Mirror Symmetry and Fukaya Categories

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0 Introduction

In this text, we have the goal to introduce the definition of the Fukaya Category and present the statement of the Homological Mirror Symmetry Conjecture, which was first stated in 1994 by Kontsevich [8].

This conjecture relates the symplectic geometry and the complex algebraic geometry of Calabi Yau manifolds. It arose when physicists came up with the concept of mirror manifolds in string theory: topologically different spaces for which the quantum field theories were equivalent.

As the physical theory of mirror manifolds showed itself to be useful to predict mathematical results, the mathematical community started to try to find a way to bring rigour to these discoveries [2]. In this context, Kontsevich proposed the conjecture that mirror manifolds should be related to an equivalence of categories involving the Fukaya Category, on one side, and Coherent Sheaves, on the other side.

In this project, our goal is to present the foundations to define the Fukaya Category and state HMS conjecture. In order to do this, we will also give some results related to algebraic complex geometry and homological algebra.

It is important to emphasize that we do not have any hope of this being a comprehensive introdution to the topic. It could be considered a survey, in which we give the most important results, definitions and intuitions but try to not get too technical.

The work is divided in three sections. In the first, we focus on defining the Floer Complex for Lagrangians submanifolds and list some of its properties. Then, in the second, we define what is the Fukaya Category of a manifold. To do this, we also study $\mathcal{A}_\infty$-categories and technicalities about the Floer Complex, which are necessary to understand Fukaya Categories. Finally, in the third, we focus on presenting some results and definitions in order to be able to present the Homological Mirror Symmetry conjecture. We try to explain some of the ideias behind definitions and results, but we do not really dive into why the conjecture should be true.

We tried to use as many references as possible in order to get a more general view of the topic. We try to cite them during the text.

We finish hoping that the text is readable and apologize for any mistakes.

Background and Notation

We start by fixing notation and recalling some basic defintions and results from Symplectic geometry. For basics on Symplectic geometry, see [15] or [9].

Recall that a *Symplectic Manifold* is a C^∞ manifold (for us, all manifolds will be C^∞ unless explicitly stated otherwise) equipped with a so-called *symplectic form* ω, that is, a closed and non-degenerate two-form. We denote a symplectic manifold as the pair (M, ω). It is important to recall that all symplectic manifolds are even dimensional.

A n-dimensional submanifold L of a $2n$-dimensional symplectic manifold (M, ω) for which $\omega|_L = 0$ is called a *Lagrangian submanifold*. Recall that locally, a Lagrangian can always be embedded in a neighborhood of the zero section of the T^*L with the tautological form. That is, all lagrangians have the same local behavior.

An *almost complex structure* on M is a tensor J which is a linear transformation $J_p : T_pM \to T_pM$ such that $J_p^2 = -I$. If M is symplectic, J is compatible with the symplectic form if $\omega(u, Jv) = g(u, v)$ is a positive definate Riemannian metric.

A J-holomorphic map $f : N \to M$ from a complex manifold to M is a map which satisfy the Cauchy Riemann equations with respect to J (just interchange i and J).

1 Lagrangian Floer Homology

In this first part of the text, we have the goal of defining Lagrangian Floer Homology and present some of its properties, specially the ones which are going to be important for the second and third parts of the text.

For this section, our main references were [10], [1] and [3].

Lagrangian Floer Homology was created in order to better understand how the intersection of two lagrangians should behave. More specifically, it was developed to prove the following conjecture first presented by Arnold in the sixties. A somewhat detailed exposition on this conjecture for simple cases can be found in [9].

Conjecture 1.1 *Let (M, ω) be a closed symplectic manifold and let φ be a Hamiltonian diffeomorphism for which all fixed points are non degenerate. Then:*

$$\#\{p \in M \mid \varphi(p) = p\} \geq \sum_{j=0}^{2n} rkH_j(M, \mathbb{Z}_2).$$

But how does it relates to Lagragians? It is simple! The fixed points of φ correspond to the intersection points of the graph $\Gamma(\varphi)$ and the diagonal Δ in the symplectic manifold given by the cartesian product $(M \times M, \omega \oplus -\omega)$. So, to study the fixed points of a Hamiltonian diffeomorphism, it suffices to study how two lagragians intersect and, more specifically, how a lagrangian intersects with a deformation of itself.

This inspired Floer to define the Lagrangian Floer homology. The construction of it is very similar to the construction of the Morse Homology. But, instead of considering the critical points of a Morse function on a manifold, we consider the critical points of a functional on a infinite dimensional space related to a pair of lagrangians. These critical points will turn out to be the intersection points of these two lagrangians. It is important, nevertheless, to note that we need some restrictions for these ideas to work, what means that Lagrangian Floer homology is not always defined. Anyway, this restrictions are natural and are satisfied in many cases. In this section, in order to explain the main ideas, we are going to assume that (M, ω) is *symplectic aspherical* (this term is used in [10]), that is, for any map $f : S^2 \to M$, we have $\int_{S^2} f^*\omega = 0$. Floer was able to prove the following in [3]

Theorem 1.2 (in [3]) *Under these assumptions, we have:*

$$\#(L \cap \varphi(L)) \geq \sum_{j=0}^{n} rkH_j(L, \mathbb{Z}_2),$$

which would imply the conjecture in our case.

1.1 The action functional and J-holomorphic strips

Consider a pair of compact lagrangians L_0 and L_1 inside (M, ω) which intersect transversally. Define the space of smooth paths between them as:

$$\mathcal{P}(L_0, L_1) = \mathcal{P} := \{\gamma : [0,1] \to M \text{ smooth} \mid \gamma(0) \in L_0 \text{ and } \gamma(1) \in L_1\}.$$

This space has a natural topological structure given by the C^∞ topology. Although, this is space is not necessarily connected. To get a nicer space, we fix any path $\gamma_0 \in \mathcal{P}$ and consider the universal covering of its connected component: $\tilde{\mathcal{P}}$.

This universal covering is given by pairs $(\gamma, [u])$, where γ is a path in $\mathcal{P}$ and u is the homotopy class of maps:

$$u : [0,1] \times [0,1] \to M, \qquad (s,t) \mapsto u_s(t),$$

for which $u_0 = \gamma_0$, $u_1 = \gamma_1$, $u_s(0) \in L_0$ and $u_s(1) \in L_1$.

These maps u, of course, represent "paths" in the space $\mathcal{P}$, so this construction is the usual construction of the universal covering.

Now, we would like to define a functional on $\tilde{\mathcal{P}}$. This functional will work as our "morse function", as we will see:

$$A : \tilde{\mathcal{P}} \to \mathbb{R}$$
$$A(\gamma, [u]) = -\int u^*\omega.$$

Proposition 1.3 *A is well defined if M is symplectically aspherical and if the lagrangians are simply connected.*

Proof. Suppose that we have two pairs (γ, u) and (γ, v), with u and v in the same homotopy class. This means we have a cylinder $c : S^1 \times [0,1] \to M$ "joining" u and v (see the picture) with boundary given by two loops (one in L_1 and another in L_0).

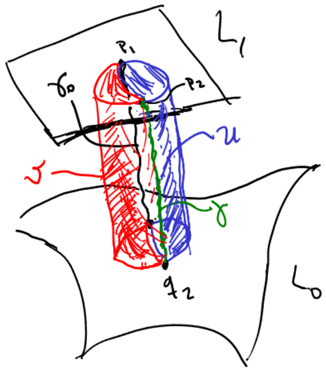


Figure 1: Homotopies u and v forming a cylinder

Now, as we assumed that L_0 and L_1 are simply connected, we can consider the loops as boundaries of disks, and therefore, consider a topological sphere $\hat{c}$ as the cylinder glued with the two disks. Now, using the hypothesis and the fact that ω is zero on the lagrangians:

$$\int_{\hat{c}} \omega = 0 \implies \int_{[0,1]\times[0,1]} u^*\omega - \int v^*\omega = 0,$$

since we have to reverse the direction of v.

We conclude that A is well defined under this hypothesis.

■

We could define a functional in another space different from $\tilde{P}$ and with different assumptions about our objects. We are not going to enter into much detail here, since the goal of this article is not to give a detailed introduction to Floer Homology, but to understand some of the relations between it and Mirror Symmetry.

Now, to compute the derivative of A and get the critical points as it is done in Morse Homology, we have to consider tangent vectors at $\tilde{\mathcal{P}}$. These, at a point (γ, u), are vector fields along γ obtained by looking at how γ vary infinitesimally (remember the tangent space of $\tilde{\mathcal{P}}$ is the same as the tangent space of $\mathcal{P}$. Then, taking X to be such a tangent vector (that, is, a vector field along γ):

$$dA_{(\gamma,u)}(X) = -\int_0^1 \omega(\gamma'(t), X(t))dt,$$

since those are the partial derivatives of u at a point t.

Taking an almost complex structure J which is compatible with ω and denoting by g the Riemannian metric associated, we get:

$$dA_{(\gamma,u)}(X) = -\int_0^1 \omega(\gamma'(t), X(t))dt = -\int_0^1 g_{\gamma(t)}(J_{\gamma(t)}(\gamma'(t)), X(t))dt.$$

As this integral is a metric on the tangent space of $\tilde{\mathcal{P}}$, we conclude that, for the expression above to be zero for any X, we would have to have $\gamma'(t) = 0$ for all t, meaning that γ is constant. Observing that the constant paths are exactly the intersection points between L_0 and L_1, we see that the construction is giving us the results we expected.

The next step in the construction is, as it is done for Morse homology, to consider the flow lines of the gradient field connecting two critical points.

First, we compute the gradient. In our case, we can consider the metric above and write:

$$dA_{(\gamma,u)}(X) = \langle -J\gamma', X\rangle,$$

denoting the integral by $\langle\ ,\ \rangle$.

Thus, we have:

$$\text{grad}A(\gamma, u) = -J\gamma'.$$

Here, we see γ' as a infinitesimal variation of the point $[(\gamma, u)]$ and, therefore, as an element of the tangent space of $\tilde{\mathcal{P}}$.

After this, recall that a flow line connecting two critical points p and q is a curve on the space: in our case, a map $f : \mathbb{R} \to \tilde{\mathcal{P}}$ satisfying the following equations:

$$\frac{df}{ds} = \text{grad}A(f(s)) \qquad \lim_{s\to\infty} f(s) = q \qquad \lim_{s\to-\infty} f(s) = p$$

If we use that a map f as above is actually a map $f : \mathbb{R} \times [0,1] \to M$ and that the first equation is

$$\frac{\partial f}{\partial s} = -J\frac{\partial f}{\partial t} \implies \frac{\partial f}{\partial s} + J\frac{\partial f}{\partial t} = 0,$$

we would get exactly the definition of a J-holomorphic map!

We call such J-holomorphic maps *J-holomorphic strips* and they will play the role played by the gradient lines in Morse homology.

As the domain $\mathbb{R} \times [0,1] \subset \mathbb{C}$ is biholomorphic to a disk without two points on the boundary, one could consider the domain of the maps into consideration to be the disk.

Now, the ideia will be to define a complex generated by the points in the intersection $L_0 \cap L_1$ and define a differential by counting holomorphic strips.

We finish this section with one last assumption we make about these J-holomorphic strips: if M is not compact, it is important that they have finite symplectic volume (finite energy), that is:

$$\int u^*\omega < \infty$$

for $u : [0,1] \times \mathbb{R} \to M$ J-holomorphic.

1.2 Maslov Index and the Floer Complex

Before defining the Floer Complex and its differential, we will define the so called *Maslov Index*, which will play the role that the difference of Morse indices plays. We follow mostly the ideas given in [1].

Consider the following result on Lagrangian Grassmanians.

Proposition 1.4 *Let $\Lambda(n)$ be the space of all lagrangian linear subspaces of $\mathbb{R}^{2n}$ with the standard symplectic form (n-dimensional subspaces for which the form restricts to zero). Then we can identify $\Lambda(n)$ with the quotient $U(n)/O(n)$, i. e., the quotient of the unitary group by the orthonormal group.*

Proof. The ideia is simple. Firstly, observe that each element of $\Lambda(n)$ can be represented by a $2n \times n$ matrix modulo the action of $GL(n)$ (think about the matrices as inclusions $\mathbb{R}^n \to \mathbb{R}^{2n}$: if we change coordinates on the domain, the image will be the same!). The columns of this matrices are, of course, a basis for their images, that is, a basis for the lagrangian subspace it represents.

As the standard symplectic form is given by the matrix $J_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$, if $A = \begin{bmatrix} X \\ Y \end{bmatrix}$ is a $2n \times n$ matrix with columns given by a basis of an element of $\Lambda(n)$, we have:

$$A^t J_0 A = 0 \implies \begin{bmatrix} X^t & Y^t \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = 0 \implies X^t Y = Y^t X$$

So, besides the action of $GL(n)$, we have another restriction, on the columns, which is the equation above.

We have therefore a map $U(n) \to \Lambda(n)$ that takes a matrix $U = X + iY$ and sends it to the lagrangian subspace represented by the matrix $\left[\begin{smallmatrix} X \\ Y \end{smallmatrix}\right]$. Note that it is well defined since $(X+iY)^*(X+iY) = I \implies (X^t - iY^t)(X+iY) = I \implies -Y^t X + X^t Y = 0$ (taking the imaginary part). Factoring by the action of $GL_n(\mathbb{R}) \cap U(n) = O(n)$, we have the result.

∎

Now, we can define a map $\rho : \Lambda(n) \to S^1$ given by $\rho(A) = det(A^2)$ (it is well defined since A is an unitary matrix) and the elements of $O(n)$ have determinant ± 1. We arrive at the definition:

Definition 1.5 *Let γ be a loop in $\Lambda(n)$. We define the Maslov index of γ as the degree of the map $\rho \circ \gamma$. Note it is independent of the homotopy class of the loop.*

A nice way to get some intuition about this index is to consider its Poincaré Dual:

Proposition 1.6 (see [4]) *By the definition above, the Maslov index may be seen as a cocycle $\mu \in H^1(\Lambda(n))$. Then, its Poincaré dual is given by:*

$$\Sigma = \{\Lambda \in \Lambda(n) \mid dim\Lambda \cap \Lambda_0 > 0\},$$

where Λ_0 is a fixed reference lagrangian.

How can we apply this definition to the case we are considering? The ideia is to see how the tangent spaces of our to lagrangians move in the lagrangian grassmanian as a strip goes from p to q. In practice, we will associate a path in $\Lambda(n)$ for each strip.

Let $u : \mathbb{R} \times [0, 1] \to M$ be a J-holomorphic strip. Consider the pullback bundle u^*TM over $\mathbb{R} \times [0, 1]$. It can be trivialized, since the base is simply connected. Now, we get two paths in $\Lambda(n)$: by considering the variation of u^*TL_0 over $\mathbb{R} \times 0$ and u^*TL_1 over $\mathbb{R} \times 1$. By choosing a nice trivializiation, we can assume that the first path is constant and equals to some lagrangian Λ_0 and that the second one induces a path φ in $\Lambda(n)$.

We would like to define the Maslov index of the strip as being the number of times (with sign and multiplicity) that φ intersects Λ_0 non-tranversely, that is, the number of intersections of the path and Σ (defined in the proposion above).

But, as the φ is not a loop, we have a small technical problem. This can be solved by simply considering a small deformation Λ_ε of Λ_0 and connect this new lagragian to φ without intersecting Σ. Now, we can compute the Maslov index of this loop and, by the proposition above and the fact that Σ is not intersected, the index will be exact the number of intersections cited above.

After this discussion, we can make the following definition:

Definition 1.7 *For a J-holomorphic strip $u : \mathbb{R} \times [0, 1] \to M$, the Maslov index of the strip is given by the number of non-transverse intersections (counted with sign and multiplicity) of $u^*TL_0|_{\mathbb{R}\times 0}$ and $u^*TL_1|_{\mathbb{R}\times 1}$ in the lagrangian grassmanian after a trivialization of u^*TM.*

Now that we have the Maslov index, we are set to define the Floer complex and state the main properties of its homology.

Definition 1.8 *The Novikov field Λ over a ground field k is given by:*

$$\Lambda = \{\sum_{i=1}^{\infty} a_i T^{\beta_i} \mid a_i \in k, \beta_i \in \mathbb{R} \textit{ and } lim_{i\to\infty}\beta_i = \infty\}.$$

This field will be the base field for the definition of our complex. Now, given two lagrangians L_0 and L_1 in M which intersect transversally, we define:

$$CF(L_0, L_1) = \bigoplus_{p \in L_0 \cap L_1} \Lambda p,$$

the free Λ-module generated by the intersection points.

In order to get a complex, we need to define a differential in this space, that is, a map $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$ for which we have $\partial^2 = 0$.

Again, we are inspired by Morse homology. There, we defined ∂ by counting the number of gradient lines connecting two critical points with index diference equal to 1. Here, our objective is to count J-holomorphic strips connecting two points in the intersection $L_0 \cap L_1$. Consider, then, the following space:

$$\mathcal{M}(p, q, J, A) = \{u \text{ is J-holomorphic connecting } p \text{ and } q \mid [u] = A \in \pi_2(M, L_0 \cup L_1)\}.$$

Lemma 1.9 (see [3]) *For a generic family $J = \{J_t\}$ of ω-compatible complex structures depending on t, we have that $\mathcal{M}(p, q, J_t, A)$ is a manifold with dimension given by $\mu(A) - 1$.*

The proof of this fact is mainly analitical, involving some results about Fredholm operators and can be found in the original papers by Floer. It is important to consider a family of complex structures to ensure we get surjectivity of certain operators.

It is clear now, after the Lemma, that the Maslov index will play the same role as the diference of the indices of the critical points. Therefore, we define, for $p \in L_0 \cap L_1$ seen as a generator of $CF(L_0, L_1)$:

$$\partial(p) = \sum_{\substack{q \in L_0 \cap L_1 \\ A \in \pi_2(M, L_1 \cup L_2) \\ \mu(A) = 1}} \# M(p, q, J, A) \cdot T^{\int_A \omega} q.$$

The problem now is to prove that $\partial^2 = 0$ and to show that the formula above is well defined. For example, it is non trivial to find a nice orientation in order to count holomorphic strips, and that is one of the reasons this theory was first developed for $\mathbb{Z}_2$. Also, there are some assumptions that need to made to get $\partial^2 = 0$. A possible setting would be the one below:

Theorem 1.10 *If M, L_0 and L_1 are symplectic aspherical w.r.t. ω and $\omega|_{L_i}$ respectively and J is a generic t-dependent family of almost complex structures, then $\partial^2 = 0$. This allow us to define a homology $HF(L_0, L_1, J) = ker\partial / im\partial$.*

Idea of the proof. Let p, q be generators of the floer complex and B be an homotopy class on $\pi_2(M)$ with Maslov index 2. Then, we can consider the space of J-holomorphic strips with homotopy type B. This is a smooth manifold of dimension 1 by the lemma above. By Gromov compactness [6], we have that any sequence of J-holomorphic curves with bounded energy admits a subsequence converging to what is called a "nodal tree" of holomorphic curves. In our cases, we could have three types of limit: *strip breaking, disk bubbling* and *sphere bubbling*. Those happen when the energy concentrates in a point (limit point, boundary point, interior point, respectively).

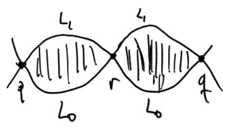


Figure 2: Strip Breaking

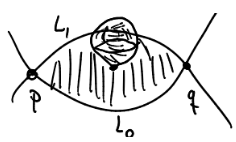


Figure 3: Disk Bubbling

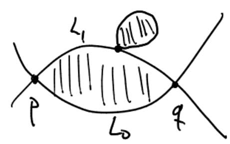


Figure 4: Sphere Bubbling

If we compactify the space of index 2 strips, we are simply adding these limit "nodal trees". As we are assuming disks and spheres have no energy (zero symplectic volume), we conclude that the only limit that can happen is strip breaking. But,in this case, we simply get two index 1 J-holomorphic strips with homotopy classes that sum B.

So, the boundary of the compactified manifold is the set of all pairs of index 1 J-holomorphic strips with homotopy types summing B. If we consider our field to be $\mathbb{Z}_2$ or get orientations on the lagrangians, it would be possible to define a way to count pointsspace of strips. Now, we use that for a 1-dimensional manifold the signed sum of the boundary points is zero (for $\mathbb{Z}_2$, it suffices to say that the number is always even). We conclude that

$$\sum_{\substack{r \in L_0 \cap L_1 \\ A + A' = B \\ \mu(A) = \mu(A') = 1}} \# M(p, r, J, A) \cdot \# M(r, q, J, A') = 0,$$

since it is the number of broken strips we can get for a fixed B. Now, if we sum over all B's, and multiply by $T^{\omega(B)}$, we get exactly the coefficent of q after ∂^2. So $\partial^2 = 0$.

∎

Now we have a definition for Floer Homology. The main properties of it are listed above. They were proved in various papers and we cite texts in which they are stated in a more organized list.

Theorem 1.11 (see [1] or [10]) *Let $L_0, L_1 \subset M$ be lagrangians in a symplectic manifold. Assume the same hypothesis of the theorem above. Then:*

- *$HF(L_0, L_1, J)$ does not depend on J, and so we write just $H(L_0, L_1)$.*

- $HF(L_0, L_1) = HF(L_0, \varphi(L_1))$, *where φ is a hamiltonian diffeomorphism.*
- $HF(L, L) := HF(L, \varphi(L)) = H_*(L, \Lambda)$ *(singular homology with coefficients in* Λ.

Now, the third result above show us how Floer could prove the particular case of the conjecture: if we take ground field $\mathbb{Z}_2$, the sum of ranks in $H_*(L, \mathbb{Z}_2)$ is the sum of dimensions (over Λ) in $H_*(L, \Lambda)$ which is isomorphic to $HF(L, L)$. But, of course the sum of the dimensions is less than the number of intersection points by definition of the $HF(L, L)$.

The Lagrangian Floer Homology will be the basis for defining the Fukaya Category, which will be of great importance in the goal of understanding the statement of the HMS conjecture.

2 The Fukaya Category

The Fukaya Category is one of the possible forms to study, at the same time, all Lagrangians and its Floer homologies. It was first introduced by Fukaya in 93 and it became a very important object in the study of symplectic geometry, since it encodes many important properties of a Symplectic manifold, specially about its lagrangians and the way they intersect.

Later, it became important in the study of the HMS conjecture, as we will see when we state it in this text. The ideia is that the relationship between Fukaya Categories and Coherent Sheaves has shown itself to be the correct way to formally define the concept of Mirror Manifolds used in Physics.

Here, especially when talking about A_∞-categories, we follow [14] but also use many ideas from [1] and [10] to define Fukaya categories.

2.1 A_∞-categories

In this section, we assume the reader is familiar with the classical concepts of category, functor and natural transformation. We start, then, by defining A_∞-categories:

Definition 2.1 *Fix a field k. An A_∞-category $\mathcal{C}$ consists of a set of objects (or a class, but we do not want to get into set-theoric details here), graded vector spaces over k denoted by $hom_{\mathcal{C}}(X, Y)$ for any pair of objects in $\mathcal{C}$ and composition maps*

$$\mu^d_{\mathcal{C}} : hom(X_0, X_1) \otimes \cdots \otimes hom(X_{d-1}, X_d) \to hom(X_0, X_d)[2 - d]$$

for every $d \geq 1$, where $V[n]$ means that the grading is shifted down by n, that is, if $V = \bigoplus V_k$, $V[n]_m = V_{m-n}$. In our case, that means that the map μ^d has degree $2 - d$, *i.e., elements of degree k of : $hom(X_0, X_1) \otimes \cdots \otimes hom(X_{d-1}, X_d)$ go to elements of degree $k+2-d$ of $hom(X_0, X_d)$.*

Moreover, these composition maps sastify, for any $a_i \in$ hom(X_{i-1}, X_i):

$$\sum_{\substack{1\le m\le d\\ 0\le n\le d-m}} (-1)^{\Xi_n}\mu^{d-m+1}(a_1, a_2, \ldots, \mu^m(a_{n+1}, \ldots, a_{n+m}), a_{n+m+1}, \ldots, a_d),$$

where $\Xi_n = (\sum_{i=1}^n deg(a_i)) - n$ and $\Xi_0 = 0$.

It is important to observe that $\mathcal{A}_\infty$-categories **are not** categories, since the composition of morphisms do not satisfy associativity and since there may be no indentity morphisms. Also, note that, for $d = 1$ and $d = 2$, we have the following equations (based, of course, on the composition relation above):

- $d = 1$

  In this case, we have only one possible value for each m and n, which are $m = 1$ and $n = 0$. Therefore, the equation become

  $$\mu^1(\mu^1(a)) = 0$$

  for any $a \in$ hom(X, Y) for any pair of objects X, Y. In other words, this means that the hom spaces are actually chain complexes with differential given by μ^1.

- $d = 2$

  Here, we have three cases: $(m, n) = (1, 0)$, $(m, n) = (1, 1)$ and $(m, n) = (2, 0)$. To avoid sign problems, we just write $\pm$. The equation then, says:

  $$\pm\mu^2(\mu_1(a_1), a_2) \pm \mu^2(a_1, \mu^1(a_2)) \pm \mu^1(\mu_2(a_1, a_2)) = 0$$

  for any $a_i \in$ hom(X_{i-1}, X_i) and $X_0, X_1, X_2 \in \mathcal{C}$.

  Looking at μ^2 as a product $\cdot$ and μ^1 as a differential ∂, the above equation implies that ∂ satisfies the Leibniz rule w.r.t $\cdot$ (modulo sign):

  $$\pm\partial(a_1 \cdot a_2) = \pm\partial(a_1) \cdot a_2 \pm a_1 \cdot \partial(a_2).$$

These two equations inspire us to define the *cohomological category* associated to the $\mathcal{A}_\infty$-category $\mathcal{C}$.

Definition 2.2 *Let $\mathcal{C}$ be an $\mathcal{A}_\infty$-category. Then, its cohomological category $\mathcal{H}(\mathcal{C})$ is a category (perhaps without identity morphisms) with objects given by the same objects of $\mathcal{C}$ and hom-sets given by the cohomology groups $H^*(hom(X,Y),\mu^1)$ of the complex $hom(X,Y)$ for any pair of objects. The composition is defined by μ^2 as follows:*

$$[a_1]\circ[a_2] = (-1)^{deg(a_1)}[\mu^2(a_1,a_2)].$$

Lemma 2.3 *The composition defined above is associative.*

Proof. Let X_i, $i=1,2,3,4$, be objects and let $[a_i]\in H^*(\hom(X_i,X_{i+1}),\mu^1),\ i=1,2,3$. We have:

$$([a_1]\circ[a_2])\circ[a_3] = \left((-1)^{|a_1|}[\mu^2(a_1,a_2)]\right)\circ[a_3] = (-1)^{|a_1|+|a_1|+|a_2|}\left[\mu^2(\mu^2(a_1,a_2),a_3)\right]$$

denoting the degree by $|\cdot|$ and using that μ^2 has degree zero.

On the other hand:

$$[a_1]\circ([a_2]\circ[a_3]) = [a_1]\circ\left((-1)^{|a_2|}[\mu^2(a_2,a_3)]\right) = (-1)^{|a_1|+|a_2|}\left[\mu^2(a_1,\mu^2(a_2,a_3))\right].$$

We need to relate the classes $[\mu^2(\mu^2(a_1,a_2),a_3)]$ and $[\mu^2(a_1,\mu^2(a_2,a_3))]$. For this, we use the composition relation from the definition of $\mathcal{A}_\infty$-categories. For $d=3$ and $m=1$, every summand will be zero, since our elements a_i are cocycles (and thus have $\mu^1=0$). For $d=3$ and $m=3$, we get only on summand $(-1)^{\Xi_0}\mu^1(\mu_3(a_1,a_2,a_3))$ which is zero on cohomology.

Hence, we get:

$$(-1)^{\Xi_0}\left[\mu^2(\mu^2(a_1,a_2),a_3)\right] + (-1)^{\Xi_1}\left[\mu^2(a_1,\mu^2(a_2,a_3))\right] = 0,$$

which implies

$$\left[\mu^2(\mu^2(a_1,a_2),a_3)\right] = (-1)^{|a_1|}\left[\mu^2(a_1,\mu^2(a_2,a_3))\right],$$

using the definition of Ξ_n.

Combining the equations, we have:

$$([a_1]\circ[a_2])\circ[a_3] = (-1)^{|a_1|+|a_1|+|a_2|}\left[\mu^2(\mu^2(a_1,a_2),a_3)\right] =$$
$$= (-1)^{|a_2|}(-1)^{|a_1|}\left[\mu^2(a_1,\mu^2(a_2,a_3))\right] =$$
$$= (-1)^{|a_1|+|a_2|}\left[\mu^2(a_1,\mu^2(a_2,a_3))\right] = [a_1]\circ([a_2]\circ[a_3]).$$

Note that, in the second step, we used that $(-1)^{2|a_1|}=1$ to consider only $|a_2|$ in the exponent.

This shows associativity and thus concludes the proof.

■

It is good to emphasize again that $\mathcal{A}_\infty$-categories and its cohomological counterparts do not always have identity morphisms. This means that we do not have the concept of isomorphic objects or equivalence of categories. Nevertheless, we still have functors:

Definition 2.4 *A $\mathcal{A}_\infty$-functor $\mathcal{F}$ between two A_∞-categories $\mathcal{C}$ and $\mathcal{D}$ consists of a map $\mathcal{F} : Ob(\mathcal{C}) \to Ob(\mathcal{D})$ and multilinear maps for all $d \geq 1$:*

$$\mathcal{F}^d : hom_{\mathcal{C}}(X_0, X_1) \otimes \cdots \otimes hom_{\mathcal{C}}(X_{d-1}, X_d) \to hom_{\mathcal{D}}(\mathcal{F}X_0, \mathcal{F}X_d)[1-d],$$

which satisfy

$$\sum_r \sum_{s_0+\cdots+s_r=d} \mu^r_{\mathcal{D}}(\mathcal{F}^{s_1}(a_1, \ldots, a_{s_1}), \ldots, \mathcal{F}^{s_r}(a_{d-s_r+1}, \ldots, a_d)) =$$

$$= \sum_{m,n} (-1)^{\Xi_n} \mathcal{F}^{d-m+1}(a_1, \ldots, \mu^m_{\mathcal{C}}(a_{n+1}, \ldots, a_{n+m}), \ldots, a_d).$$

These equations imply that if we have a $\mathcal{A}_\infty$-functor, we can induce an usual functor in the cohomological category. The idea is exactly the same as the one used to define the cohomological categories.

Proposition 2.5 *If $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ is a $\mathcal{A}_\infty$-functor, then there is a (usual) functor $H(\mathcal{F}) : H(\mathcal{C}) \to H(\mathcal{D})$, for which the action on morphisms is given by $[a] \mapsto \mathcal{F}^1([a])$.*

Proof. We just need to check that $H(\mathcal{F})$ is well defined and that it commutes with composition. By definition, this means to check that $\mu^1_{\mathcal{D}}(\mathcal{F}^1(a)) = \mathcal{F}^1(\mu^1_{\mathcal{C}}(a))$(well definetess) and that $\mu^2_{\mathcal{D}}(\mathcal{F}^1(a_1), \mathcal{F}^1(a_2)) \sim \mathcal{F}^1(\mu^2_{\mathcal{C}}(a_1, a_2))$ (composition), where the second equation is up to boundaries.

It is just a direct application of the definition for $d = 1$ and for $d = 2$.

■

Observe that $\mathcal{A}_\infty$-functors can be composed. At the object level, it is the usual composition and, at morphism level, we define:

$$(\mathcal{G} \circ \mathcal{F})^d(a_1, \ldots, a_d) = \sum_r \sum_{s_1+\cdots+s_r=d} \mathcal{G}^r(\mathcal{F}^{s_1}(a_1, \ldots, a_{s_1}), \ldots, F^{s_r}(a_{d-s_r+1}, \ldots, a_d)).$$

Note that composition has a neutral element: the identity functor $\mathbb{I}$. It is given by the identity on objects, and for morphisms, we have the indentity map as $\mathbb{I}^1$ and zero for all higher degree maps.

In the same line, we can define A_∞-natural transformations in order to give the set of $\mathcal{A}_\infty$-functors a structure of $\mathcal{A}_\infty$-category and also to induce natural transformations on the cohomological category. In order to get less technical, we choose to omit this definition.

An important point we need to adress before proceeding is the identity issue.

Definition 2.6 *An $\mathcal{A}_\infty$-category $\mathcal{C}$ is called strictly unital if, for each object $X \in \mathcal{C}$, there is a morphisms $e_X \in hom(X, X)$ of degree 0 satisfying:*

1. $\mu^1(e_X) = 0$
2. $(-1)^{|a|}\mu^2(e_{X_0}, a) = a = \mu^2(a, e_{X_1})$ *for any $a \in hom(X_0, X_1)$*
3. $\mu^d(a_1, \ldots, a_d) = 0$ *if $a_i = e_{X_i}$ for some i and $d > 2$.*

Definition 2.7 *An $\mathcal{A}_\infty$-category $\mathcal{C}$ is called cohomologically unital if $H(\mathcal{C})$ is a category in the ordinary sense. This means that $H(\mathcal{C})$ has identity morphisms $[e_X] \in H^*(hom(X, X))$ for each object X.*

We now speak about two constructions that can be made over $\mathcal{A}_\infty$-categories: exact triangles and the twisted complexes. Those will be important in the task of understanding the statment of HMS conjecture.

The reader less interested in categorical details may skip the two sections below. What they need to keep in mind is that if we have an $\mathcal{A}_\infty$-category, the *twsited category* associated to it (whatever this means) has a *triangulated structure* given by the *exact triangles*. This structure gives us properties which are analogous to properties from abelian categories, even though we have only additive categories.

Derived categories of abelian categories are triangulated so it will be natural to compare the twisted category of the Fukaya Category with the derived category of coherent sheaves: this is exactly what is done in HMS conjecture.

2.1.1 Exact Triangles

The idea of exact triangles comes from the concept of triangulated categories, which is an important defintion from standard category theory that we will give in section 3. In the case of $\mathcal{A}_\infty$-categories, such structures can be defined naturally in the cohomological categories.

Definition 2.8 *Let $Z = \{Z_0, Z_1, Z_2\}$ be the set with three elements. We define the Triangle $\mathcal{A}_\infty$-Category $\mathcal{Z}$ to be strictly unital $\mathcal{A}_\infty$-category with $Ob(\mathcal{Z}) = Z$ and morphisms as follows:*

- *For each Z_i, we have $hom(Z_i, Z_i) = k \cdot e_{Z_i}$, where e_{Z_i} is the identity morphism.*
- *$hom(Z_i, Z_{i+1}) = k \cdot x_{i+1}$, with $|x_1| = |x_2| = 0$ and $|x_3| = 1$. Note that the indices of the objects are taken mod 3.*
- *We define $\mu^d = 0$ except for μ^2 when applied at the identity morphisms (in acordance with the definition of a strictly unital category) and for μ^3, which satisfy $\mu^3(x_i, x_{i+1}, x_{i+2}) = e_{Z_i} \in hom(Z_i, Z_i)$. Again, we use indices mod 3.*

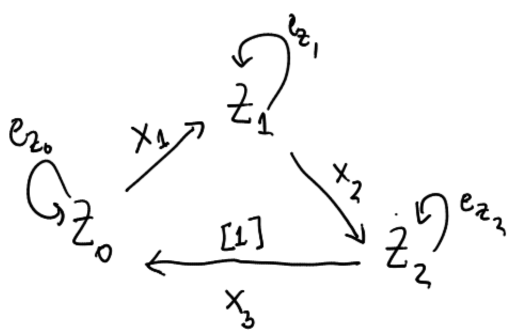


Figure 5: The category $\mathcal{Z}$. Here $[1]$ represents that the morphism has degree 1.

This category model what we call *exact triangles*:

Definition 2.9 *Let $\mathcal{C}$ be an $\mathcal{A}_\infty$-category. We say a diagram of the form below lying in*

$$\begin{tikzcd} Y_0 \arrow[rr, "{[c_1]}"] & & Y_1 \arrow[dl, "{[c_2]}"] \\ & Y_2 \arrow[ul, "{[c_3]}", "{[1]}"'] & \end{tikzcd}$$

the cohomological category $H(\mathcal{C})$ is called an exact triangle if there is a $\mathcal{A}_\infty$-functor $\mathcal{F} : \mathcal{Z} \to \mathcal{C}$ such that $\mathcal{F}(Z_i) = Y_i$ and $[\mathcal{F}^1 x_i] = [c_i]$ in the cohomological category.

Moreover, we say that $\mathcal{C}$ is triangulated if it is non empty and:

- *every morphism $[c] \in H^0(\mathcal{C})$ can be completed to an exact triangle;*
- *for any object Y, there is a $\tilde{Y}$ such that $S\tilde{Y} = Y$ in $H^0(\mathcal{C})$.*

Here, S is the shift functor, defined via tensor products (see [14]) and satisfy:

$$\begin{gathered} Hom_{H(\mathcal{C})}(Y_0, SY_1) \cong Hom_{H(\mathcal{C})}(Y_0, Y_1)[1] \\ Hom_{H(\mathcal{C})}(Y_0, SY_1) \cong Hom_{H(\mathcal{C})}(SY_0, SY_1) \\ Hom_{H(\mathcal{C})}(SY_0, Y_1)[1] \cong Hom_{H(\mathcal{C})}(Y_0, Y_1) \end{gathered}$$

.

As we said, we later will compare this definition of triangulated category with the usual definition we will present in the third section. We will see that if $\mathcal{C}$ is triangulated, $H^0(\mathcal{C})$ has a usual triangulation.

2.1.2 Twisted Category

Another important construction that will be useful in order to understant the statement of HMS conjecture is the Twisted Category. This construction has the goal of, given any $\mathcal{A}_\infty$-category $\mathcal{C}$, associate to it a new category Tw($\mathcal{C}$) which is trangulated in the sense defined above.

Here we will give the main ideas for the construction of these objects and their category for an arbitrary $\mathcal{A}_\infty$-category.

Definition 2.10 *The additive enlargment $\Sigma\mathcal{C}$ of an $\mathcal{A}_\infty$-category is formed by objects $X = \bigoplus V^i \otimes X^i$, where $X_i \in \mathcal{C}$ and V_i are graded vector spaces. The hom space is simply the sum of the tensor products of the hom spaces from $\mathcal{C}$ and hom spaces from the category of graded vector spaces. The composition is given by the ordinary composition of graded linear maps tensored with the maps μ^d from $\mathcal{C}$.*

These enlarged categories are created in order to define Twisted complexes.

Definition 2.11 *A pre twisted complex in $\mathcal{C}$ is an object X of $\Sigma\mathcal{C}$ with a differential map $\delta_X \in hom^1(X, X)$. It is called a twisted complex if δ is strictly lower triangular (each time it is applied, its image is inside a smaller subspace of V^i) and if it satisfies $\sum_r \mu^r(\delta, \ldots, \delta) = 0$.*

For more details on these definitions, one should consult [14].

These complexes form a category. The reader with a background in triangulated categories may notice that, as above, defining this complex with lower triangular differential and direct sums as above is similar to what is done to define the triangulated structure in the derived category of an abelian category.

It is natural, then, to expect the following:

Theorem 2.12 (see [14]) *The category of Twisted Complexes on $\mathcal{C}$, Tw$\mathcal{C}$ has an $\mathcal{A}_\infty$-structure and is triangulated.*

As we already observed before defining exact triangles and twisted categories, we will use these constructions to get a triangulated category associated to the Fukaya Category, that will be compared with the (also triangulated) derived category of coherent sheaves in the statement of HMS conjecture.

2.2 Definition of the Fukaya Category

Now that we have presented some of the categorical background, we can define the Fukaya Category. It, of course, will be an A_∞-category with objects being the lagrangians of a symplectic manifold and $hom(L_1, L_2) = CF(L_1, L_2)$. Although we defined Floer *homology* instead of *cohomology* and in the previous section we study *cohomological* categories, there is no contradiction: recall that we did not define a grading in $CF(L_1, L_2)$ yet. So, being a homology or a cohomology is just a matter of changing the sign of this grading.

We have some issues to adress:

1. How to get a grading on $CF(L_1, L_2)$? How to make ∂ a *coboundary* operator?
2. How to define the higher degree composition maps μ^d?
3. Which restrictions do we have to consider on M to be able to define the Fukaya Category of M, $\mathcal{F}(M)$?

2.2.1 Grading

First, the definition of the grading. When we defined the Floer Complex $CF(L_0, L_1)$, we did not say what should be the degree of an element, we just defined the Maslov index. If, again, we recall Morse homology, it is reasonable to define this grading in a way the difference of the degrees of two intersection points is the Maslov index of the strips connecting them. Hence, in order to define this grading, we have to ensure that the Maslov index of a strip will depend only on the points it connects and not on its homotopy class.

Suppose that (M, ω) is a symplectic manifold and L_0, L_1 are lagrangians. Consider the bundle over M given by $\Lambda(TM)$, that is, consider the lagrangian grassmanian of each tangent space. Assume that this bundle can be lifted to a new bundle given fiberwise by the universal cover $\Lambda(\tilde{TM})$ of each grassmanian.

If the sections $f_i : L_i \to \Lambda(n)$ given by $p \mapsto T_pL_i \subset T_pM$ can be lifted to $\Lambda(\tilde{TM})$, then we define the grading as follows:

Take a point $p \in L_0 \cap L_1$ and consider the images $f_i(p) \in \Lambda(T_pM)$ and $\tilde{f}_i(p) \in \Lambda(\tilde{T_p}M)$.Then, any path in the universal covering γ connecting $\tilde{f}_0(p)$ and $\tilde{f}_1(p)$ and also take the canonical path λ connecting $f_0(p)$ and $f_1(p)$, which is defined by considering (after a change of coordinates) one of the two lagrangians as the $\mathbb{R}^n \subset \mathbb{C}^n$ and then take the rotation joining the two lagragians.

Projecting, we get a loop in $\Lambda(T_pM)$ and compute its Maslov index. This is the degree of p (see figure 6).

It is possible to prove that if we have a strip, the Maslov index of the strip only depends on the difference of its degrees. In particular, if we have a strip connecting p and q, the Maslov index would be $\deg(q) - \deg(p)$ and the boundary map defined on section 1 would have degree -1, as expected for homology. In order to work cohomologically, we change the sign of the grading, as we commented in the opening of 2.2.

Also, we need to set some restrictions to be able to lift the paths as we did above. For this, as its stated in [1], it suffices to consider that the firts chern class of TM vanishes,

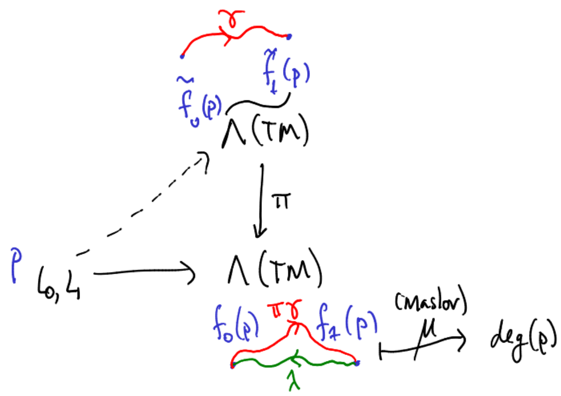


Figure 6: Definition of the grading

$c_1(TM) = 0$ and that the *Maslov class* of L_i are zero. This class is the obstruction to lifting paths as above and it is defined in [1].

2.2.2 Composition maps

Now, our objective is to define the composition maps. The idea is pretty simple. Instead of considering two lagrangians and counting the holomorphic strips (which can be regarded as maps from $\mathbb{D} \setminus \{-1, 1\}$) joining their intersection points, we consider d lagrangians and count maps from $\mathbb{D} \setminus \{\zeta^i : i = 1, \ldots, d\}$ where ζ is a d-root of unity.

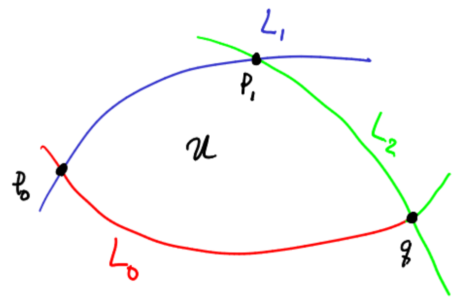


Figure 7: Three Lagrangians intersecting

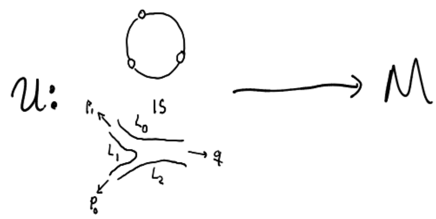


Figure 8: Map u and its two biholomorphic domains

So, in the case of three lagrangians, we can define, under nice conditions (no disk bubbling, symplectically aspherical, etc):

$$\mu^2 : CF(L_0, L_1) \otimes CF(L_1, L_2) \to CF(L_0, L_2)$$

$$\mu^2(p_1, p_2) = \sum_{\substack{q \in L_0 \cap L_2 \\ A \in \pi_2(M, L_0 \cup L_1 \cup L_2) \\ \mathrm{ind}(A)=0}} \#M(p_1, p_2, q, J, A) \cdot T^{\int_A \omega} q,$$

where ind(A) is the Maslov index of the loop given by concatenating paths on $\Lambda(TM)$ induced by the restrictions of u to L_i (where $[u] = A$) and $M(p_1, p_2, q, J, A)$ is the space of J-holomorphic maps from the disk without three boundary points to M which have homotopy class A.

Again, it is necessary to prove that M is indeed a manifold. It is done in a similar way it as in the case of two lagrangians and the dimension is given by the index.

It is important to note that if we have nice conditions as $c_1(TM) = 0$ and Maslov class vanishing for all three lagrangians, we can write ind$(A) =$ deg$(q) -$ deg$(p_1) -$ deg(p_2) independently of the homotopy class.

We now proceed with the main property of this operation:

Proposition 2.13 *If we have nice hypothesis (e.g. symplectically aspherical, $c_1(M) = 0$, Maslov class zero etc), μ^2 satisfy the Leibniz rule (with suitable signs) with respect to ∂:*

$$\pm\partial(\mu^2(p_1, p_2)) = \pm\mu^2(p_1, \partial p_2) \pm \mu^2(\partial p_1, p_2).$$

Idea of the Proof. Since we are making the right assumptions, we proceed in the same way we did to prove that $\partial^2 = 0$. We consider a homotopy class B with index 1. By Gromov compacteness, as we are excluding bubbling, we arrive with only strip breaking, as we did for the differential.

In this cases our map is broken in a strip connecting two points and in another strip connecting three points. We thus have three possibilities, permuting p_1, p_2 and q:

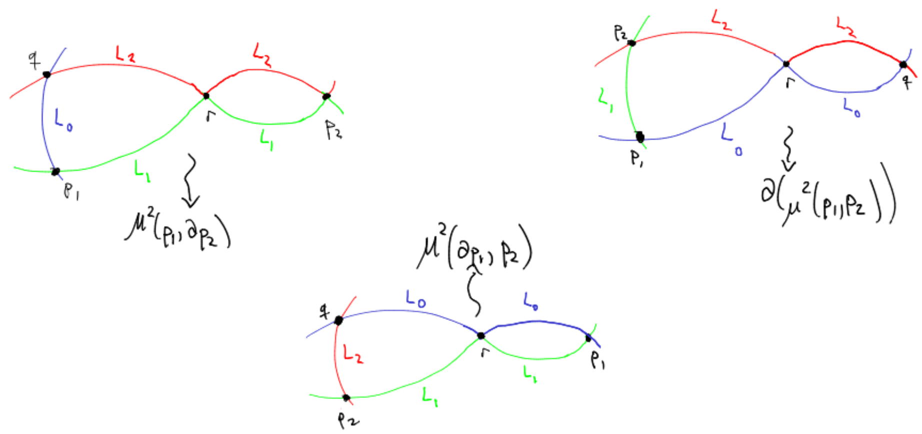


Figure 9: Possible cases of strip-breaking for three lagrangians

Each of the possibilities contribute to the coefficient of q in the expressions appering in the Leibniz rule (see figure 9 below). For example, in the third case (from left to right), the triple strip before the point r contributes to the computation of $\mu^2(p_1, p_2)$ and the double strip after the point r contribute to its boundary, that is, $\partial(\mu^2(p_1, p_2))$. The other two cases are analogous.

To end, we use the argument that the oriented sum of the points of the boundary of the compactified 1-dimensional manifold of strips has to be zero.

∎

In the same spirit, it is natural to define the higher degree composition maps (see figure 10):

Definition 2.14 *For d lagrangians, we can define (under the same assumptions of no bubbling, symplectically aspherical, grading, orientation, etc):*

$$\mu^d : CF(L_0, L_1) \otimes \cdots \otimes CF(L_{d-1}, L_d) \to CF(L_0, L_d)$$

$$\mu^d(p_1, p_2, \ldots, p_d) = \sum_{\substack{q \in L_0 \cap L_d \\ A \in \pi_2(M, L_1 \cup L_2 \cup \cdots \cup L_d) \\ ind(A) = 2-d}} \#M(p_1, p_2, \ldots, p_d, q, J, A) \cdot T^{\int_A \omega} q,$$

where the index is again obtained by concatenating paths and M is the manifold of J holomorphic maps with homotopy class A joining $p_1, \ldots, p_d$, and q.

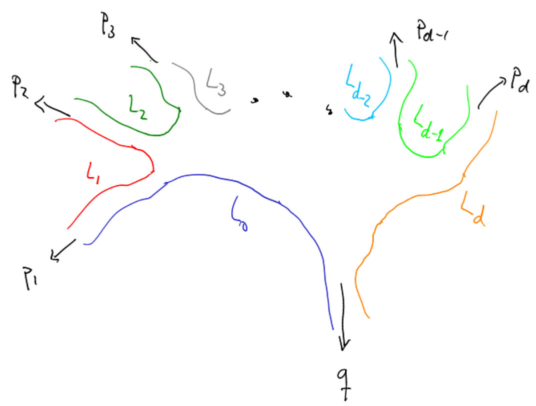


Figure 10: The domain of the J-holomorphic map connecting d lagrangians. The arrows means the limit taken in that direction.

Theorem 2.15 (see [14]) *The maps μ^k satisfy the $\mathcal{A}_\infty$-relations.*

The proof of the above statement follows the same ideas used in the proofs that $\partial^2 = 0$ for the Floer Complex and that μ^2 satisfy the Lebiniz rule.

With this result in hands, we can finally define the Fukaya Category (as it is done in [1]):

Definition 2.16 *Let (M, ω) be a symplectic manifold with $c_1(TM) = 0$. The objects of the Fukaya category $\mathcal{F}(M, \omega)$ are compact closed, oriented, Lagrangian submanifolds $L \subset M$ such that $[\omega] \cdot \pi_2(M, L) = 0$ and with vanishing Maslov class (in order to define a grading and have no bubblig).*

For every pair of objects (L_0, L_1), (not necessarily distinct), we consider a family of almost complex structures and hamiltonians (to achieve transversality) in order to define $CF(L_0, L_1)$. We also need to get some perturbation data to have transversality of more than two lagrangians in order to define the maps μ^d as above.

Given this, we set $hom(L_0, L_1) = CF(L_0, L_1)$; with the floer differential to be μ^1 and the higher composition maps given by counts of perturbed J-holomorphic disks as in the definition above. By the theorem 2.15, this makes $\mathcal{F}(M, \omega)$ a Λ-linear, $\mathbb{Z}$-graded, non-unital (but cohomologically unital) $\mathcal{A}_\infty$ -category.

Although many of the hypothesis we assumed in order to make the definition are not detailed here, we think the main ideias are clear: the Fukaya Category is an $\mathcal{A}_\infty$-category with objects given by lagrangians and with hom sets given by the floer complexes. The composition maps are given by counting holomorphic strips.

With this object in hands, we can present the Kontsevich HMS conjecture.

3 Homological Mirror Symmetry

In this third part of the text, we focus on introducing some algebraic facts related to homological algebra and coherent sheaves. We do not give much details, as our objective here is just to give the main definitions and results necessary to state HMS conjecture. We also define what are the Calabi-Yau manifolds and give a brief explanation of why they are important, although we do not present many results about them.

3.1 Some Algebraic Geometry and Homological Algebra

In this section, we give some basic results concerning algebraic geometry and homological algebra, specially to understand the derived category of Coherent Sheaves if a variety, which is the one appearing in the HMS conjecture. For a more detailed exposition on sheaves and coherent sheaves, we recommend [7] or [17]

3.1.1 Derived Categories and Triangulations

Our goal here is to define derived categories and explain what we mean by a triangulated category. Also, we plan to compare the triangulation defined here with triangulation for cohomological categories associated to $\mathcal{A}_\infty$-categories presented in 2.1.2. We follow the ideias of [5].

This section is pretty technical and we assume the reader is familiar with abelian categories and its related concepts. This section can be skipped if the reader is not comfortable with those ideas.

However, they should keep in mind that derived categories are the correct place to consider functors as the sections of a sheaf, tensor products, etc and that the triangulated structure play a very important role on all that. Also, the reader should remember theorem 3.5, that relates triangulated $\mathcal{A}_\infty$-categories and the usual triangulation.

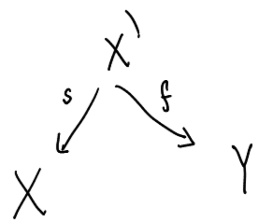
Definition 3.1 *Let $\mathcal{A}$ be an abelian category. We can define the derived category of $\mathcal{A}$ in some steps:*

1. *Define the category of complexes Kom$(\mathcal{A})$ as the category whose objects are chain complexes in A and morphisms are morphisms of chain complexes (i.e., families of morphisms commuting with the differential).*

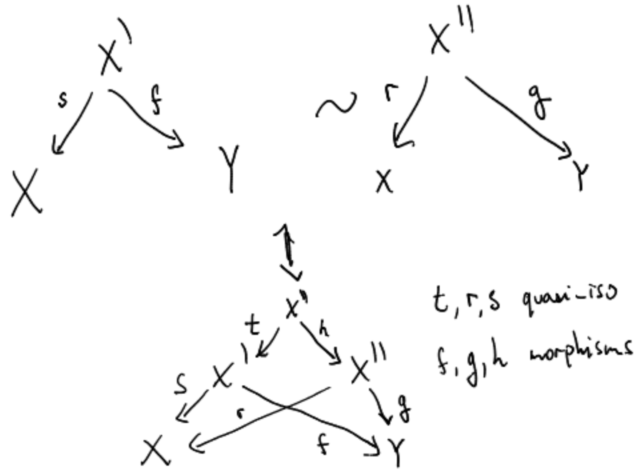
2. *Define the category of homotopy chain complexes $K(\mathcal{A})$, whose objects are chain complexes, but morphisms are identified if they are homotopic (in the sense that there is a family of maps h for which $f - g = dh + hd$).*

3. *Consider the quasi-isomorphisms (morphisms which induce isomorphisms on cohomology) and add inverses to all of them (localization). This can be done by simply*

defining a new category with the same objects but morphisms defined by equivalence classes of oriented edges on a graph (with the new morphisms added as edges of opposite orientation). In practice, such morphisms can be seen as classes of "roofs", which are diagrams of the form:



where f is a morphism and s is a quasi isomorphism. The equivalence relation is given by the existence of "roof between the roofs" and it is possible to define composition easily by the properties of quasi isomorphisms.



It is important to observe that one could define the derived category for $\mathrm{Kom}^b(\mathcal{A})$, i.e., considering only bounded complexes (the ones for which we get zero for higher degrees).

Derived categories are useful because they, in some sense, identify objects with its resolutions. This helps us to make non exact functors exact in some sense and also helps us to deal better with quasi isomorphisms (which are important if we want to work with cohomology). So, for example, as we cited in the opening of the section, the functor of sections of a sheaf, cohomology of sheaves, tensor products and others should be treated considering derived categories. That is one of the reasons to consider the derived category of Coherent Sheaves, as it is done in the next subsection.

Note, although, that derived categories are not abelian categories! This means that we do not have the nice properties about exact sequeces we have over Abelian Categories. This is the main reason to define a triangulation structure.

Before diving into Coherent sheaves, we need to discuss a last topic in category theory: triangulation.

Definition 3.2 *Let $\mathcal{A}$ be an additive category.*

A triangulated structure on $\mathcal{A}$ is given by the following data:

- *An additive automorphic functor $S : \mathcal{A} \to \mathcal{A}$, called shift. We denote $S^n(X) =: X[n]$ and $S^n(f) =: f[n]$ for any object X and morphism f.*

  This functor allows us to define $\mathcal{A}$-triangles, which are sequences of the form

  $$X \to Y \to Z \to X[1],$$

  and its morphisms, which are diagrams as below.

  $$\begin{array}{ccccccc} X & \to & Y & \to & Z & \to & X[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \to & Y' & \to & Z' & \to & X'[1] \end{array}$$

- *A set (or class) of distinguished $\mathcal{A}$-triangles satisfying:*

  1. *(a) $X \to X \to 0 \to X[1]$ is distinguished.*

     (b) Any triangle which is isomorphic to a distinguished triangle is distinguished.

     (c) Any morphism can be completed to a disguinshed triangle.

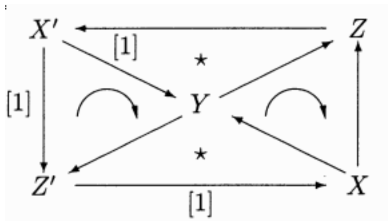
  2. *A triangle $X \to Y \to Z \to X[1]$ is distinguished if and only if $Y \to Z \to X[1] \to Y[1]$ (with the same morphisms) is distinguished.*

  3. *If we have two triangles $X \to Y \to Z \to X[1]$ and $X' \to Y' \to Z' \to X'[1]$ and morphisms $X \to X'$ and $Y \to Y'$, there exists a morphism (not necessarily unique) $Z \to Z'$ for which the diagram*

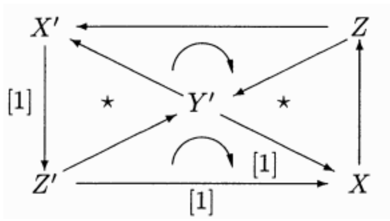
     $$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, dashed] & X[1] \arrow[d] \\ X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & X'[1] \end{tikzcd}$$

     commutes.

  4. *The last axiom concerns the so called octhaedral diagrams. Suppose we have a diagram like the one below:*



Here, $Y \to Z \to X' \to Y[1]$ and $X \to Y \to Z' \to X[1]$ are distinguished (marked with a star). The other two triangles commute.
Then there exists an object Y' and morphisms as below such that the triangles $Z' \to Y' \to X' \to Z'[1]$ and $X \to Z \to Y' \to X[1]$ (marked with star) are distinguished, the other two commute, the composite morphisms $Y \to Y'$ (through Z and through Z') coincide and the composite morphisms $Y' \to Y[1]$ (through $X[1]$ and X') coincide.



Derived categories of abelian categories are triangulated. We will not enter in many details here, but we refer the reader to [5] if they want to learn more.

Proposition 3.3 **(see [5])** *If $\mathcal{A}$ is an abelian category, $D^*(\mathcal{A})$ is triangulated. S is given by the natural shift of the complexes and the distinguished triangles are defined as triangles isomorphic to triangles of the form:*

$$K^\bullet \to K^\bullet \oplus K[1]^\bullet \oplus L^\bullet \to K[1]^\bullet \oplus L^\bullet \to K[1]^\bullet$$

defined for each morphism $K^\bullet \to L^\bullet$.

The next theorem justifies the comparision with exact sequences:

Theorem 3.4 *Let*

$$K \to L \to M \to K[1]$$

be an distinguished triangle of complexes in $D(\mathcal{A})$. Then the sequence:

$$\cdots \to H^i(K) \to H^i(L) \to H^i(M) \to H^i(K[1]) = H^{i+1}(K) \to \dots$$

is long exact.

Both results above are standard and do not have complicated proofs. We opt here to refer the reader to [5], where they can find these proofs and more results about derived and triangulated categories.

To conclude, we state the last result of this subsection:

Theorem 3.5 *Let $\mathcal{C}$ be an $\mathcal{A}_\infty$-category. Suppose $\mathcal{C}$ is triangulated in the sense of 2.1.1. Then $H^0(\mathcal{C})$ is triangulated in the sense defined above.*

Idea of Proof. The ideia is simply define exact triangles to be the distinguished triangles and the induced shift functor $H^0(\mathcal{S})$ to be the shift functor.

The axioms are easily verified. For example, we verify 1. Axiom 1a is verified by defining a functor $\mathcal{Z} \to \mathcal{C}$ for which $Z_1 \mapsto X$, $Z_2 \mapsto X$ and $Z_3 \mapsto 0$ and all the morphisms go to zero (except for the identities). 1b is verified by the functorial definition and 1c is verified by the hypothesis of $\mathcal{C}$ being triangulated.

We wont do every computation here, but the idea is clear. For a complete approach see [14].

■

3.1.2 Basics on Coherent Sheaves

We assume the reader has familiarity of sheaves, ringed spaces and basics on complex manifolds. We recomend the reader to give a look at [16] for more details on complex analytic geometry and at [17] or [7] in order to learn more algebraic geometry.

Definition 3.6 *A sheaf of $\mathcal{O}_X$-modules $\mathcal{S}$ over a ringed space $(X, \mathcal{O}_X)$ is called coherent if it satisfies:*

- *$\mathcal{S}$ is of finite type over $\mathcal{O}_X$, that is, every point in X has an open neighborhood U in X such that there is a surjective morphism $\mathcal{O}_X^n|_U \to \mathcal{S}|_U$ for some natural number n;*
- *for any open set $U \subset X$, any natural number n and any morphism $\varphi : \mathcal{O}_X^n|_U \to \mathcal{S}|_U$ of $\mathcal{O}_X$-modules, the kernel of φ is of finite type.*

These sheafs are important because they are locally cokernels of free finite $\mathcal{O}_X$-modules. So, locally, its sections may be seen as elements of $\mathcal{O}_X^{\oplus n}$ (after a quotient).

In the case S is locally $\mathcal{O}_X^{\oplus n}$ (locally free) it can be regarded as a vector bundle after considering, at a small neighborhood U, a tuple $(s_1, \ldots, s_n)$ as a section of $E = U \times k^n$ over U. As vector bundles encode many important geometric properties, it surely makes sense to study the locally free sheaves.

The problem is that they do not form an abelian category! It is natural then to seek a more general class of objects that have this properties. Results as the Oka's theorem and many GAGA style theorems (relating algebraic geometry and analytic geometry) also can be very important reasons to study coherent sheaves.

Theorem 3.7 see [16],Ch. 8 *Let X be a compact complex manifold and let $\mathbf{Coh}(X)$ be the category of coherent sheaves over X. Then $\mathbf{Coh}(X)$ is an abelian category.*

Here, recall that a complex manifold comes with a structural sheaf $\mathcal{O}_X$ of holomorphic functions, and it is therefore a ringed space.

Definition 3.8 *For a compact complex manifold X we define $\mathcal{D}^b(X)$ to be the bounded derived category $\mathcal{D}^b(\mathbf{Coh}(X))$, as defined in the previous subsection.*

3.2 The Homological Mirror Symmetry Conjecture

Finally we are all set to state the HMS conjecture. The idea of this conjecture is to relate the symplectic geometry (Fukaya Category) and the algebraic geometry (Coherent Sheaves) of mirror manifolds.

Our objects of study in this conjecture (although today many people go further to more general spaces) are the *Calabi-Yau* manifolds.

Definition 3.9 *A compact Kähler manifold of complex dimension n is called a Calabi-Yau manifold if it has trivial canonical bundle, i.e., if there exists a nowhere vanishing holomorphic n-form.*

On complex dimension 1, Calabi-Yau manifolds are just elliptic curves and, on dimension two, they are the famous K3 surfaces. On dimension 3 there are many topologically different Calabi-Yau manifolds, but we still do not know if this number is finite or not.

There is not a formal definition for what the “mirror map” should be. What generally is known is that mirror manifolds have related numerical invariants. At least their Hodge numbers are swaped.

This map comes from symmetries that appears in Physics: we have the so called A and B models for string theory, and what happen is that the formalism of the A-model is based on symplectic geometry and the formalism of the B-model is based on algebraic geometry. As they should physically result in the same observations, we arrive with mirror symmetry, which relates the two geometries.

In order to state the conjecture, we have to give a little more structure to our Fukaya Category.

Definition 3.10 *Consider a symplectic manifold and associate to each lagrangian a local system, which are flat vector bundles $\mathcal{E} \to L$ with unitary holonomy over the Novikov field over $\mathbb{C}$. Then, our objects will be pairs $(\mathcal{E}, L)$ and we will define our hom spaces to be*

$$CF\left((\mathcal{E}_0, L_0), (\mathcal{E}_1, L_1)\right) =$$

$$= \bigoplus_{p \in L_0 \cap L_1} hom(\mathcal{E}_0|_p, \mathcal{E}_1|_p).$$

With composition maps given by taking $(\rho_1 \otimes \cdots \otimes \rho_d) \in \bigotimes_i hom(\mathcal{E}_{i-1}|_{p_i}, \mathcal{E}_i|_{p_i})$ *and, for each* $q \in L_0 \cap L_d$ *and a fixed homotopy class* $[u]$*, sending it to the map defined by composing this morphisms and the pararallel transports over the boundaries* $\gamma_i \in hom(\mathcal{E}_i|_{p_i}, \mathcal{E}_i|_{p_{i+1}})$ *(see figure 11)*

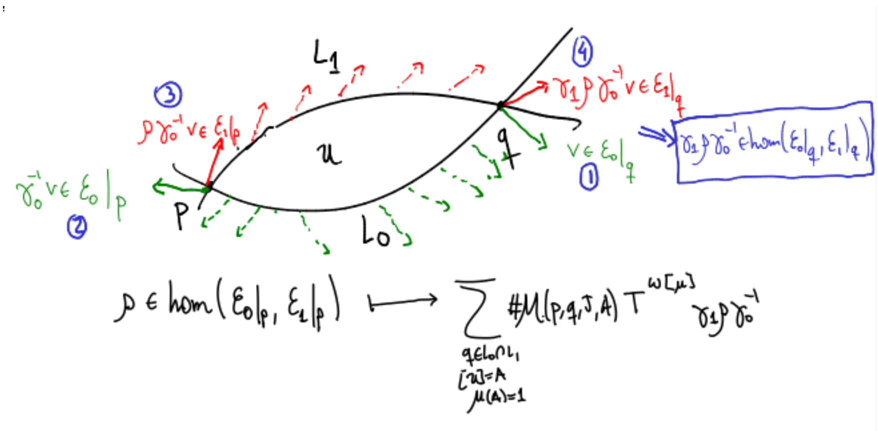


Figure 11: *Definition of the differential* μ^1 *in this setting. The numbering on the figure represents the order the morphisms* ρ *and* γ *are applied. The dashed arrows represent the vector being parallel transported.*

Now, in this slightly modified Fukaya category $Fuk(M)$, we have the result:

Theorem 3.11 *The twisted* $\mathcal{A}_\infty$*-category of* $Fuk(M)$ *is a triangulated* $\mathcal{A}_\infty$*-category. Its index zero cohomological category is denoted by* $D^\pi(M)$*.*

Now, we have two triangulated categories. One enconding the symplectic geometry of the manifold M ($D^\pi(M)$) and the other encoding the complex analytical/algebraic geometry of the manifold M ($D^b(M)$). We can then state the conjecture:

Conjecture 3.12 *(***HMS Conjecture***)***(see [8])**. *For mirror Calabi-Yau manifolds,* X *and* $\hat{X}$*, we have that* $D^\pi(X)$ *and* $D^b(\hat{X})$ *are equivalent triangulated categories.*

Today, homological Mirror Symmetry is proved for some simple cases in which the mirror map is easy to define (like elliptic curves, see [12] and [11]) and also for some specific manifolds (see [13]).

Besides that, we have many results concerning other spaces beyond Calabi Yau manifolds, as Fano Varieties, Abelian Varieties and others.

We can give some of the insights Kontsevich had in order to state the conjecture. One of the main points is that both categories have a "duality" $(\hom(X, Y))^* \cong \hom(Y, X[n])$

which is given by Serre's duality in the algebraic geometric side and follows from the symmetry of the definitions (we are not going into details) on the sympletic side.

Another point is the physics: A-branes and D-branes are physical entities which are related to the objects in the (modified as above) Fukaya Category and in the Derived category of Coherent Sheaves. Also, they should be equivalent since both models give rise to the same physics.

Finally, we emphazise that the conjecture is wide open and there is much research to be done. Mirror Symmetry is a fascinating subject that connect many areas of mathematics. I hope this survey could give the reader a general view of the HMS conjecture, especially the concepts related to the symplectic side.

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