geq k+2$, and the common limiting value $\pi_k(\mathbb{S})$ is known as the *k-th stable homotopy group of spheres*.

The generalized homology theory $\pi_*^S(X) = \pi_*(\mathbb{S} \wedge X)$ and the generalized cohomology theory $\pi_S^{-*}(X) = \pi_* \operatorname{Map}(X, \mathbb{S})$ associated to the sphere spectrum are called *stable homotopy* and *stable cohomotopy*, respectively. As a consequence of the proven *Segal conjecture* [3], stable cohomotopy has the exceptional property that $\pi_S^{-*}(BG_+)$ vanishes for all $* < 0$. Here BG is the classifying space of an arbitrary finite group. In the Atiyah–Hirzebruch spectral sequence

$$E^2_{s,t} = H^{-s}(BG; \pi_t(\mathbb{S})) \Longrightarrow \pi_S^{-(s+t)}(BG_+)$$

there is a complicated differential interplay between group cohomology and the stable homotopy groups of spheres, making $E^\infty_{s,t} = 0$ for $s + t < 0$. For $G = \mathbb{Z}/p$ the Segal conjecture also provides a copy of $\pi_*(\mathbb{S})^\wedge_p$ as a direct summand in the abutment, so each class $x \in \pi_k(\mathbb{S})^\wedge_p$ is represented at $E^\infty_{s,t}$ by some coset $M(x) \subset \pi_t(\mathbb{S})^\wedge_p$, with $t \geq k$, called the *Mahowald root invariant* of x. Empirically, when x is part of a periodic family in $\pi_*(\mathbb{S})^\wedge_p$ detected by the n-th Morava K-theory $K(n)$, then $M(x)$ is part of a family detected by the next Morava K-theory $K(n+1)$ [9].

A map of sequential spectra $f \colon E \to F$ is a sequence of based maps $f_n \colon E_n \to F_n$ commuting with the structure maps. It induces a homomorphism $f_* \colon \pi_*(E) \to \pi_*(F)$ of homotopy groups, and is called a *stable equivalence* if f_* is an isomorphism in each degree. The *stable homotopy category* is the category obtained from the category of spectra by inverting the stable equivalences. Let $\mathcal{S}p$ denote the category of sequential spectra, and let $\mathcal{B}$ (for Boardman) denote its associated stable homotopy category. The sphere spectrum $\mathbb{S}$ generates $\mathcal{B}$ in the sense that for each spectrum E there exists a *cell spectrum* E^c, which has been assembled from integer suspensions of $\mathbb{S}$ in the same way that a cell complex is built from non-negative suspensions of S^0, and a stable equivalence $E^c \to E$.

The smash product of two maps $S^{k+m} \to S^m$ and $S^{\ell+n} \to E_n$ composes with the iterated structure map $S^m \wedge E_n \to E_{m+n}$ to produce a map $S^{k+m+\ell+n} \to E_{m+n}$. With some care, especially about the ordering of the various circle factors in these smash products, this rule induces a pairing $\pi_k(\mathbb{S}) \otimes \pi_\ell(E) \to \pi_{k+\ell}(E)$. In the case $E = \mathbb{S}$ this product makes $\pi_*(\mathbb{S})$ a graded commutative ring, and in general $\pi_*(E)$ is a graded module over $\pi_*(\mathbb{S})$.

In the stable homotopy category $\mathcal{B}$ there is a functorial *smash product* $E \wedge F$ of spectra [1], well-defined up to stable equivalence, so that these commutative ring and module structures are realized by morphisms $\mathbb{S} \wedge \mathbb{S} \to \mathbb{S}$ and $\mathbb{S} \wedge E \to E$. However, in the category $\mathcal{S}p$ of sequential spectra there is no definition of a smash product $E \wedge F$ such that the product on $\mathbb{S}$ is commutative. It is at best associative, and sequential spectra E and F may be regarded as left (or right) $\mathbb{S}$-modules, but no natural $\mathbb{S}$-module structure remains on their smash product $E \wedge F$. The situation is reminiscent of that of modules over a non-commutative ring.

To overcome this defect, modern stable homotopy theory takes place in one of several possible modified categories $\mathcal{S}p'$ of spectra, three of which are reviewed below. In each of these there is a smash product $E \wedge F$ defined within the category of spectra, that is so well-behaved that the sphere spectrum $\mathbb{S}$ admits a commutative product $\mathbb{S} \wedge \mathbb{S} \to \mathbb{S}$ in $\mathcal{S}p'$. More precisely, the smash product is a symmetric monoidal pairing (= coherently unital, associative and commutative) with $\mathbb{S}$ as the unit object. The spectra E, F are naturally modules over $\mathbb{S}$ with this product, i.e., $\mathbb{S}$-modules, and the smash product $E \wedge F$ over $\mathbb{S}$ of two $\mathbb{S}$-modules is again an $\mathbb{S}$-module, because $\mathbb{S}$ is commutative. Furthermore, there is a notion of stable equivalence on $\mathcal{S}p'$, so chosen that the associated homotopy category is equivalent to $\mathcal{B}$.

This makes the sphere spectrum $\mathbb{S}$ the initial ground "ring" for stable homotopy theory, much like the integers $\mathbb{Z}$ is the initial ground ring for algebra. The categories of $\mathbb{S}$-modules, resp. associative or commutative $\mathbb{S}$-algebras, can be thought of as enriched versions of the categories of $\mathbb{Z}$-modules (= abelian groups), resp. associative or commutative $\mathbb{Z}$-algebras (= rings). This is a fruitful point of view for promoting ideas from algebra, algebraic geometry or number theory to the algebraic-topological context. The Eilenberg–Mac Lane functor embeds algebra into topology, and the enrichment amounts to a change of ground ring along the Hurewicz map $h \colon \mathbb{S} \to \mathbb{Z}$. The earlier theories of A_∞ and E_∞ ring spectra [11] provide many more examples of associative and commutative $\mathbb{S}$-algebras in topology, beyond those coming from algebra. These are therefore "brave new rings," a term coined by Waldhausen.

Several modern reinterpretations $\mathcal{S}p'$ of the category of spectra appeared shortly after 1994. The principal three are (a) the S-modules $\mathcal{M}_S$ of Elmendorf, Kriz, Mandell and May [4], (b) the symmetric spectra $\mathcal{S}p^\Sigma$ of Hovey, Shipley and Smith

[6], and (c) the Γ-spaces $\Gamma\mathcal{S}_*$ of Segal [13] and Lydakis [8]. The essential equivalence of these and other approaches is discussed in [10] and [12].

(a) The *S-modules* of May et al. were introduced in [4]. To start, a *coordinate-free spectrum* E is a rule that assigns a based space EV to each finite-dimensional vector subspace $V \subset \mathbb{R}^\infty$, together with a compatible system of homeomorphisms $EV \cong \Omega^{W-V} EW$ whenever $V \subset W$. Here $W - V$ is the orthogonal complement of V in W, S^{W-V} is its one-point compactification, and $\Omega^{W-V} X = F(S^{W-V}, X)$ is the mapping space. The coordinate-free sphere spectrum S is the rule with $SV = \operatorname{colim}_{V \subset W} \Omega^{W-V} S^W$. Its 0th space $S0$ is also known as $Q(S^0)$.

An $\mathbb{L}$-*spectrum* is a coordinate-free spectrum equipped with a suitable action by the space $\mathcal{L}(1)$ of linear isometries $\mathbb{R}^\infty \to \mathbb{R}^\infty$, which is part of the linear isometries operad $\mathcal{L}$. The sphere spectrum S is canonically an $\mathbb{L}$-spectrum, and there is an operadic smash product $E \wedge_{\mathcal{L}} F$ of $\mathbb{L}$-spectra. Finally, the *S-modules* are the $\mathbb{L}$-spectra E such that a natural map $\lambda \colon S \wedge_{\mathcal{L}} E \to E$ is an isomorphism. The sphere spectrum is an S-module, and the operadic smash product of $\mathbb{L}$-spectra $E \wedge_{\mathcal{L}} F$ restricts to the desired smash product $E \wedge F$ on the full subcategory of S-modules.

(b) The *symmetric spectra* of J. Smith et al. were introduced in [6]. First, a *symmetric sequence* E is a sequence of based simplicial sets E_n with an action by the symmetric group Σ_n, for each $n \geq 0$. The sphere symmetric sequence $\mathbb{S}$ has the n-fold smash product $S^n = S^1 \wedge \cdots \wedge S^1$ as its n-th space, with Σ_n permuting the factors. There is a symmetric monoidal pairing $E \otimes F$ of symmetric sequences, so defined that a map $E \otimes F \to G$ corresponds to a set of $(\Sigma_m \times \Sigma_n)$-equivariant maps $E_m \wedge F_n \to G_{m+n}$. Then $\mathbb{S}$ is a commutative monoid with product $\mathbb{S} \otimes \mathbb{S} \to \mathbb{S}$ corresponding to the equivariant isomorphisms $S^m \wedge S^n \cong S^{m+n}$.

A *symmetric spectrum* E is defined to be an $\mathbb{S}$-module in symmetric sequences, i.e., a symmetric sequence with a unital and associative action $\mathbb{S} \otimes E \to E$. Explicitly, the module action amounts to a set of $(\Sigma_m \times \Sigma_n)$-equivariant maps $S^m \wedge E_n \to E_{m+n}$. The sphere spectrum $\mathbb{S}$ is then a symmetric spectrum, and the desired smash product $E \wedge F$ of two symmetric spectra is defined as the coequalizer of two obvious maps $E \otimes \mathbb{S} \otimes F \to E \otimes F$. There is a notion of a stable equivalence $f \colon E \to F$ of symmetric spectra, strictly more restrictive than asking that $\pi_*(f)$ is an isomorphism, so that the associated homotopy category is equivalent to $\mathcal{B}$.

A variant of symmetric spectra, called *orthogonal spectra* [10], is obtained by replacing the symmetric group actions by orthogonal group actions. Then the π_*-isomorphisms are the correct weak equivalences to invert, in order to obtain a homotopy category equivalent to $\mathcal{B}$.

(c) Let Γ be the category of finite sets $n_+ = \{0, 1, \dots, n\}$ based at 0, for $n \geq 0$, and base-point preserving functions. Segal [13] defined a Γ-*space* E to be a functor from Γ to based simplicial sets, such that $E(0_+)$ is a point. Each Γ-space can be prolonged (degreewise) to an endofunctor of based simplicial sets, and there is an associated sequential spectrum with n-th space $E(S^n)$. Bousfield and Friedlander [2] show that the homotopy category of Γ-spaces under stable equivalences is equivalent to the stable homotopy category of connective spectra, i.e., spectra with $\pi_k(E) = 0$ for $k < 0$.

The sphere Γ-space $\mathbb{S}$ is the functor that takes n_+ to itself, considered as a based simplicial set. Its prolongation is the identity endofunctor, and the associated sequential spectrum is the sphere spectrum $\mathbb{S}$. The smash product $E \wedge F$ of two Γ-spaces is defined so that a map $E \wedge F \to G$ of Γ-spaces amounts to a natural transformation $E(k_+) \wedge F(\ell_+) \to G(k_+ \wedge \ell_+)$, for k_+ and ℓ_+ in Γ. This defines a symmetric monoidal pairing on Γ-spaces, with the sphere as the unit object. Lydakis [8] realized that this categorical construction also has good homotopical properties, in particular that it really models the smash product of spectra.

References

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