COMPACTLY GENERATED SPACES

CHARLES REZK

Contents

1. Introduction 1
2. Conventions 1
3. k-spaces 2
4. k-Hausdorff spaces 3
5. Compactly generated spaces 5
6. The projection criterion for k-open subsets of a product 6
7. Locally compact spaces 7
8. Mapping spaces 8
9. Compactly generated spaces and quotients 10
10. Some colimits and limits in compactly generated spaces 12
11. Weak Hausdorff spaces 14
References 15

1. Introduction

This document is a summary of basic facts about the "standard" category of compactly generated spaces introduced by McCord [McC69] (often referred to in the literature as "compactly generated weak Hausdorff spaces", or "weak Hausdorff k-spaces"). The main references I've used for this material are Gaunce Lewis's thesis [Lew, App. A], and Neil Strickland's note [Str09] (especially for material about colimits in compactly generated spaces). Many of the proofs given here stem from ones given in those works, though they may have mutated significantly.

There is one innovation here, namely the notion of a "k-Hausdorff" space, which is (perhaps) a bit more convenient than "weak Hausdorff". Every weak Hausdorff space is k-Hausdorff (11.2), and for a k-space, weak Hausdorff and k-Hausdorff are equivalent (11.4), so our category of "k-Hausdorff k-spaces" is identical to McCord's category. I have also tried, as much as possible, to rely directly on these definitions and not on inessential intermediate constructions. For instance, I give a direct construction of the topology on mapping spaces, rather than producing it as the k-ification of the compact-open topology. Also notable is the characterization of compactly generated spaces given as (9.10).

2. Conventions

We write Top for the category of topological spaces and continuous maps. In the following, "function" means a not-necessarily continuous function between topological spaces, and "map" means "continuous function between topological spaces", unless otherwise indicated.

Date: May 8, 2018.

When we speak of limits and colimits, and in particular products and coproducts, we always mean those in the category Top, unless otherwise explicitly specified. Similar remarks apply to subspaces and quotient spaces

We use the Bourbaki convention for compactness. A space X is **quasi-compact** if every open cover admits a finite subcover. A space X is **compact** if it is quasi-compact and Hausdorff.

3. k-spaces

3.1. **The k-topology and k-spaces.** We say that a subset C of a topological space X is **k-closed** (resp. **k-open**) if and only if for every map $f\colon K \to X$ from a compact K, the set $f^{-1}(C)$ is closed (resp. open) in K.

The complements of k-open subsets are k-closed, and vice versa, and the collection of k-open subsets of X satisfy the axioms for a topology. We write $\Bbbk X$ for the space with underlying point set X, whose open sets are the k-open sets of X. All open sets are k-open, so that the identity function $i\colon \Bbbk X \to X$ is continuous.

We say that a space X is a **k-space** if every k-open set is open (or equivalently, every k-closed set is closed). Thus X is a k-space if and only if $i\colon \Bbbk X \to X$ is a homeomorphism.

All compact spaces are k-spaces, since if $f\colon K \to L$ is a map between compact spaces and $C \subseteq L$ is a subset such that $f^{-1}C$ is closed in K, then $C = f(f^{-1}C)$ is a compact subspace in L and hence closed.

We say that a function $f\colon X \to Y$ between topological spaces is **k-continuous** if k-open sets of Y pull back along f to k-open sets of X; equivalently, f is k-continuous if and only if $\Bbbk f\colon \Bbbk X \to \Bbbk Y$ is continuous. Note that continuous maps are automatically k-continuous.

3.2. **Proposition.** *For every map $f\colon Y \to X$ from a k-space Y, there is a unique map $f'\colon Y \to \Bbbk(X)$ such that $if' = f$.*

Proof. It suffices to show that if $f\colon Y \to X$ is a map from a k-space Y, then $f' = fi^{-1}$ is continuous. For any k-closed subset $C \subseteq X$, we have that $f^{-1}(C)$ is k-closed in Y, and hence is closed in Y. Thus $f' = fi^{-1}$ is continuous. □

In fact, the map $i\colon \Bbbk(X) \to X$ is characterized up to unique isomorphism by this universal property.

3.3. **Inheritance properties of k-spaces.** Recall that a map $g\colon X \to Y$ is a **proclusion** if it is surjective, and if for every subset $V \subseteq Y$ such that $g^{-1}V$ is open in X, then V is open in Y. In other words, g is a proclusion if it factors through a homeomorphism from a quotient space of X to Y.

3.4. **Proposition.**

(1) *If X is a k-space, any closed subspace C of X is a k-space.*
(2) *If $g\colon X \to Y$ is a proclusion from a k-space X, then Y is a k-space.*
(3) *Arbitrary coproducts of k-spaces are k-spaces.*

Proof.

Proof of (1). We must show that any k-closed $D \subseteq C$ is closed in X, and hence closed in C. For any map $f\colon K \to X$ from a compact K, we see that $f^{-1}(C)$ is closed in K and hence compact, and therefore $f^{-1}(D)$ is closed in $f^{-1}(C)$ and hence closed in K. Therefore D is closed in X since X is a k-space.

Proof of (2). We must show that any k-closed $C \subseteq Y$ is closed in Y. Since g is a proclusion, it is enough to show that $g^{-1}(C)$ is closed in X. If $f\colon K \to X$ is a map from a compact K, then $f^{-1}(g^{-1}(C)) = (gf)^{-1}(C)$ is closed in K since C is k-closed in Y.

Proof of (3). Let $X = \coprod_\alpha X_\alpha$, where each X_α is a k-space. Suppose that $C \subseteq X$ is k-closed; we must show that $C \cap X_\alpha$ is closed for every α. Let $f\colon K \to X_\alpha$ be a map from a compact space. Then $f^{-1}(C \cap X_\alpha) = f^{-1}(C) \cap f^{-1}(X_\alpha)$ is closed in K, since X_α is closed in X and C is k-closed. □

3.5. **The category of k-spaces.** Let $\Bbbk$Top denote the full subcategory of k-spaces in Top. We obtain a pair of adjoint functors

$$\text{include}\colon \Bbbk\text{Top} \rightleftarrows \text{Top} : \Bbbk.$$

In particular, if X is a k-space and Y is a space, a function $f\colon X \to Y$ is continuous if and only if $i^{-1}f\colon X \to \Bbbk Y$ is continuous.

The category $\Bbbk$Top admits both limits and colimits. Colimits in $\Bbbk$Top are computed as they are in Top; (3.4) implies that a colimit (in Top) of a diagram of k-spaces is itself a k-space. Limits in $\Bbbk$Top are computed by taking the limit in Top and then applying the $\Bbbk$-functor. In particular, we write

$$X \times^{\Bbbk} Y := \Bbbk(X \times Y);$$

if X and Y are k-spaces then $X \times^{\Bbbk} Y$ is the product in the category of k-spaces. We reserve the notation $X \times Y$ for the usual product in Top, even if X and Y are k-spaces.

3.6. *Remark.* A function $K \to X \times^{\Bbbk} Y$ from a compact K is continuous if and only if it is continuous as a map $K \to X \times Y$. This implies that all ways of k-ifying a finite product coincide; e.g., $X \times^{\Bbbk} (Y \times^{\Bbbk} Z) = (X \times^{\Bbbk} Y) \times^{\Bbbk} Z$.

Observe that a general subspace of a k-space need not be a k-space. In particular, open subsets of a k-space can fail to be k-spaces. More generally, a k-open subset of a subspace A of a space X need not be the intersection of A with a k-open subset of X.

4. k-Hausdorff spaces

We say that a space X is **k-Hausdorff** if the diagonal subset $\Delta_X \subseteq X \times X$ is k-closed (in the usual product topology).

4.1. **Characterizations of k-Hausdorff spaces.**

4.2. **Proposition.** *Let X be a space. The following are equivalent.*

1. *X is k-Hausdorff.*
2. *For all maps $f\colon K \to X$ and $g\colon L \to X$ from compact K, L, the set $(f \times g)^{-1}(\Delta_X) = K \times_X L$ is closed in $K \times L$.*
3. *For all maps $f\colon K \to X$ from compact K, the set $(f \times f)^{-1}(\Delta_X) = K \times_X K$ is closed in $K \times K$.*
4. *For every map $f\colon K \to X$ from a compact K and every pair of points $k_1, k_2 \in K$ such that $f(k_1) \neq f(k_2)$, there exist open neighborhoods U_i of k_i in K such that $f(U_1)$ and $f(U_2)$ are disjoint.*

Proof. The equivalence of (1) and (2) follows from (4.3) below. The equivalence of (1) and (3) follows from (4.4) below. The equivalence of (3) and (4) is immediate. □

4.3. **Lemma.** *Let X and Y be spaces, and let $C \subseteq X \times Y$. The following are equivalent.*

1. *The subset C is k-closed in $X \times Y$.*
2. *For all maps $f\colon K \to X$ and $g\colon L \to Y$ from compact K and L, $(f \times g)^{-1}(C)$ is closed in $K \times L$.*
3. *For all maps $f\colon K \to X$ and $g\colon K \to Y$ from compact K, $(f \times g)^{-1}(C)$ is closed in $K \times K$.*

(4) *For all* $g\colon L \to Y$ *from compact* L*,* $(\mathrm{id} \times g)^{-1}(C)$ *is k-closed in* $X \times L$*.*

Proof. It is clear that (1) implies (2), since $f\colon K \times L \to X \times Y$ is a map from a compact space. It is immediate that (2) implies (3). To see that (3) implies (1), note that any $f\colon K \to X \times Y$ factors as $f = (f_1 \times f_2)d$, where $d\colon K \to K \times K$ is the diagonal map and the f_i are the projections of f to each factor of $X \times Y$.

That (1) implies (4) is immediate. To show that (4) implies (1), suppose $C \subseteq X \times Y$ satisfies (4), and let $(f, g)\colon L \to X \times Y$ be a map from a compact L. We can factor this as $L \xrightarrow{(f,\mathrm{id})} X \times L \xrightarrow{\mathrm{id} \times g} X \times Y$, and we see that since $(\mathrm{id} \times g)^{-1}C$ is k-closed in $X \times L$, then $(f, \mathrm{id})^{-1}(\mathrm{id} \times g)^{-1}C$ is closed in L, as desired. □

4.4. **Lemma.** *Let* X *be a space. A subset* $C \subseteq X \times X$ *is k-closed if and only if for all* $f\colon K \to X$ *from compact* K*,* $(f \times f)^{-1}(C)$ *is closed in* $K \times K$*.*

Proof. It is immediate that a k-closed $C \subseteq X \times X$ has the asserted property.

Now suppose that $C \subseteq X \times X$ is such that $(f \times f)^{-1}(C)$ is closed in $K \times K$ for all $f\colon K \to X$ from compact K. We will show that C is k-closed in $X \times X$ using criterion (2) of (4.3).

Given $f_i\colon K_i \to X$ from compact K_i for $i = 1, 2$, let $K = K_1 \amalg K_2$ with inclusion $j_i\colon K_i \to K$, and let $f\colon K \to X$ be the map such that $f j_i = f_i$. Note that K is compact. Since $(f \times f)^{-1}(C)$ is closed in $K \times K$ by hypothesis, we have that $(f_1 \times f_2)^{-1}(C) = (j_1 \times j_2)^{-1}(f \times f)^{-1}(C)$ is closed in $K_1 \times K_2$. This verifies condition (2) of (4.3), whence C is k-closed. □

We can characterize the k-Hausdorff property in terms of fiber products of compact spaces mapping to X.

4.5. **Proposition.** *A space* X *is k-Hausdorff if and only if for all maps* $f\colon K \to X$ *and* $g\colon L \to X$ *from compact* K *and* L*, the fiber product* $K \times_X L$ *is compact.*

Proof. The fiber product $K \times_X L = (f \times g)^{-1}\Delta_X$ is a subspace of the compact space $K \times L$, and so is itself compact if and only if it is a closed subset of $K \times L$. The claim follows by the equivalence of (1) and (2) in (4.2). □

4.6. **Inheritance properties of k-Hausdorff spaces.**

4.7. **Proposition.**

(1) *Hausdorff spaces are k-Hausdorff.*
(2) *If* $g\colon X \to Y$ *is a continuous injective map, and* Y *is k-Hausdorff, then* X *is also k-Hausdorff. In particular, subspaces of k-Hausdorff spaces are k-Hausdorff.*
(3) *Arbitrary products of k-Hausdorff spaces are k-Hausdorff.*
(4) *Arbitrary coproducts of k-Hausdorff spaces are k-Hausdorff.*
(5) *In a k-Hausdorff space, finite subsets are k-closed.*

Proof of (1). If X is Hausdorff, then Δ_X is closed in $X \times X$, and so is k-closed. □

Proof of (2). Let $f\colon K \to X$ be a map from a compact K. Then $(f \times f)^{-1}\Delta_X = (gf \times gf)^{-1}\Delta_Y$ since g is injective, and thus is closed in $K \times K$ by (4.2). □

Proof of (3). Let $\{X_\alpha\}_{\alpha \in A}$ be an indexed set of k-Hausdorff spaces, and let $X = \prod_\alpha X_\alpha$, with projection maps $p_\alpha\colon X \to X_\alpha$. We have that $\Delta_X = \bigcap_\alpha (p_\alpha \times p_\alpha)^{-1}(\Delta_{X_\alpha})$ in $X \times X$. By hypothesis Δ_{X_α} is k-closed in $X_\alpha \times X_\alpha$, and $p_\alpha \times p_\alpha\colon X \times X \to X_\alpha \times X_\alpha$ is continuous and hence k-continuous; therefore Δ_X is k-closed in $X \times X$ as desired. □

Proof of (4). Let $\{X_\alpha\}_{\alpha\in A}$ be an indexed set of k-Hausdorff spaces, and let $X = \coprod_\alpha X_\alpha$. If $f\colon K \to X$ is a map from a compact K, then the collection $\{f^{-1}(X_\alpha)\}$ forms an open cover of K. Thus there is a finite set of indices $\alpha_1, \dots, \alpha_n$ so that $K = \bigcup_{k=1}^n K_i$ where $K_i = f^{-1}(X_{\alpha_i})$ is compact. It is now straightforward to check condition (4) of (4.2). □

Proof of (5). If $x \in X$, $\{x\} = (c, \mathrm{id})^{-1}(\Delta_X)$ where $c\colon X \to X$ is the constant map sending all points to x. Thus if X is k-Hausdorff then $\{x\}$ is k-closed. □

4.8. **k-Hausdorffification.** Given a space X, we define a space $\mathbb{h}(X)$ as follows. Given an equivalence relation $\sim_\alpha$ on the point set of X, let $X_\alpha = X/\sim_\alpha$ denote the quotient space, and $q_\alpha\colon X \to X_\alpha$ the quotient map. Consider the evident map

$$h = (q_\alpha)\colon X \to \prod_\alpha X_\alpha$$

where the product is taken over all equivalence relations such that X_α is k-Hausdorff. Define $\mathbb{h}(X)$ to be the subspace $h(X)$ of $\prod_\alpha X_\alpha$, and write $q\colon X \to \mathbb{h}(X)$ for the surjective map induced by h.

4.9. **Proposition.** *The space $\mathbb{h}(X)$ is k-Hausdorff, and the map $q\colon X \to \mathbb{h}(X)$ is a proclusion. For every map $f\colon X \to Y$ to a k-Hausdorff Y, there is a unique map $f'\colon \mathbb{h}(X) \to Y$ such that $f'q = f$.*

Proof. Since $\mathbb{h}(X)$ is a subspace of a product of k-Hausdorff spaces, it is k-Hausdorff by (4.7) (2) and (3).

Let $\sim$ be the equivalence relation on X defined by the map q, and let $X' = X/\sim$ be the corresponding quotient space, with quotient map $g\colon X \to X'$. Then the map $k\colon X' \to \mathbb{h}(X)$ factoring q through g is a continuous bijection, whence X' is k-Hausdorff by (4.7)(2). This implies that g is one of the quotient maps q_α involved in the definition of $\mathbb{h}(X)$, and thus there is continuous map $\mathbb{h}(X) \to X'$ factoring g through q, and we conclude that $k\colon X' \to \mathbb{h}(X)$ is a homeomorphism, whence q is a proclusion.

Given $f\colon X \to Y$ to a k-Hausdorff space Y, let $\sim$ denote the equivalence relation on X defined by the map f, and let $\widetilde{Y} = X/\sim$ with quotient map $g\colon X \to \widetilde{Y}$. The induced map $i\colon \widetilde{Y} \to Y$ maps $\widetilde{Y}$ injectively to Y, and thus $\widetilde{Y}$ is k-Hausdorff by (4.7)(2). Since g is thus a proclusion to a k-Hausdorff space, by construction of $\mathbb{h}(X)$ there exists a map $\widetilde{g}\colon \mathbb{h}(X) \to \widetilde{Y}$ such that $\widetilde{g}q = g$, and thus $\widetilde{f} = i\widetilde{g}$ is such that $\widetilde{f}q = f$, uniquely so since q is a proclusion. □

4.10. **The category of k-Hausdorff spaces.** Let $\mathbb{k}$Haus denote the full subcategory of k-Hausdorff spaces in Top. We obtain a pair of adjoint functors

$$\mathbb{h}\colon \mathrm{Top} \rightleftarrows \mathbb{k}\mathrm{Haus} : \mathrm{include}.$$

The category $\mathbb{k}$Haus has both limits and colimits. Limits in $\mathbb{k}$Haus are computed as they are in Top; colimits in $\mathbb{k}$Haus are computed by taking the colimit in Top and then applying the $\mathbb{h}$-functor. Note though that coproducts in $\mathbb{k}$Haus can be computed exactly as in Top, by (4.7)(6).

5. Compactly generated spaces

We say that a space X is **compactly generated** if it is a k-Hausdorff k-space. We write $\mathbb{CG}$ for the full subcategory of compactly generated spaces inside Top.

5.1. **Proposition.** *If X is a k-space, then $\mathbb{h}(X)$ is compactly generated. If X is k-Hausdorff, then $\mathbb{k}(X)$ is compactly generated.*

Proof. Immediate from (3.4)(2) and (4.7)(2), since $q\colon X \to \mathbb{h}(X)$ is a proclusion and $i\colon \mathbb{k}(X) \to X$ is injective. □

The category of compactly generated spaces admits both limits and colimits. Limits in $\mathbb{CG}$ are computed by taking the limit in Top and then applying the functor $\mathbb{k}$. Colimits in $\mathbb{CG}$ are computed by taking the colimit in Top and then applying the functor $\mathbb{h}$. Note, however, that coproducts in $\mathbb{CG}$ can be computed exactly as they are in Top.

5.2. **Proposition.** *If $\{X_i\}$ is a set of compactly generated spaces, then $\coprod X_i$ is compactly generated and is the coproduct of $\{X_i\}$ in $\mathbb{CG}$.*

Proof. Immediate from (3.4)(3) and (4.7)(4). □

As is our convention, notation such as "$\amalg$" and "$\times$" refers to coproduct and product in Top. In fact, $\amalg$ is also the coproduct in $\mathbb{CG}$, while product in $\mathbb{CG}$ coincides with the k-space product $\times^{\mathbb{k}}$.

6. The projection criterion for k-open subsets of a product

We have a k-topology analogue of the tube lemma.

6.1. **Proposition.** *Let X be a space and L a compact space. Then the projection map $\pi\colon X \times L \to X$ takes closed sets to closed sets, and takes k-closed sets to k-closed sets.*

Proof. The first statement says that if C is closed in $X \times L$, then $\pi(C) = \{\, x \in X \mid (x \times L) \cap C \neq \varnothing \,\}$ is closed in X, or equivalently that $\{\, x \in X \mid (x \times L) \cap C = \varnothing \,\}$ is open in X. This is equivalent to the "tube lemma", which asserts that for such C, and any $x \in X$ such that $(x \times L) \cap C = \varnothing$ there exists an open neighborhood V of x in X such that $(V \times L) \cap C = \varnothing$. (Proof sketch: for each $y \in L$ choose $(x, y) \in V_y \times W_y \subseteq (X \times L) \smallsetminus C$ and use compactness of L to obtain a finite subcollection covering $x \times L$.)

For the second statement, suppose C is k-closed in $X \times L$, and consider any map $f\colon K \to X$ from compact K. Then

$$f^{-1}\{\, x \in X \mid (x \times L) \cap C \neq \varnothing \,\} = \{\, k \in K \mid (k \times L) \cap (f \times \mathrm{id})^{-1}C \neq \varnothing \,\}.$$

Since C is k-closed, $(f \times \mathrm{id})^{-1}C$ is closed in $K \times L$, so the tube lemma applied to $\pi\colon K \times L \to K$ shows that the above set is closed in K, and therefore that $\{\, x \in X \mid (x \times L) \cap C \neq \varnothing \,\}$ is k-closed in X as desired. □

The following gives a precise criterion for a subset of a product $X \times Y$ to be k-open. We will usually use this in the special case that both X and Y are themselves k-spaces, in which case it gives a precise description of the open subsets of $X \times^{\mathbb{k}} Y$.

6.2. **Proposition.** *Let X, Y be spaces. A subset $U \subseteq X \times Y$ is k-open if and only if*

(1) *for each $x \in X$, the set $U_x := \{\, y \in Y \mid (x, y) \in U \,\}$ is k-open in Y, and*
(2) *for each map $g\colon L \to Y$ from compact L, the set*

$$T_g(U) := \{\, x \in X \mid x \times g(L) \subseteq U \,\}$$

is k-open in X.

Proof. Note that if $C = (X \times Y) \smallsetminus U$, then (1) and (2) correspond precisely to

(1') for each $x \in X$, the set $C_x := \{\, y \in Y \mid (x, y) \in C \,\}$ is k-closed in Y, and
(2') for each map $g\colon L \to Y$ from compact Y, the set $\pi(\mathrm{id} \times g)^{-1}C$ is k-closed in X, where $\pi\colon X \times L \to X$ is the projection.

We will prove this formulation, i.e., that C is k-closed if and only if (1') and (2') hold.

If C is k-closed, then (1') is immediate, while (2') follows by (6.1).

Conversely, suppose $C \subseteq X \times Y$ satisfies (1') and (2'). By (4.3)(4), to show that C is k-closed, it suffices to show that $D := (\mathrm{id} \times f)^{-1}C$ is k-closed in $X \times M$, for all maps $f\colon M \to Y$ from compact M.

Suppose given such an $f\colon M\to Y$, and consider a point $(x_0,m_0)\notin D$. By (1'), C_{x_0} is k-closed in Y, and thus

$$\{\, m\in M \mid (x_0,m)\in D\,\} = f^{-1}C_{x_0}$$

is closed in the compact space M. Therefore there exists an open neighborhood V of m_0 in M such that $(x_0\times\overline{V})\cap D=\varnothing$, where $\overline{V}$ is the closure of V in M (compact spaces are locally compact).

The closure $\overline{V}$ of V in M is compact. Writing $j\colon \overline{V}\to M$ for the inclusion, we see that

$$E := \pi(\mathrm{id}\times j)^{-1}D = \pi(\mathrm{id}\times(fj))^{-1}C$$

is k-closed in X, where $\pi\colon X\times\overline{V}\to X$ is the projection, by (2'), taking $g=fj$. Since $(x_0\times\overline{V})\cap D=\varnothing$, we have that $x_0\notin E$, and thus there exists a k-open subset U in X such that $x_0\in U$ and $U\cap E=\varnothing$.

Thus, starting with $(x_0,m_0)\notin D$, we get a k-open subset $U\times V$ of $X\times M$ such that $(x_0,m_0)\in U\times V$ and $(U\times V)\cap D=\varnothing$. We have thus proved that D is k-closed, as desired. □

7. Locally compact spaces

We say that a space X is **locally compact** if it is Hausdorff, and if every point of X has a compact neighborhood. Every compact space (in our sense) is locally compact. Furthermore, if X is locally compact then for every open neighborhood U of a point $x\in X$, there exists an open V in X such that $x\in V\subseteq\overline{V}\subseteq U$ with $\overline{V}$ compact.

We will show that locally compact spaces are a class of CG spaces which satisfy convenient inheritance properties with respect passage to open subspaces and finite products.

7.1. **Proposition.** *Every locally compact space is compactly generated.*

Proof. Let X be a locally compact space. Thus we have that $X=\bigcup_K \mathrm{Int}_X K$, where the union ranges over all compact subspaces of X, and $\mathrm{Int}_X K$ denotes the interior of K relative to X.

We claim that the surjective map $f=(f_K)\colon \coprod_K K\to X$ obtained from the collection of all inclusions of compact subspaces is a proclusion. Suppose $U\subseteq X$ is a subset such that each $f_K^{-1}U=U\cap K$ is open in K. Then given $x\in U$ there exists a compact subset K of X such that $x\in\mathrm{Int}_X K$, whence $U\cap\mathrm{Int}_X K$ is open in $\mathrm{Int}_X K$ and hence is an open neighborhood of x in X which is itself a subset of U. Thus U is open in X.

It now follows that X is a k-space using (3.4), (2) and (3). Because X is Hausdorff, it is also k-Hausdorff by (4.7)(1). Thus X is compactly generated as desired. □

7.2. **Proposition.** *Every open subset of a locally compact space, viewed as a subspace, is compactly generated.*

Proof. By (7.1) it suffices to note that open subsets of locally compact spaces are themselves locally compact. □

7.3. **Proposition.** *If X is a k-space and Y is locally compact, then $X\times Y$ is a k-space*

Proof. Suppose $S\subseteq X\times Y$ is a k-open subset; we will show that S is open. By the projection criterion (6.2) and the fact that both X and Y are k-spaces (7.1), we have that

1. for each $x\in X$, the set $S_x=\{\, y\in Y\mid (x,y)\in S\,\}$ is open in Y, and
2. for each map $g\colon L\to Y$ from compact L, the set $T_g(S)=\{\, x\in X\mid x\times g(L)\subseteq S\,\}$ is open in X.

Consider $(x_0, y_0) \in S$. By (1), we have that S_{x_0} is open in Y. As Y is locally compact, there exists an open V in Y such that $y_0 \in V \subseteq \overline{V} \subseteq S_{x_0}$ with $\overline{V}$ compact. Let $g\colon \overline{V} \to Y$ denote the inclusion map. Then by (2) the set $T_g(S) = \{\, x \in X \mid x \times \overline{V} \subseteq S \,\}$ is open in X. Since $x_0 \in T_g(S)$, there exists an open set U in X with $x_0 \in U \subseteq T_g(S)$. We have thus found an open subset $U \times V$ of $X \times Y$ such that $(x_0, y_0) \in U \times V \subseteq S$. Thus, we have proved that S is open in $X \times Y$, and therefore that $X \times Y$ is a k-space. □

7.4. **Corollary.** *If X is any space and Y is locally compact, then the "identity" map $\Bbbk(X) \times Y \to \Bbbk(X \times Y)$ is a homeomorphism.*

Proof. By the universal property of $i\colon \Bbbk(X) \to X$ (3.2), composition with $i \times \mathrm{id}_Y$ induces a bijective correspondence $\mathrm{Map}(T, \Bbbk(X) \times Y) \to \mathrm{Map}(T, X \times Y)$ between sets of continuous maps for any compact T. Since $\Bbbk(X) \times Y$ is itself a k-space (7.3), we have that $i \times \mathrm{id}_Y\colon \Bbbk(X) \times Y \to X \times Y$ has the same universal property as $i\colon \Bbbk(X \times Y) \to X \times Y$, so the claim follows. □

7.5. **Proposition.** *If X is a compactly generated space and Y is locally compact, then $X \times Y$ is compactly generated.*

Proof. Given (7.3), it suffices to show that $X \times Y$ is k-Hausdorff. We want to show that the diagonal subset $\Delta_{X \times Y}$ is k-closed in $(X \times Y)^{\times 2}$, or equivalently that $\Delta_X \times \Delta_Y$ is k-closed in $X^{\times 2} \times Y^{\times 2}$ via the evident homeomorphism which switches middle factors. Since X is compactly generated, Δ_X is closed in $\Bbbk(X \times X)$. Since Y is locally compact it is Hausdorff, and thus Δ_Y is closed in $Y \times Y$. Therefore

$$\Delta_X \times \Delta_Y = [\Delta_X \times Y \times Y] \cap [\Bbbk(X \times X) \times \Delta_Y]$$

is closed in $\Bbbk(X \times X) \times Y \times Y$, which is identical to $\Bbbk(X \times X \times Y \times Y)$ by (7.4). □

8. Mapping spaces

Given spaces X and Y, let $\mathrm{Map}(X, Y)$ denote the set of continuous maps $X \to Y$. For a function $f\colon T \times X \to Y$ and $t \in T$, write $f_t\colon X \to Y$ for the restriction of f to the slice at t, i.e., $f_t(x) = f(t, x)$. Then functions $\widetilde{f}\colon T \to \mathrm{Map}(X, Y)$ from a set T correspond exactly to functions $f\colon T \times X \to Y$ such that each f_t is continuous.

We will show below that if X and Y are compactly generated spaces, then $\mathrm{Map}(X, Y)$ can be equipped with a compactly generated topology, so that it is an internal function object in $\mathbb{CG}$: continuous maps $T \to \mathrm{Map}(X, Y)$ correspond exactly to continuous maps $T \times^{\Bbbk} X \to Y$ for all compactly generated T.

8.1. **A topology on spaces of continuous maps.** We define a topology on $\mathrm{Map}(X, Y)$ as follows. We declare a subset $S \subseteq \mathrm{Map}(X, Y)$ to be open if, for every map $f\colon K \times X \to Y$ with K a compact space, the set

$$\widetilde{f}^{-1} S = \{\, k \in K \mid f_k \in S \,\}$$

is open in K; here $\widetilde{f}\colon K \to \mathrm{Map}(X, Y)$ is the evident adjoint function to $f\colon K \times X \to Y$. It is straightforward to see that this is indeed a topology on $\mathrm{Map}(X, Y)$.

8.2. **Proposition.** *The construction of* Map *gives rise to a functor* $\mathrm{Map}\colon \mathrm{Top}^{\mathrm{op}} \times \mathrm{Top} \to \mathrm{Top}$.

Proof. Let $g\colon X' \to X$ be a continuous map. We want to show that $\mathrm{Map}(g, Y)\colon \mathrm{Map}(X, Y) \to \mathrm{Map}(X', Y)$ is continuous. Let $S \subseteq \mathrm{Map}(X', Y)$ be an open set, and let $f\colon K \times X \to Y$ be a map with K compact. Then

$$\widetilde{f}^{-1} \mathrm{Map}(g, Y)^{-1} S = \{\, k \in K \mid f_k g \in S \,\} = \{\, k \in K \mid f'_k \in S \,\} = \widetilde{f'}^{-1} S,$$

where $f' = f(\mathrm{id} \times g)\colon K \times X' \to Y$, and $\widetilde{f'}^{-1} S$ is open in K by definition. This proves that $\mathrm{Map}(g, Y)$ is continuous, as desired.

Let $h\colon Y \to Y'$ be a continuous map. We want to show that $\mathrm{Map}(X,h)\colon \mathrm{Map}(X,Y) \to \mathrm{Map}(X,Y')$ is continuous. Let $S \subseteq \mathrm{Map}(X,Y')$ be an open set, and let $f\colon K \times X \to Y$ a map with K compact. Then

$$\widetilde{f}^{-1}\,\mathrm{Map}(X,h)^{-1}S = \{\, k \in K \mid hf_k \in S \,\} = \{\, k \in K \mid f'_k \in S \,\} = \widetilde{f'}^{-1}S,$$

where $f' = hf\colon K \times X \to Y'$, and $\widetilde{f'}^{-1}S$ is open in K by definition. This proves that $\mathrm{Map}(X,h)$ is continuous, as desired. □

8.3. **Construction of some open subsets in** $\mathrm{Map}(X,Y)$**.**

8.4. **Lemma.** *Let $g\colon L \to X$ be a map from a compact space L, and V an open subset of Y, and define*

$$S(g,V) := \{\, \phi \in \mathrm{Map}(X,Y) \mid \phi(g(L)) \subseteq V \,\}.$$

Then $S(g,V)$ is an open subset of $\mathrm{Map}(X,Y)$.

Proof. Given a map $f\colon K \times X \to Y$ with K compact, we see that

$$\widetilde{f}^{-1}S(g,V) = \{\, k \in K \mid f(k \times g(L)) \subseteq V \,\} = \{\, k \in K \mid k \times g(L) \subseteq f^{-1}V \,\}.$$

That is, we see that $\widetilde{f}^{-1}S(g,V) = T_g(f^{-1}V)$ as described in (6.2)(2), which asserts that it is k-open in K (and hence open) since $f^{-1}V$ is k-open (and in fact open) in $K \times X$. □

8.5. **Mapping space adjunction for k-spaces.**

8.6. **Proposition.** *If X and Y are k-spaces and Z is any space, then a function $f\colon X \times^{\Bbbk} Y \to Z$ is continuous if and only if its adjoint $\widetilde{f}\colon X \to \mathrm{Map}(Y,Z)$ is defined and continuous.*

Proof. First, suppose given a continuous map $f\colon X \times^{\Bbbk} Y \to Z$. Then each slice $f_x\colon Y \to Z$ is continuous, since it is isomorphic to the composite $Y = x \times^{\Bbbk} Y \to X \times^{\Bbbk} Y \xrightarrow{f} Z$ as Y is a k-space. Thus we have a well-defined adjoint function $\widetilde{f}\colon X \to \mathrm{Map}(Y,Z)$.

To show that $\widetilde{f}$ is continuous, consider an open subset $S \subseteq \mathrm{Map}(Y,Z)$. Since X is a k-space, we must show that for any continuous $g\colon K \to X$ with K compact the set

$$g^{-1}\widetilde{f}^{-1}S = g^{-1}\{\, x \in X \mid f_x \in S \,\} = \{\, k \in K \mid f_{g(x)} \in S \,\} = \{\, k \in K \mid f'_k \in S \,\} = \widetilde{f'}^{-1}S$$

is open in K, where $f' = f(g \times \mathrm{id})\colon K \times Y \to Z$. That this is so is immediate from the definition of the topology on $\mathrm{Map}(Y,Z)$.

Next, suppose given $\widetilde{f}\colon X \to \mathrm{Map}(Y,Z)$ a continuous map, and let $f\colon X \times Y \to Z$ denote the adjoint function. We want to show that, as a function $X \times^{\Bbbk} Y \to Z$ it is continuous. Thus, given V an open set in Z, we need to show that $f^{-1}V$ is k-open in $X \times Y$.

By the criterion of (6.2), and the fact that both X and Y are k-spaces, we must show

(1) that each $f_x^{-1}V = \{\, y \in Y \mid (x,y) \in f^{-1}V \,\}$ is open in Y, and
(2) that for each map $g\colon L \to Y$ from compact L, the set $T_g(f^{-1}V) = \{\, x \in X \mid x \times g(L) \subseteq f^{-1}V \,\}$ is open in X.

Statement (1) is immediate from the fact that each $f_x\colon Y \to Z$ is continuous. Statement (2) follows because $T_g(f^{-1}V) = \widetilde{f}^{-1}S(g,V)$, and that $S(g,V)$ is open in $\mathrm{Map}(Y,Z)$ by (8.4). □

8.7. **Corollary.** *If X and Y are spaces with X a k-space, then* $\mathrm{Map}(X,Y)$ *is a k-space.*

Proof. A subset S of $\mathrm{Map}(X,Y)$ is k-open if and only if $\widetilde{f}^{-1}S$ is open in K for each continuous map $\widetilde{f}\colon K \to \mathrm{Map}(X,Y)$ from compact K, while by definition S is open if and only if $\widetilde{f}^{-1}S$ is open in K for each adjoint $\widetilde{f}$ of a continuous map $f\colon K \times X \to Y$ with K compact. By (7.3), $K \times X$ is a k-space, whence $K \times X = K \times^{\Bbbk} X$. The result follows by (8.6), which shows that continuous $\widetilde{f}\colon K \to \mathrm{Map}(X,Y)$ correspond exactly to continuous $f\colon K \times^{\Bbbk} X \to Y$. □

8.8. **Proposition.** *For all spaces Z and k-spaces X and Y, the evident bijection* $\mathrm{Map}(X, \mathrm{Map}(Y, Z)) \to \mathrm{Map}(X \times^{\mathbb{k}} Y, Z)$ *implied by* (8.6) *is a homeomorphism.*

Proof. Since both mapping spaces are k-spaces (8.7), it suffices to show that for T a k-space this map induces a bijective correspondence between maps $T \to \mathrm{Map}(X, \mathrm{Map}(Y, Z))$ and maps $T \to \mathrm{Map}(X \times^{\mathbb{k}} Y, Z)$. That this is so is a straightforward consequence of (8.6), and the fact (3.6) that $(T \times^{\mathbb{k}} X) \times^{\mathbb{k}} Y = T \times^{\mathbb{k}} (X \times^{\mathbb{k}} Y)$. □

8.9. **Maps into k-Hausdorff spaces.**

8.10. **Proposition.** *If X is a space and Y is a k-Hausdorff space, then* $\mathrm{Map}(X, Y)$ *is k-Hausdorff.*

Proof. Write $M := \mathrm{Map}(X, Y)$. We need to show that the diagonal Δ_M is k-closed in $M \times M$. For $x \in X$, let $e_x \colon M \to Y$ denote the evaluation map $e_x(f) = f(x)$, which is continuous by (8.2). Then

$$(e_x \times e_x)^{-1}\Delta_Y = \{\, (f_1, f_2) \in M \times M \mid f_1(x) = f_2(x) \,\}$$

is k-closed in $M \times M$ since Y is k-Hausdorff, and hence $\Delta_M = \bigcap_{x \in X} (e_x \times e_x)^{-1}\Delta_Y$ is k-closed in $M \times M$ as desired. □

8.11. **Internal function objects in $\mathbb{CG}$.** Putting together what we have done, we see that if X and Y are compactly generated spaces, then $\mathrm{Map}(X, Y)$ is compactly generated (8.7), (8.10). We obtain the following from (8.8).

8.12. **Corollary.** *The mapping space construction exhibits $\mathbb{CG}$ as a Cartesian closed category. In particular, for compactly generated spaces X, Y, Z there are natural homomorphisms*

$$\mathrm{Map}(X, \mathrm{Map}(Y, Z)) \approx \mathrm{Map}(X \times^{\mathbb{k}} Y, Z)$$

of compactly generated spaces.

9. Compactly generated spaces and quotients

9.1. **Products of proclusions of k-spaces.**

9.2. **Proposition.** *Let $f \colon X \to X'$ be a proclusion between k-spaces, and let Y be a k-space. Then the map $f \times^{\mathbb{k}} \mathrm{id} \colon X \times^{\mathbb{k}} Y \to X' \times^{\mathbb{k}} Y$ is a proclusion.*

Proof. To show that the surjective map $f \times^{\mathbb{k}} \mathrm{id}$ is a proclusion, it suffices to show that if $g' \colon X' \times^{\mathbb{k}} Y \to Z$ is a function to any space Z such that $g := g'(f \times \mathrm{id}) \colon X \times^{\mathbb{k}} Y \to Z$ is continuous, then g' is itself continuous.

First note that for any $x' \in X$, the slice function $g'_{x'} \colon Y \to Z$ defined by $g'_{x'}(y) = g'(x, y)$ is continuous. To see this, choose any $x \in X$ such that $f(x) = x'$, and note that $g'_{x'} = g_x$, which is continuous because Y is a k-space and so g_x is the composite $Y = x \times^{\mathbb{k}} Y \to X \times^{\mathbb{k}} Y \xrightarrow{g} Z$.

Thus, the adjoint function $\widetilde{g'} \colon X' \to \mathrm{Map}(Y, Z)$ to g' is well-defined, and $\widetilde{g} = \widetilde{g'} f$ is the adjoint function to g. By (8.6) $\widetilde{g}$ is continuous, and thus $\widetilde{g'}$ is continuous since f is a proclusion. Apply (8.6) again to see that g' is continuous, as desired. □

9.3. **Proposition.** *Let $f \colon X \to X'$ and $g \colon Y \to Y'$ be proclusions between k-spaces. Then $f \times^{\mathbb{k}} g \colon X \times^{\mathbb{k}} Y \to X' \times^{\mathbb{k}} Y'$ is a proclusion.*

Proof. Factor $f \times^{\mathbb{k}} g$ as $X \times^{\mathbb{k}} Y \xrightarrow{f \times^{\mathbb{k}} \mathrm{id}} X' \times^{\mathbb{k}} Y \xrightarrow{\mathrm{id} \times^{\mathbb{k}} g} X' \times^{\mathbb{k}} Y'$ and apply (9.2). □

9.4. **Compactly generated quotients of k-spaces.**

9.5. **Proposition.** *Let $f\colon X \to Y$ be a proclusion from a k-space X. The following are equivalent.*

(1) *Y is compactly generated.*
(2) *$(f\times f)^{-1}\Delta_Y$ is k-closed in $X\times X$.*

Proof. (1) implies (2). Δ_Y is k-closed in $Y\times Y$ by definition of k-Hausdorff, whence $(f\times f)^{-1}\Delta_Y$ is k-closed in $X\times X$.

(2) implies (1). By (3.4)(2), Y is a k-space. By (9.3), $f\times^{\Bbbk} f\colon X\times^{\Bbbk} X\to Y\times^{\Bbbk} Y$ is a proclusion. By (2), $(f\times^{\Bbbk} f)^{-1}\Delta_Y$ is closed in $X\times^{\Bbbk} X$, whence Δ_Y is closed in $Y\times^{\Bbbk} Y$, so that Y is k-Hausdorff. □

9.6. **Proposition.** *Let X be a k-space, and suppose $E\subseteq X\times X$ is a k-closed set which is also an equivalence relation on X. Then the quotient space $Y := X/\sim_E$ is compactly generated.*

Proof. If $f\colon X\to Y$ is the quotient map, then $E=(f\times f)^{-1}\Delta_Y$, and the claim follows from (9.5). □

9.7. **Corollary.** *If X is a k-space, then $\mathbb{h}(X)$ is the quotient of X by the smallest k-closed equivalence relation in $X\times X$.*

Proof. Such a smallest k-closed equivalence relation exists, since k-closed subsets are closed under intersection. The result is a formal consequence of (9.5) and (9.6). □

9.8. **CG spaces as quotients of coproducts of compact spaces.**

9.9. **Proposition.** *A topological space X compactly generated if and only if it is a k-space, and if for all $f\colon K\to X$ and $g\colon L\to X$ from compact K, L, the fiber product $K\times_X L$ is compact.*

Proof. Suppose X a k-space. By (4.2) we see that X is k-Hausdorff if and only if and only if $(f\times g)^{-1}\Delta_X = K\times_X L$ is closed in $K\times L$ for all $f\colon K\to X$, $g\colon L\to X$ from compact K, L. Since $K\times L$ is compact, $K\times_X L$ is closed in $K\times L$ if and only if $K\times_X L$ is compact. □

We obtain the following pleasant characterization of compactly generated spaces.

9.10. **Proposition.** *Let X be a topological space. Then X is compactly generated if and only if there exists a set of maps $\{f_\alpha\colon K_\alpha\to X\}_{\alpha\in A}$ such that*

(1) *$f=(f_\alpha)\colon \coprod_{\alpha\in A} K_\alpha\to X$ is a proclusion,*
(2) *each K_α is compact, and*
(3) *each $K_\alpha\times_X K_\beta$ is compact.*

Proof. First suppose we are given $\{f_\alpha\}$ satisfying (1)–(3). We have that $U:=\coprod_\alpha K_\alpha$ is compactly generated by (2) and (5.2). Observe that $U\times U=\coprod_{\alpha,\beta} K_\alpha\times K_\beta$ is also compactly generated for the same reason (finite products in Top distribute over coproducts), so that $U\times U = U\times^{\Bbbk} U$. To show that X is compactly generated, it thus suffices by (1) and (9.5) to show that $(f\times f)^{-1}\Delta_X$ is closed in $U\times U$. This follows since $(f_\alpha\times f_\beta)^{-1}\Delta_X = K_\alpha\times_X K_\beta$ is compact by (3) and hence is a closed subset of the compact space $K_\alpha\times K_\beta$.

Now suppose X is compactly generated. To obtain a set of maps $\{f_\alpha\}$, choose for each non-closed subset $S\subseteq X$ a map $f\colon K\to X$ from a compact space K such that $f^{-1}S$ is not closed in K. Taken together with inclusions $\{x\}\to X$ of singletons, the resulting collection $\{f_\alpha\}$ satisfies (1) and (2). Condition (3) follows by (9.9). □

9.11. **Corollary.** *Every CW-complex is compactly generated.*

Proof. Let X be a CW-complex, and consider the collection $\{f_\alpha\colon K_\alpha\to X\}$ of all inclusions of finite CW-subcomplexes. This collection satisfies conditions (1)–(3) of (9.10). □

10. Some colimits and limits in compactly generated spaces

As we have already noted, colimits in $\mathbb{CG}$ are obtained by taking the colimit in Top, and then applying $\mathbb{h}$, while limits in $\mathbb{CG}$ are obtained by taking the limit in Top, and then applying $\mathbb{k}$. Furthermore, coproducts in $\mathbb{CG}$ coincide with coproducts in Top. We observe some more situations in which certain limits and colimits in $\mathbb{CG}$ and Top coincide.

10.1. **Closed subspaces.** Note that subspaces of a compactly generated space are not necessarily themselves compactly generated. However, it is reasonable to speak of "closed subspaces" of a compactly generated space.

10.2. **Proposition.** *A closed subspace of a compactly generated space is compactly generated.*

Proof. Immediate from (3.4)(1) and (4.7)(2). □

10.3. **Lemma.** *Let $i\colon A \to X$ be a closed inclusion of spaces. Then i is a k-closed map, i.e., if C is k-closed in A, then $i(C)$ is k-closed in X.*

Proof. Let C be a k-closed subset of A, and suppose $f\colon K \to X$ is a map from a compact K. Form the pullback square

$$\begin{array}{ccc} A \times_X K & \xrightarrow{j} & K \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle f} \\ A & \xrightarrow[i]{} & X \end{array}$$

in Top, and note that $f^{-1}i(C) = jg^{-1}(C)$. Because i is a closed inclusion, so is j, and thus $A \times_X K$ is a closed subspace of K and hence compact. It follows that $jg^{-1}(C)$ is closed in K. □

10.4. **Corollary.** *If X is a topological space, and if $i\colon A \to X$ is inclusion of a closed subspace, then $\mathbb{k}i\colon \mathbb{k}A \to \mathbb{k}X$ is also inclusion of a closed subspace.*

10.5. **Pushouts along closed inclusions.**

10.6. **Proposition.** *Let $i\colon A \to X$ and $f\colon A \to B$ be maps of compactly generated spaces spaces, and let $Y = \operatorname{colim}(X \xleftarrow{i} A \xrightarrow{f} B)$ denote the pushout in* Top. *If i is the inclusion of a closed subspace, then Y is compactly generated, and thus Y is the pushout in $\mathbb{CG}$.*

Proof. Note that Y is also the colimit of the diagram

$$X \amalg B \xleftarrow{i \amalg \mathrm{id}} A \amalg B \xrightarrow{(f,\mathrm{id})} B.$$

Since $i \amalg \mathrm{id}$ is a closed inclusion, and (f, id) is a proclusion, we can replace i and f with these maps, as coproducts of compactly generated spaces are compactly generated. Thus without loss of generality we may assume that f is a proclusion. We will use (9.5) applied to $g\colon X \to Y$ to show that Y is compactly generated.

Because i is injective we have that $(g \times g)^{-1}\Delta_Y = \Delta_X \cup (i \times i)(f \times f)^{-1}\Delta_B$. We know that Δ_X is k-closed in $X \times X$ since X is k-Hausdorff, and $(f \times f)^{-1}\Delta_B$ is k-closed in $A \times A$ since B is k-Hausdorff. Since i is a closed inclusion of spaces, so is $i \times i$, and thus $(i \times i)(f \times f)^{-1}\Delta_B$ is k-closed in $X \times X$ by (10.3). We conclude that $(g \times g)^{-1}\Delta_Y$ is k-closed in $X \times X$, and thus $(g \times^{\mathbb{k}} g)^{-1}\Delta_Y$ is closed in $X \times^{\mathbb{k}} X$, proving statement (2) of (9.5), which implies that Y is compactly generated since f is a proclusion. □

10.7. **Corollary.** *If $A \subseteq X$ is a closed subset of a compactly generated space X, then X/A is compactly generated.*

10.8. **Pullbacks along closed inclusions.**

10.9. **Proposition.** *Suppose* $X \xrightarrow{f} Y \xleftarrow{i} B$ *are maps between compactly generated spaces, and that* i *is a closed inclusion. Then the pullback* $A := X \times_Y B$ *in* Top *is also compactly generated, and* $A \to \Bbbk(X \times_Y B)$ *is a homeomorphism, i.e.,* A *is the pullback of* $X \xrightarrow{f} Y \xleftarrow{i} B$ *in* $\mathbb{CG}$.

Proof. Since i is a closed inclusion, its pullback $j\colon A \to X$ is also a closed inclusion, and therefore A is compactly generated (10.2). That $A \to \Bbbk(X \times_Y B)$ is a homeomorphism follows by verifying that both objects satisfy the same universal property in $\mathbb{CG}$. ☐

The following gives a direct construction of an arbitrary pullback in $\mathbb{CG}$.

10.10. **Proposition.** *Given maps* $X \xrightarrow{f} Y \xleftarrow{p} B$ *between compactly generated spaces, let*

$$\begin{array}{ccc} A & \xrightarrow{\quad g \quad} & B \\ {\scriptstyle (q,g)}\downarrow & & \downarrow{\scriptstyle (p,\mathrm{id})} \\ X \times^{\Bbbk} B & \xrightarrow[f\times \mathrm{id}]{} & Y \times^{\Bbbk} B \end{array}$$

be a commutative square which is a pullback square in Top. *Then* A *is compactly generated, and* $(q,g)\colon A \to \Bbbk(X \times_Y B)$ *is a homeomorphism, i.e.,* A *is the pullback of* $X \xrightarrow{f} Y \xleftarrow{p} B$ *in* $\mathbb{CG}$. *Furthermore, both vertical maps are closed inclusions.*

Proof. (See [Str09, 2.36].) Let $\Gamma = \{\, (p(b), b) \mid b \in B \,\}$ denote the graph of p in $Y \times B$. As $\Gamma = (\mathrm{id} \times p)^{-1}\Delta_Y$ and Y is k-Hausdorff, we have that Γ is a closed subset of $Y \times^{\Bbbk} B$. If $C \subseteq B$ is any subset, then $(p, \mathrm{id})(C) = \Gamma \cap \pi^{-1}C$, and thus we see that $(p, \mathrm{id})\colon B \to Y \times^{\Bbbk} B$ is a closed injective map, i.e., an inclusion of a closed subspace. Therefore its pullback $(q, g)\colon A \to X \times^{\Bbbk} B$ is also a closed inclusion, and by (10.9), A is compactly generated and the diagram is a pullback in $\mathbb{CG}$. That $(q, g)\colon A \to \Bbbk(X \times_Y B)$ is a homeomorphism follows by verifying that both objects satisfy the same universal property in $\mathbb{CG}$. ☐

10.11. **Pullbacks of proclusions.**

10.12. **Proposition.** *In the category of compactly generated spaces, proclusions are closed under base change.*

Proof. Let $X \xrightarrow{f} Y \xleftarrow{p} B$ be maps of compactly generated spaces such that f is a proclusion. By (9.2), $f \times \mathrm{id}\colon X \times^{\Bbbk} B \to Y \times^{\Bbbk} B$ is a proclusion. Using (10.10), it thus suffices to show that the pullback in Top of a proclusion along a closed map is a proclusion, which is straightforward. ☐

Warning. This does *not* imply that arbitrary colimits in $\mathbb{CG}$ are preserved under base change: Although all colimits in $\mathbb{CG}$ can be described using proclusions, colimits in $\mathbb{CG}$ do not necessarily coincide with set-theoretic colimits. See [Str09, 6.9] for a counterexample.

Remark. The above proposition is not true if "compactly generated space" is replaced with "k-space". This is an advantage of compactly generated spaces over k-spaces.

10.13. **Sequential colimits.**

10.14. **Proposition.** *Let* $X_0 \to X_1 \to \cdots$ *be a countable direct sequence of compactly generated spaces, and suppose that each map* $f_{n,n+1}\colon X_n \to X_{n+1}$ *is the inclusion of a closed subspace. Let* X *denote the direct limit in* Top. *Then*

(1) X *is compactly generated,*

(2) *each* $X_k \to X$ *is a closed inclusion, and*
(3) *every map* $f\colon K \to X$ *from a compact* K *factors through* X_n *for some* n.

Proof. (See [Str09, §3.2].) First we prove (1). It is clear that X is a k-space, being a quotient of the k-space $U := \coprod_k X_k$ (3.4).

To show that X is k-Hausdorff, $E \subseteq U \times U$ denote the equivalence relation defining X. As $U \times U = \coprod_{i,j} X_i \times X_j$, we have $E = \coprod_{i,j} E_{i,j}$ where $E_{i,j} = E \cap (X_i \times X_j)$ as a subspace of $X_i \times X_j$.

Since each $f_{i,k}\colon X_i \to X_k$ is injective for $i \leq k$, we have that $E_{i,j} = (f_{i,k} \times f_{j,k})^{-1}\Delta_{X_k}$ for any $k \geq \max(i, j)$. Since the X_k are k-Hausdorff, we have that $E_{i,j}$ is k-closed in $X_i \times X_j$, and thus E is k-closed in $U \times U$. It follows that $X = U/\sim_E$ is k-Hausdorff by (9.6). (Note: it seems that to prove that (1), we only needed the maps $X_n \to X_{n+1}$ to be injective, not necessarily closed.)

To prove (2), note that this is merely a property of colimits in Top along sequences of closed inclusions.

Now we prove (3). Suppose $f\colon K \to X$ is a map from a compact K, and suppose that f does not factor through any X_n; we will derive a contradiction. For each n choose an $x_n \in (X \smallsetminus X_n) \cap f(K)$, and let $T = \{x_n\}$; this must be an infinite set. Each intersection of $S \cap X_k$ of any subset $S \subseteq T$ is finite, and thus is closed in X_k by (4.7)(5). Since $X_k \to X$ is a closed inclusion, each such S is closed in X, and thus T is a closed and discrete subset of X, and so a closed and discrete subset of $f(K)$. Since $f(K)$ is quasi-compact, T is necessarily finite, a contradiction. □

11. Weak Hausdorff spaces

A space X is said to be **weak Hausdorff** if for every map $f\colon K \to X$ with K compact, the image $f(K)$ is closed in X.

11.1. **Proposition.** *If* X *is weak Hausdorff, then for every map* $f\colon K \to X$ *with* K *compact, the image* $f(K)$ *is a compact subspace of* X.

Proof. It is clear that the subspace $f(K)$ is quasi-compact, so we only need to show that it is Hausdorff. It is immediate that points in a weak Hausdorff space are closed. Thus if $x_1, x_2 \in f(K)$ are distinct points, then $f^{-1}(x_1)$ and $f^{-1}(x_2)$ are closed in K and hence are compact subsets of K. Since compact spaces are normal, there exist open neighborhoods U_i of $f^{-1}(x_i)$ in K such that U_1 and U_2 are disjoint. Since the subspaces $K \smallsetminus U_i$ are compact, their images $f(K \smallsetminus U_i)$ are closed in the weak Hausdorff space X, and hence closed in $f(K)$. Thus the sets $V_i = f(K) \smallsetminus f(K \smallsetminus U_i)$ are open in $f(K)$, and provide disjoint neighborhoods of x_1 and x_2 in $f(K)$. □

11.2. **Proposition.** *Every weak Hausdorff space is k-Hausdorff.*

Proof. Suppose X is weak Hausdorff, and let $f\colon K \to X$ be a map from compact K. Let $k_1, k_2 \in K$ be points such that $f(k_1) \neq f(k_2)$. Since $f(K)$ is compact by (11.1), there exist disjoint open neighborhoods V_i of $f(k_i)$ in $f(K)$, and thus $U_i = f^{-1}V_i$ provide open neighborhoods of k_i such that the $f(U_i)$ are pairwise disjoint. It follows that X is k-Hausdorff by (4.2)(4). □

11.3. **Lemma.** *If* X *is k-Hausdorff and* $f\colon K \to X$ *is a map from a compact* K*, then the subset* $f(K)$ *is k-closed in* X.

Proof. Let $g\colon L \to X$ be any map from a compact L, and let $\pi\colon K \times_X L \to L$ denote the projection. Then $g^{-1}f(K) = \pi(f \times g)^{-1}\Delta_X$ as a subset of L. Since X is k-Hausdorff, $(f \times g)^{-1}\Delta_X$ is closed in $K \times L$ by (4.2), and thus its image under π in the compact space L is closed. We conclude that $f(K)$ is a k-closed subset of X. □

11.4. **Proposition.** *Let X be a k-space. The following are equivalent.*

(1) *X is k-Hausdorff.*
(2) *X is a weak Hausdorff k-space.*

Thus, the compactly generated spaces (in our sense) are precisely the weak Hausdorff k-spaces.

Proof. Immediate using (11.3) and (11.2). □

11.5. **Proposition.** *Let X be weak Hausdorff. The following are equivalent.*

(1) *X is compactly generated.*
(2) *A subset $C \subseteq X$ is closed if and only if its intersection with every compact subset of X is closed.*

Proof. Let X be weak Hausdorff. We will show that a subset $C \subseteq X$ is k-closed if and only if all its intersection with compact subsets of X are closed. Given this, the equivalence of (1) and (2) is immediate.

Suppose $C \subseteq X$ is k-closed. Then it is immediate that intersections of C with compact subsets K of X are closed in K, and thus closed in X by the weak Hausdorff property.

Conversely, suppose that $C \subseteq X$ is such that all intersections with compact subsets of X are closed in X. Let $f\colon K \to X$ be a map from a compact K. Then $f^{-1}C = f^{-1}(C \cap f(K))$. The weak Hausdorff property says that $f(K)$ is compact in X by (11.4), whence $f^{-1}C$ is closed in K. Thus, C is k-closed. □

References

[Lew] L. G. Lewis Jr., *The stable category and generalized Thom spectra: Appendix A*, http://www.math.uchicago.edu/~may/MISC/GaunceApp.pdf. Ph.D. Thesis, University of Chicago, 1978.

[McC69] M. C. McCord, *Classifying spaces and infinite symmetric products*, Trans. Amer. Math. Soc. **146** (1969), 273–298.

[Str09] N. P. Strickland, *The category of CGWH spaces* (2009), preprint, available at http://neil-strickland.staff.shef.ac.uk/courses/homotopy/cgwh.pdf.

Department of Mathematics, University of Illinois at Urbana-Champaign, Urbana, IL
E-mail address: rezk@math.uiuc.edu