CHERN-WEIL THEORY

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Abstract. We give an introduction to the Chern-Weil construction of characteristic classes of complex vector bundles. We then relate this to the more classical notion of Chern classes as described in [2].

Contents

1. Introduction

One of the fundamental problems of topology is that of the classification of topological spaces. Algebraic topology helps provide useful tools for this quest by means of functors from topological spaces into algebraic spaces (e.g. groups, rings, and modules). By ensuring that these functors are suitably invariant under topologically irrelevant deformation (e.g homeomorphism, or often just the weaker notion of homotopy equivalence), these functors provide us with a means by which to classify spaces—if the targets of two spaces don't match, then they are surely not homeomorphic (precisely due to this invariance property).

Characteristic classes provide us with a tool of the same nature, but for the more specialized notion of fibre bundles. A characteristic class is a map c from a bundle $\xi = (E, \pi, B)$ into the cohomology of B, $c : \xi \to H^*(B)$ that satisfies a naturality

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condition, $c(\xi) = f^*c(\xi')$ for any bundle map $f : \xi \to \xi'$. It is this naturality condition which ensures that characteristic classes are invariant under vector bundle isomorphism, and thus capture information about the isomorphism class of a vector bundle. In this way they provide us with a new classification tool—if two bundles do not have the same sets of characteristic classes, they are surely not isomorphic.

It was shown by Chern and Weil in the late 1940's that one can in fact construct such characteristic classes in the case of complex plane bundles using geometrical data—in particular connections and their curvatures. This is of remarkable use in physics, where complex vector bundles arise as the playground of quantum fields (operator valued sections of complex vector bundles). These bundles are often prescribed with a connection A, called the "gauge field", which has a curvature F called the "field strength". The work of Chern and Weil tells us that we can use F alone to construct the characteristic classes of this bundle. Thus every quantum gauge theory comes prescribed with a way to understand its underlying topology. The properties of the characteristic classes of these bundles, in turn, constrain the physical theory, giving rise to such phenomena as the Bohm-Aharonov effect. Conversely, any gauge theory which does not depend on the metric of the underlying spacetime manifold must have a Lagrangian which depends only on "topological terms". These "topological terms" are precisely the characteristic classes of the relevant bundles.

In this paper we will review the construction of Chern and Weil, and then show that we can in fact construct all characteristic classes of complex plane bundles (satisfying certain axioms) in this way. We begin in Section 2 by reviewing the construction of characteristic classes from curvatures. We then review the relevant topological theory of characteristic classes to show that the previous construction does indeed give us what we want. In Section 3 we consider the Euler class of an oriented real bundle and in Section 4 use this to construct the Chern classes of a complex plane bundle. In this same section we show that *any* characteristic class of a complex plane bundle satisfying a certain set of simple axioms must be a Chern class. Armed with this uniqueness theorem, we proceed in Section 5 to relate our initial geometrically constructed invariants to the Chern classes.

Throughout this paper we will assume familiarity with the theory of smooth manifolds and some elementary algebraic topology. In particular, we will assume familiarity with vector bundles and cohomology theory. Readers unfamiliar with the notion of connections and their associated curvatures may refer to Appendix B. Appendix A reviews some basic vector bundle constructions, and Appendix C reviews the theory of complex Grassmann manifolds and their utility in providing classifying spaces for complex plane bundles.

1.1. **Conventions.** All manifolds will be smooth, Hausdorff, and paracompact. Bundles will be denoted by triples $\zeta = (E, \pi, M)$ or by the standard $E \xrightarrow{\pi} M$. We will use $\Gamma(\zeta)$ to denote the space of smooth sections of a bundle ζ. Unless otherwise specified, all functions should be assumed smooth.

For any bundle $\zeta = (E, \pi, M)$, E_0 will denote the space obtained by removing the zero section from E.

We will use the sign convention of [2] (Milnor & Stasheff) for the definition of the coboundary δ, i.e.

$$\langle \delta c,\, \alpha \rangle = (-1)^{n+1} \langle c,\, \partial\alpha \rangle$$

for $[c] \in H^n(X; \Lambda)$, $[\alpha] \in H_{n+1}(X; \Lambda)$ (X some space and Λ some ring).

With this convention, for example, Stokes' theorem reads

$$\langle d\omega,\, \mu \rangle = (-1)^{\text{dim M}} \langle \omega,\, \partial\mu \rangle$$

and thus our volume forms will differ from the standard volume forms by a sign

$$\text{vol}_{\text{our}} = (-1)^{\text{dim}M} \text{vol}_{\text{std}}$$

To return to the standard conventions for coboundaries and volumes, send $\Omega \to -\Omega$ in all characteristic class formulas involving the curvature that follow.

2. Chern-Weil Theory: Invariants from Curvature

Let $\zeta = (E, \pi, M)$ be a smooth, complex n-plane bundle. We denote the complexification of the cotangent bundle of M by $(T^*M)_{\mathbb{C}}$ as in Appendix B. We will here use geometrical data (connections and curvatures) to construct useful topological invariants of the bundle ζ.

Consider a connection ω on ζ with curvature Ω. Then Ω is a ($\mathfrak{gl}(n, \mathbb{C})$-valued) two-form on M. Our goal is to use the two-form character of Ω to obtain a map from ζ into the de Rham cohomology of M. We want this construction to be independent of the choice of connection (since the space of all connections on ζ is contractible, and the de Rham cohmology functor is homotopy invariant), and thus produce invariants of the bundle itself.

Our first task is to isolate the differential form character of Ω from its matrix character.

Let $M(n, \mathbb{C})$ denote the space of $n \times n$ complex matrices.

Definition 2.1. An **invariant polynomial** on $M(n, \mathbb{C})$ is a function

$$P : M(n, \mathbb{C}) \to \mathbb{C}$$

which is basis invariant; i.e., which satisfies

$$P(TAT^{-1}) = P(A)$$

for every nonsingular matrix T.

Proposition 2.2. *The above property of basis independence is equivalent to cyclicity. In particular, for any invariant polynomial P, $P(AB) = P(BA)$.*

Proof. Let P be basis independent and let A, B be any matrices. If B is nonsingular, then

$$P(AB) = P\left(B(AB)B^{-1}\right) = P(BA)$$

by basis independence. If B *is* singular, the result follows by the continuity of P, since we may approximate B to arbitrary accuracy by nonsingular matrices. Thus

P is cyclic.

Conversely, let P be cyclic and T be a nonsingular matrix. Then

$$P(A) = P(AI) = P(AT^{-1}T) = P(TAT^{-1})$$

and thus P is basis independent.

Thus any basis independent matrix is cyclic and vice-versa □

Example 2.3. Both the determinant and the trace are invariant polynomials.

It is this machinery of invariant polynomials which precisely provides us with a way to isolate the differential form character of Ω. This occurs as follows:

Given any curvature form Ω and any invariant polynomial P, we may define a differential form $P(\Omega)$ in the following way. Consider an open cover of M, and in each open set select a local basis of sections $\{s_i\}$. We may define the components Ω_{ij} of our curvature form in this basis via

$$\Omega(s_i) = \sum_j \Omega_{ij} \otimes s_j$$

where each Ω_{ij} is a 2-form. Regarding the curvature form as a matrix $\Omega = [\Omega_{ij}]$ whose entries lie in the commutative algebra of even-dimensional forms over $\mathbb{C}$, we can evaluate P on Ω (precisely because of the commutativity of all elements involved) to obtain a sum of forms of even degree. Since M is Hausdorff and para-compact, we may patch together the local representatives $P(\Omega_{ij})$ into a global form using a partition of unity. Since P is invariant, this form does not depend on our choices of local basis.

Note that we may just as easily replace our invariant polynomial P by a formal power series

$$P = P_0 + P_1 + P_2 + \dots$$

where each P_k is an invariant homogeneous polynomial of degree k.

The evaluation $P(\Omega)$ will always be well defined, since $P_k(\Omega) = 0$ for $2k > \dim M$ (since $P_k(\Omega)$ will be a differential form of degree $2k$).

The following lemma shows that $P(\Omega)$ provides us with exactly what we sought—an element of the de Rham cohomology.

Lemma 2.4. *For any invariant polynomial P, the form $P(\Omega)$ is closed.*

Proof. Let P be an invariant polynomial and Ω_{ij} the components of Ω in some local basis. We have that

$$dP(\Omega) = \sum_{ij} \frac{\partial P}{\partial \Omega_{ij}}\, d\Omega_{ij} = \operatorname{tr}\left(P'(\Omega)\, d\Omega\right)$$

where we have introduced the matrix

$$[P'(\Omega)]_{ij} = \frac{\partial P}{\partial \Omega_{ji}}$$

Using that

$$\Omega = d\omega - \omega \wedge \omega$$

we arrive at the **Bianchi identity**

$$d\Omega = \omega \wedge \Omega - \Omega \wedge \omega$$

which gives us the matrix of three-forms $[d\Omega]$.

We note that $P'(\Omega)$ and Ω must commute, which may be seen by differentiating both sides of the following identity and evaluating at $t = 0$ (where $[E_{ij}]_{kl} = \delta_{ik}\delta_{jl}$ is the matrix unit with (i, j)th component 1 and all other components zero)

$$P\left((I + tE_{ij})\Omega\right) = P\left(\Omega(I + tE_{ij})\right)$$

Taking the derivatives yields

$$\sum_k \frac{\partial P}{\partial \Omega_{ik}} \Omega_{jk} = \sum_k \frac{\partial P}{\partial \Omega_{kj}} \Omega_{ki}$$

Thus, in cohomology, we must have that

$$[P'(\Omega)] \cdot [\Omega] = [\Omega] \cdot [P'(\Omega)] \implies \Omega \wedge P'(\Omega) = P'(\Omega) \wedge \Omega$$

Substituting the Bianchi identity into our formula for $dP(\Omega)$ and using the graded cyclicity of the trace ($\text{tr } (\alpha \wedge \beta) = (-1)^{|\alpha||\beta|}\text{tr } (\beta \wedge \alpha)$) gives us that

$$\begin{aligned} dP(\Omega) &= \text{tr } \left(P'(\Omega) \wedge \omega \wedge \Omega - P'(\Omega) \wedge \Omega \wedge \omega\right) \\ &= \text{tr } \left(P'(\Omega) \wedge \Omega \wedge \omega - P'(\Omega) \wedge \Omega \wedge \omega\right) = 0 \end{aligned}$$

Thus $P(\Omega)$ is closed and is a (complex) de Rham cocycle, as we wished to show. □

Thus every invariant polynomial P allows us to assign cohomology classes to curvatures on ζ:

$$P : \Omega \mapsto [P(\Omega)] \in H^*(M; \mathbb{C})$$

We have the following lemma

Corollary 2.5. *The cohomology class $[P(\Omega)]$ is independent of the connection ω.*

Proof. Let ω_0 and ω_1 be two different connections on ζ. Considering the map

$$p_1 : M \times \mathbb{R} \to M \qquad (p, t) \mapsto p$$

we may form the induced bundle $\zeta' \coloneqq p_1^* \zeta$ over $M \times \mathbb{R}$ and induced connections $\omega_i' \coloneqq p_1^* \omega_i$ on ζ'. Note that the linear combination

$$\omega \coloneqq t\omega_1' + (1 - t)\omega_0'$$

is also a connection on ζ'. Let Ω be the curvature of ω and Ω_i the curvature of each ω_i, and note that $P(\Omega)$ will be a de Rham cocycle on $M \times \mathbb{R}$.

We now consider the family of maps $i_t : M \to M \times \mathbb{R}$ defined by $i_t : p \mapsto (p, t)$, $t \in [0, 1]$. For $t = 0, 1$, we can identify the induced connections $(i_t)^*\omega$ on $(i_t)^*\zeta'$ with the connections ω_t on ζ. Thus

$$(i_t)^*[P(\Omega)] = [P(\Omega_i)]$$

Since the mapping i_0 is homotopic to i_1, it follows that

$$[P(\Omega_1)] = [P(\Omega_2)]$$

□

Thus we confirm what we initially suspected, that the mapping $P : \Omega \mapsto [P(\Omega)]$ determined a **characteristic (cohomology) class** of the bundle ζ in $H^*(M;\mathbb{C})$ (i.e. this class is characteristic of the bundle structure itself). Since any isomorphism of bundles gives an isomorphism of base spaces (which in turn induces an isomorphism of cohomology), we see that this class can only depend on the isomorphism class of ζ. Indeed, we have from the above that for any map $g : M' \to M$, we must have that

$$[P(\Omega')] = g^*[P(\Omega)]$$

where Ω' is the curvature of the induced connection on $g^*\zeta$.

It is now natural to ask how this "newly discovered" invariant corresponds to other topological notions that have already been well studied (e.g. as the de Rham cohomology of a manifold corresponds via de Rham's theorem to its singular cohomology and topological Betti numbers). We will see later that any such curvature "characteristic class" for a complex vector bundle can be expressed in terms of what will be called "Chern classes".

We conclude this section by classifying these curvature invariants, later returning to the problem of expressing these invariants in terms the Chern classes.

2.1. **Constructing Curvature Invariants.** For any square $(n \times n)$ matrix A, define $\sigma_k(A)$ to be the kth elementary symmetric function of the eigenvalues of A

$$\sigma_k(A) = \sum_{1 \le i_1 < \ldots i_n \le n} a_{i_1} \ldots a_{i_k}$$

where the $\{a_i\}$ are the eigenvalues of A. We then have that

$$\det(I + tA) = \sum_{k=0}^{\infty} t^k \sigma_k(A)$$

Note that $\sigma_k(A) = 0$ for $k > n$ so that the sum is finite.

Lemma 2.6. *Any invariant polynomial on $M(n,\mathbb{C})$ can be expressed as a polynomial function of $\sigma_1, \ldots, \sigma_n$.*

Proof. Let $A \in M(n,\mathbb{C})$ and B be the similarity transformation which takes A to its Jordan canonical form—i.e. BAB^{-1} is in Jordan canonical form and is thus, in particular, upper triangular. Replace B with $\mathrm{diag}(\epsilon, \ldots, \epsilon^n) \cdot B$ for some ϵ arbitrarily close to 0 (so that the off diagonal terms of BAB^{-1} are correspondingly arbitrarily close to 0). By the continuity of P, it follows that $P(A) = P(BAB^{-1})$ only depends on the diagonal entries of BAB^{-1}, i.e. on the eigenvalues of A. Since $P(A)$ must by symmetric in these eigenvalues since if $A = \mathrm{diag}(a_1, \ldots, a_n)$ (as we may switch the ordering of any two of the eigenvalues by a change of basis) it must be expressible as a polynomial in the elementary symmetric functions (this is a classical result from the theory of symmetric functions). Since $\sigma_k(A) = 0$ for $k > n$, the result follows. □

We thus have found the building blocks, the elementary symmetric polynomials σ_k, for *any* curvature invariant we can imagine (i.e. for any invariant polynomial we can imagine). We will see that the "Chern classes" $c_r(\zeta)$ will be related to our curvature invariants via

$$c_k(\zeta) = \frac{\sigma_k(\Omega)}{(2\pi i)^k}$$

We now take a detour into the classical theory of Chern classes before returning to the issue of understanding our curvature invariants fully.

3. The Euler Class

We now introduce the Euler class, both as a preliminary step towards the definition of Chern classes, and as an independent demonstration of the construction and utility of characteristic classes.

For any oriented manifold, there is a privileged element of the top cohomology group (with coefficients in $\mathbb{Z}$): the fundamental cohomology class $[u]$ (i.e. the dual to the fundamental homology class $[\mu]$ defined by $\langle u, \mu\rangle = 1$). In particular, this is true for any oriented real vector space. Thus it is not unreasonable to believe that, if we can suitably extend the concept of orientation to real vector bundles, we may be able to find cohomological invariants of these oriented bundles. This is precisely the idea of the Euler class.

Definition 3.1. Let $E \xrightarrow{\pi} B$ be a real n-plane bundle. An orientation on E is a choice of orientation for each fiber such that each $b \in B$ has a neighborhood U_b with a local basis of sections $\{s_i\}$ such that the orientation determined by $\{s_i(b)\}$ agrees with that on F_b. Equivalently, we may require that b be contained in a local trivialization (U_b, τ) such that the linear mapping $v \mapsto \tau^{-1}(b, v)$ from $\mathbb{R}^n$ to F_b is orientation preserving.

In terms of cohomology, this says that each fiber is assigned a preferred generator $u_F \in H^n(F, F_0; \mathbb{Z})$ and for each $b \in B$, there exists a neighborhood U_b and a cohomology class $u_b \in H^n(\pi^{-1}(U_b), \pi^{-1}(U_b)_0; \mathbb{Z})$ such that for every fiber F over U_b, the restriction $u_b|_{(F,F_0)} \in H^n(F, F_0; \mathbb{Z})$ is equal to u_F.

We have the following theorem:

Theorem 3.2 (Thom isomorphism)**.** *Let $E \xrightarrow{\pi} B$ be a a real, oriented n-plane bundle. The cohomology group $H^i(E, E_0; \mathbb{Z})$ is zero for $i < n$ and $H^n(E, E_0; \mathbb{Z})$ contains a unique cohomology class u (the* ***Thom class****) such that the restrictions*

$$u|_{(F,F_0)} \in H^n(F, F_0; \mathbb{Z})$$

are equal to the fundamental classes u_F for each fiber F. The correspondence $y \mapsto y \cup u$ provides an isomorphism between $H^k(E; \mathbb{Z})$ and $H^{k+n}(E, E_0; \mathbb{Z})$ for each integer k—i.e. $H^*(E, E_0; \mathbb{Z})$ is a free $H^*(E; \mathbb{Z})$-module on one degree-n generator u.

For a proof, refer to chapter 10 of [2].

Since E deformation retracts onto its zero section (which is homeomorphic to B) via $(t, v) \mapsto tv$, we have that $H^k(B; \mathbb{Z}) \simeq H^k(E; \mathbb{Z})$ (via π^*), and so the above gives us further that

$$H^k(B; \mathbb{Z}) \simeq H^{n+k}(E, E_0; \mathbb{Z}) \qquad x \mapsto (\pi^* x) \cup u$$

We can now go forward with the definition of the Euler class.

Given an oriented n-plane bundle $E \xrightarrow{\pi} B$, the inclusion $(E, \emptyset) \subseteq (E, E_0)$ gives rise to the restriction homomorphism

$$H^*(E, E_0; \mathbb{Z}) \to H^*(E; \mathbb{Z}) \qquad y \mapsto y|_E$$

Applying this homomorphism to the Thom class $u \in H^n(E, E_0; \mathbb{Z})$ gives an element $u|_E$ of $H^n(E; \mathbb{Z})$.

Definition 3.3. The **Euler class** of an oriented (real) n-plane bundle $\xi = E \xrightarrow{\pi} B$ is the cohomology class

$$e(\xi) \in H^n(B; \mathbb{Z})$$

given by

$$\pi^* e(\xi) = u|_E$$

The name "Euler characteristic" comes from the fact that for any smooth, compact, oriented manifold M, the "Kronecker index" $\langle e\,(TM)\,,\,\mu\rangle$ recovers the Euler characteristic, $\chi(M)$ of M.

The Euler class has the following properties:

Proposition 3.4 (Naturality)**.**
For all $f : B \to B'$ covered by an orientation preserving bundle map $\xi \to \xi'$,

$$e(\xi) = f^* e(\xi')$$

Proposition 3.5 (Parity)**.**
If the orientation of ξ is reversed, the Euler class changes sign.

These results follow from the uniqueness of the Thom class and the parity property of the fundamental class (and hence the Thom class) respectively.

Example 3.6. Let ξ be a trival n-plane bundle $M \times \mathbb{R}^n \xrightarrow{\pi} M$. Consider the bundle map taking ξ to ξ', the trivial n-plane bundle over a point

$$f : M \times \mathbb{R}^n \to \{0\} \times \mathbb{R}^n$$

giving by mapping M to 0. This is clearly orientation preserving (it does not affect the fibers). This covers $\pi \circ f$, and so we must have that

$$e(\xi) = (\pi \circ f)^* e(\xi') = (\pi \circ f)^* 0 = 0$$

We also have the following proposition

Proposition 3.7.
The Euler class of a Whitney sum or Cartesian product of bundles is given by

$$e(\xi \oplus \xi') = e(\xi) \cup e(\xi') \qquad e(\xi \times \xi') = e(\xi) \times e(\xi')$$

Remark 3.8. Note that the orientation of $F \oplus F'$ for two oriented vector spaces F, F', is given by taking an oriented basis for F followed by an oriented basis for F'.

Proof. Let ξ have fibers of dimension m and ξ' have fibers of dimension n. With our sign conventions, the fundamental class (and thus the Thom class) obeys

$$\mu(\xi \times \xi') = (-1)^{mn} \mu(\xi) \times \mu(\xi')$$

(under the isomorphism $H^{m+n}(B \times B') \simeq H^m(B) \times H^n(B')$). Applying the restriction homomorphism

$$H^{m+n}(E \times E', (E \times E')_0) \to H^{m+n}(E \times E') \simeq H^{m+n}(B \times B')$$

we get

$$e(\xi \times \xi') = e(\xi) \times e(\xi')$$

Now suppose that $B = B'$. Pulling back both sides of the above equation via the diagonal map $B \to B \times B$ to $H^{m+n}(B; \mathbb{Z})$ gives us that

$$e(\xi \oplus \xi') = e(\xi) \cup e(\xi')$$

□

We now give two examples of properties of the Euler class which can extract useful information about our bundles.

Lemma 3.9. *If the fiber dimension n of ξ is odd,* $2e(\xi) = 0$

Proof. The Thom isomorphism maps $e(\xi)$ to

$$\pi^* e(\xi) \cup u = (u|_E) \cup u = u \cup u$$

i.e.

$$e(\xi) = \phi^{-1}(u \cup u)$$

where ϕ is the Thom isomorphism. But

$$u \cup u = (-1)^{(\dim u)^2} u \cup u$$

so that $u \cup u$, and hence $e(\xi)$, is its own additive inverse whenever the dimension of our space is odd. □

Using this and the Whitney sum property, we can, for example, see that any bundle for which $2e(\xi) \neq 0$ cannot be split as the Whitney sum of two oriented odd dimensional vector bundles.

The following lemma encapsulates one of the most important properties of the Euler class

Lemma 3.10. *Let $\xi = (E, \pi, B)$ be real oriented vector bundle which possesses a nowhere zero section. Then $e(\xi) = 0$.*

Proof. Let $s \in \Gamma(\xi)$ be a nonzero cross section. Since $\pi \circ s = \mathrm{id}$, the composition

$$H^n(B) \xrightarrow{\pi^*} H^n(E) \longrightarrow H^n(E_0) \xrightarrow{s^*} H^n(B)$$

must be the identity map of $H^n(B)$. We note that under the first two maps, we must have that

$$e(\xi) \mapsto u|_E \mapsto (u|_E)|_{E_0}$$

which must be zero since the composition

$$H^n(E, E_0) \to H^n(E) \to H^n(E_0)$$

is zero. Thus $e(\xi) = s^* \left[(u|_E)\, |_{E_0} \right] = s^*(0) = 0$. □

Thus the existence of a nonzero Euler class provides an "obstruction" to the existence of a nonzero section for an oriented real vector bundle ξ. This can be seen as the starting point for what is known as "obstruction theory", which seeks to understand characteristic classes as obstructions to the existence of certain sections or structures on our bundles.

4. The Chern Class

Armed with the notion of the Euler class, we now proceed to define the Chern classes that we will use to classify our curvature invariants.

We will begin with a proof of a simple statement

Lemma 4.1. *If ζ is a complex vector bundle, then the underlying real bundle $\zeta_{\mathbb{R}}$ has a canonical orientation.*

Proof. Let V be a finite dimensional complex vector space with basis $\{e_j\}$ over $\mathbb{C}$. Then the set $\{e_1, ie_1, e_2, ie_2, \dots\}$ gives a real basis for the underlying real vector space $V_{\mathbb{R}}$. We note that this ordering (which determines the required orientation for $V_{\mathbb{R}}$) cannot depend on the initial choice of basis $\{e_i\}$ since $GL(n, \mathbb{C})$ is connected and thus all changes of basis continuous. We may apply this construction to each fiber of ζ to obtain the required orientation of $\zeta_{\mathbb{R}}$. □

Example 4.2. Let M be a complex manifold. Then, by the above, $(TM)_{\mathbb{R}}$ is canonically oriented. Since every orientation of the tangent bundle gives rise to a unique orientation of its base space, it follows that $M_{\mathbb{R}}$ is canonically oriented as well. But $M_{\mathbb{R}}$ is topologically identical to M, and so M is canonically oriented as well. Thus every complex manifold come equipped with a preferred orientation.

We note that the above implies that for any complex n-plane bundle $\zeta = (E, \pi, B)$, the Euler class

$$e(\zeta_{\mathbb{R}}) \in H^{2n}(B; \mathbb{Z})$$

is automatically well defined. If ζ' is a complex m-plane bundle over the same base space B, the Whitney sum property tells us that

$$e\left(\zeta \oplus \zeta'\right)_{\mathbb{R}} = e(\zeta_{\mathbb{R}}) \cup e(\zeta'_{\mathbb{R}})$$

since $\zeta_{\mathbb{R}} \oplus \zeta'_{\mathbb{R}}$ is isomorphic to $(\zeta \oplus \zeta')_{\mathbb{R}}$ as oriented bundles (a preferred orientation $\{a_1, ia_1, \dots\}$ for $\pi_{\mathbb{R}}^{-1}(b)$ followed by a preferred orientation $\{b_1, ib_1, \dots\}$ for $\pi_{\mathbb{R}}'^{-1}(b)$ gives a preferred orientation $\{a_1, \dots, ia_n, b_1, \dots\}$ for $(\pi \oplus \pi')_{\mathbb{R}}^{-1}(b))$.

Recall that **Hermitian metric** on a complex vector space is a positive definite sesquilinear form. It may be defined as the unique inner product on the underlying real vector space which satisfies $|iv| = |v|$ and is linear in the "first slot" (i.e. $\langle w, \cdot \rangle$ is linear). As our base manifold B is paracompact, every complex vector bundle over B will admit a Hermitian metric (we may cover ζ with a local trivialization, define a metric on each element of the cover, and patch these together with a partition of unity).

4.1. **Constructing Chern Classes: Existence.**

Let $\zeta = (E, \pi, M)$ be a complex n-plane bundle equipped with a Hermitian metric h. We will preliminarily construct a canonical $(n-1)$-plane bundle, ζ_0, over the "deleted total space" E_0 of all nonzero vectors in E, whose fiber over a point (p, v) in E_0 is given by the orthogonal complement of v in $\pi^{-1}(p)$ under h.

We note that for any n-plane bundle $E \xrightarrow{\pi} M$ there is associated a long exact sequence in cohomology with integer coefficients (denoting $\pi_0 \coloneqq \pi|_{E_0}$):

$$\cdots \to H^i(B) \xrightarrow{\cup e} H^{i+n}(B) \xrightarrow{\pi_0^*} H^{i+n}(E_0) \to H^{i+1}(B) \xrightarrow{\cup e} \ldots$$

the **Gysin sequence**. A proof of the existence of this sequence can be found in chapter 8 of [2].

For $i < 2n-1$, both the groups $H^{i-2n}(B)$ and $H^{i-2n+1}(B)$ zero, and so we must have that $\pi_0^* : H^{(}B) \to H^i(E_0)$ is an isomorphism.

We may now use the above isomorphism to recursively define the Chern classes.

Definition 4.3. We define the **Chern classes** $c_i(\zeta) \in H^{2i}(B;\mathbb{Z})$ recursively as follows, inducting on the complex dimension n of ζ.

For $i > n$, we set $c_i(\zeta) = 0$. The **top Chern class** $c_n(\zeta)$ is equal to the Euler class of the underlying real bundle, i.e.

$$c_n(\zeta) = e(\zeta_\mathbb{R}) \in H^{2n}(B;\mathbb{Z})$$

For $i < n$, recalling the map $(\pi_0^*)^{-1} : H^{2i}(E_0) \to H^{2i}(B)$, we set

$$c_i(\zeta) = (\pi_0^*)^{-1}\, c_i(\zeta_0)$$

For example, this sets

$$c_{n-1}(\zeta) = (\pi_0^*)^{-1}\, e(\zeta_{0,\mathbb{R}})$$

and so on. Note that in particular this gives $c_0(\zeta) = 1$.

Definition 4.4. The formal sum

$$c(\zeta) = \sum_{i=1}^{n} c_i(\zeta) \in \bigoplus_i H^i(B;\mathbb{Z})$$

is called the **total Chern class** of ζ. This is a unit with well defined inverse.

4.2. **Properties.** The Chern classes have the following properties:

Proposition 4.5 (Naturality)**.** *If $f : B \to B'$ is covered by a bundle map from a complex n-plane bundle ζ to another complex n-plane bundle ζ', then*

$$c(\zeta) = f^* c(\zeta')$$

Proof. We know that this holds for the top Chern class, since this is just the Euler class. To prove naturality for the lower classes, we note that the bundle map $\zeta \to \zeta'$ gives rise to a unique map $f_0 : E_0(\zeta) \to E_0(\zeta')$, which is in turn covered by a bundle map $\zeta_0 \to \zeta_0'$. Thus we must have that $c_i(\zeta_0) = f_0^* c_i(\zeta_0')$, again by the naturality of the Euler class. We now appeal to the following commutative diagram

$$\begin{array}{ccc} E_0(\zeta) & \xrightarrow{f_0} & E_0(\zeta') \\ \downarrow{\scriptstyle \pi_0} & & \downarrow{\scriptstyle \pi_0'} \\ B & \xrightarrow[f]{} & B' \end{array}$$

as well as the identities $c_i(\zeta_0) = \pi_0^* c_i(\zeta)$ and $c_i(\zeta_0') = \pi_0'^* c_i(\zeta')$ where π_0^* is an isomorphism for $i < n$. Thus we must have that $c_i(\zeta) = f^* c_i(\zeta')$ as required. □

Proposition 4.6 (Sum Formula). *Let ζ and ξ be two complex vector bundles over a common paracompact base space B. Then*

$$c(\zeta \oplus \xi) = c(\zeta) \cup c(\xi)$$

As a first step towards proving this, we prove the following lemma

Lemma 4.7. *There exists one and only one polynomial*

$$p_{m,n}(c_1, \ldots, c_m; c'_1, \ldots, c'_n)$$

with integer coefficients in $m+n$ indeterminates such that

$$c(\zeta \oplus \xi) = p_{m,n}(c_1(\zeta), \ldots, c_m(\zeta); c_1(\xi), \ldots, c_n(\xi))$$

holds for every complex m-plane bundle ζ and n-plane bundle ξ over a common paracompact base space.

Proof. This proof is important since it gives an example of the utility of the Grassmann manifolds and canonical bundles discussed in Appendix C The reader unfamiliar with the use of the canonical bundles over the complex Grassmann manifolds as classifying spaces is encouraged to reference that appendix before continuing on with this proof.

We will construct the following two bundles, γ_1^m and γ_2^n over $G_m \times G_n$ as universal models for pairs of complex vector bundles. We define $\gamma_1^m := \pi_1^* \gamma^m$ and $\gamma_2^n := \pi_2^* \gamma^n$ where $\pi_1 : G_m \times G_n \to G_m$ is projection onto the first factor and similarly for π_2. Thus the Whitney sum $\gamma_1^m \oplus \gamma_2^n$ can be identified with the Cartesian product.

Noting that the external cohomology cross-product operation

$$a, b \mapsto a \times b = \pi_1^* a \cup \pi_2^* b$$

induces an isomorphism between $H^*(G_m; \mathbb{Z}) \otimes H^*(G_n; \mathbb{Z})$ and $H^*(G_m \times G_n; \mathbb{Z})$ (which follows from the Künneth isomorphism).

Using (C.5), we see that $H^*(G_m \times G_n)$ is a polynomial ring over $\mathbb{Z}$ on generators $\pi_1^* c_i(\gamma^m) = c_i(\gamma_1^m)$ and $\pi_2^* c_j(\gamma^n) = c_j(\gamma_2^n)$. Thus the total Chern class of $\gamma_1^m \oplus \gamma_2^n$ can be expressed *uniquely* as a polynomial

$$c(\gamma_1^m \oplus \gamma_2^n) = p_{m,n}\left(c_1(\gamma_1^m), \ldots, c_m(\gamma_1^m); c_1(\gamma_2^n), \ldots, c_n(\gamma_2^n)\right)$$

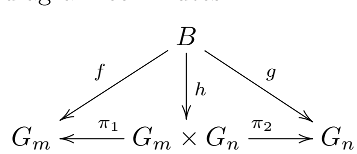
Now as ζ is a complex m-plane bundle and ξ a complex n-plane bundle,both over B, we can choose maps $f : B \to G_m$ and $g : B \to G_n$ (recall (C.3)) so that

$$f * (\gamma^m) \simeq \omega \qquad g^*(\gamma^n) \simeq \xi$$

Defining the map $h : B \to G_m \times G_n$ by

$$h(b) = (f(b), g(b))$$

we see that the following diagram commutes



so that

$$h^*(\gamma_1^m) \simeq \zeta \qquad h^*(\gamma_2^n) \simeq \xi$$

and thus

$$\begin{aligned} c(\zeta \oplus \xi) &= h^* c(\gamma_1^m \oplus \gamma_2^n) \\ &= p_{m,n}(c_1(\zeta), \ldots, c_m(\zeta); c_1(\xi), \ldots, c_n(\xi)) \end{aligned}$$

□

We now compute these polynomials, proceeding by induction of $m+n$. We make use of the following lemma

Lemma 4.8. *If ϵ^k is the trivial complex k-plane bundle over B, then $c(\zeta \oplus \epsilon^k) = c(\zeta)$ for all complex vector bundles ζ over B.*

Proof. It suffices to prove the lemma for the case $k = 1$ since the rest will follow by induction. Letting $\xi = \zeta \oplus \epsilon^1$, we note that ξ has a nonzero section $s : p \mapsto (0, \alpha)$ where α is some fixed complex number. Thus we must have that (letting ζ have fiber dimension n)

$$c_{n+1}(\xi) = e(\xi_{\mathbb{R}}) = 0 = c_{n+1}(\xi)$$

We note that $s : B \to E_0(\xi)$ is covered by a bundle map $\zeta \to \xi_0$ so that

$$s^* c_i(\xi_0) = c_i(\zeta)$$

by naturality. Substituting $\pi_0^* c_i(\xi)$ for $c_i(\xi_0)$ and using that $s^* \circ \pi_0^* = \mathrm{id}$ gives us that $c_i(\xi) = c_i(\zeta)$ as required. □

We now complete our computation, which will in turn prove the initial proposition.

Proof of Proposition 5.6. Suppose inductively that $c(\gamma_1^{m-1} \oplus \gamma_2^n) = c(\gamma_1^{m-1}) \cup c(\gamma_2^n)$ (the base case $m, n = 1$ is clear). We then consider the bundles $\gamma_1^{m-1} \oplus \epsilon^1$ and γ_2^n over $G_{m-1} \times G_n$. On one hand we have that

$$c(\gamma_1^{m-1} \oplus \epsilon^1 \oplus \gamma_2^n) = p_{m,n}\left(c_1(\gamma_1^{m-1} \oplus \epsilon^1), \ldots, c_m(\gamma_1^{m-1} \oplus \epsilon^1); c_1(\gamma_2^n), \ldots, c_n(\gamma_2^n)\right)$$

whereas on the other, in light of the above lemma, we have that

$$c(\gamma_1^{m-1} \oplus \epsilon^1 \oplus \gamma_2^n) = p_{m,n}\left(c_1(\gamma_1^{m-1}), \ldots, 0; c_1(\gamma_2^n), \ldots, c_n(\gamma_2^n)\right) = c(\gamma_1^{m-1} \oplus \gamma_2^n)$$

Letting $c_i \coloneqq c_i(\gamma_1^{m-1})$ and $c'_j \coloneqq c_j(\gamma_2^n)$ and comparing with the induction hypothesis, we find that

$$p_{m,n}(c_1, \ldots, c_{m-1}, 0; c'_1, \ldots, c'_n) = c(\gamma_1^{m-1}) c(\gamma_2^n) = (1 + c_1 + \cdots + c_{m-1})(1 + c'_1 + \cdots + c'_n)$$

(cup products understood). Introducing an indeterminate c_m we have that

$$p_{m,n}(c_1, \ldots, c_m; c'_1, \ldots, c'_n) = (1 + c_1 + \cdots + c_m)(1 + c'_1 + \cdots + c'_n) \qquad \mod c_m$$

A similar inductive argument shows that the two polynomials are also congruent mod c'_n, i.e.

$$p_{m,n}(c_1, \ldots, c_m; c'_1, \ldots, c'_n) = (1 + c_1 + \cdots + c_m)(1 + c'_1 + \cdots + c'_n) + f c_m c'_n$$

for some polynomial z. We note that f must be zero degree (i.e. an integer) since otherwise $\gamma_a^m \oplus \gamma_2^n$ would have a nonzero Chern class in dimensions greater than $2(m+n)$. □

The above formula implies that

$$c_{m+n}(\zeta \oplus \xi) = (1+z)c_m(\zeta) \cup c_n(\xi)$$

but the top Chern class is just the Euler class of the underlying real bundle, which already has the sum property

$$\begin{aligned} c_{m+n}(\zeta \oplus \xi) &= e((\zeta \oplus \xi)_{\mathbb{R}}) \\ &= e(\zeta_{\mathbb{R}} \oplus \xi_{\mathbb{R}}) \\ &= e(\zeta_{\mathbb{R}}) \cup e(\xi_{\mathbb{R}}) \\ &= c_m(\zeta) \cup c_n(\xi) \end{aligned}$$

Hence $z = 0$ and we have proved the sum formula.

Corollary 4.9. *Let ζ and ξ be two complex vector bundles over a common paracompact base space B. Then*

$$c_i(\zeta \oplus \xi) = \sum_{j+k=i} c_j(\zeta) \cup c_k(\xi)$$

Proof. On one hand we have that

$$c(\zeta \oplus \xi) = \sum_{i=1}^{n+m} c_i(\zeta \oplus \xi)$$

whereas on the other we have that

$$c(\zeta \oplus \xi) = c(\zeta) \cup c(\xi) = [1 + c_1(\zeta) + \cdots + c_n(\zeta)] \cup [1 + c_1(\xi) + \cdots + c_m(\xi)]$$

Equating orders of elements in cohomology gives

$$c_i(\zeta \oplus \xi) = \sum_{j+k=i} c_j(\zeta) \cup c_k(\xi)$$

as desired. □

4.3. **Uniqueness of the Chern Classes.** We now turn to the question of the uniqueness of the Chern classes. We begin by noting that the Chern classes can be characterized by the following axioms:

Let $\zeta = (E, \pi, B)$ a complex n-plane bundle.

(1) For each natural number i, there is a Chern class $c_i \in H^{2i}(B; \mathbb{Z})$ with $c_0(\zeta) = 1$ and $c_i(\zeta) = 0$ for $i > n$.
(2) Naturality
(3) Sum Formula
(4) For the universal line bundle γ^1 over $G_1 = \mathbb{C}P^\infty$, we have that

$$c_1(\gamma^1) = e\left((\gamma^1)_{\mathbb{R}}\right)$$

We now wish to show that it can *uniquely* be characterized by these four axioms, i.e. that the Chern classes are uniquely characterized by the above axioms.

We must introduce a bit of machinery to do this.

Definition 4.10. Let ζ be a vector bundle over B. A **splitting map** for ζ is a map $f : B' \to B$ such that $f^*\zeta$ is a Whitney sum of line bundles and $f^* : H^*(B) \to H^*(B')$ is injective.

We note that such maps actually do exist.

Proposition 4.11. *Any n-plane bundle ζ over B has a splitting map.*

A proof may be found in [1].

We have the following corollary

Corollary 4.12. *If $\{\zeta_i\}_{i=1}^k$ are vector bundles over B, then there exists a map $f : B' \to B$ which is a splitting map for all the vector bundles ζ_i*

Proof. We prove the above by induction on k. Clearly the above holds for the case of one bundle. We assume that it holds for the case of $k-1$ bundles and consider the case of k bundles. Let each bundle ζ_i have fiber dimension $n(i)$. There exists a map $g : B' \to B$ such that $g^* : H^*(B) \to H^*(B')$ injective and with $g^*\zeta_i \simeq \lambda_{i,1} \oplus \cdots \oplus \lambda_{i,n(i)}$ for $i < k$ where the λ's are line bundles. There also exists a splitting map $f : B'' \to B'$ for the bundle $g^*\zeta_k$. But then $g \circ f : B'' \to B$ is a splitting map for all the $\{\zeta_i\}_{i=1}^k$. □

We now prove that the above axioms completely characterize the Chern classes.

Theorem 4.13 (Uniqueness)**.** *Let c_i' be another sequence of characteristic classes which satisfies the above axioms $1) - 4)$. Then*

$$c_i'(\zeta) = c_i(\zeta)$$

for any complex vector bundle ζ.

Proof. By axiom 4), we must have that $c_1(\gamma^1) = c_1'(\gamma^1)$. Thus by naturality, we must have that $c_1(\lambda) = c'(\lambda)$ for any line bundle λ. Now if ζ is a complex n-bundle over a paracompact base space B, we can find a splitting map $f : B' \to B$ so that f^* is injective on cohomology and

$$f^*\zeta \simeq \lambda_1 \oplus \cdots \oplus \lambda_n$$

for some line bundles λ_i. Applying the Whitney sum formula and the fact that c and c' coincide on line bundles gives that

$$c_i(f^*\zeta) = c_i'(f^*\zeta)$$

for all i and hence

$$f^*c_i(\zeta) = f^*c_i'(\zeta)$$

for all i. Since f^* is injective, we must have that

$$c_i(\zeta) = c_i'(\zeta)$$

for all i. Hence the Chern class is unique. □

Corollary 4.14. *The first Chern class is the unique (modulo scalar multiplication) nontrivial characteristic class of a complex line bundle.*

We now return to the issue of understanding our curvature invariants.

5. An Example: The Gauss-Bonnet Theorem

With the Chern class now in hand, we return to the Chern-Weil theory to explicitly construct a curvature invariant. Along the way, we will give a new proof for a familiar theorem from the classical differential geometry of surfaces.

Consider a closed, oriented Riemannian two-manifold S. In two dimensions, the Levi-Civita curvature of M is given by

$$\Omega_{ij} = K\,dA$$

where K is the **Gaussian curvature** of M.

In any neighborhood U of S we may introduce **geodesic coordinates** (x, y) in which the metric takes the form

$$\mathbf{g} = dx \otimes dx + g(x,y)^2\, dy \otimes dy$$

Setting

$$\theta_1 = dx \qquad \theta_2 = g(x,y)dy$$

gives us a local orthonormal basis of forms over U, for which we can consider the equations of structure

$$0 = d\theta_1 = \omega_{12} \wedge \theta_2 = g(x,y)\,\omega_{12} \wedge dy$$

$$\frac{\partial g(x,y)}{\partial x} dx \wedge dy = d\theta_2 = \omega_{21} \wedge \theta_1 = -\omega_{12} \wedge \theta_1 = -\omega_{12} \wedge dx$$

Thus the connection form is given by

$$\omega_{12} = \frac{\partial g}{\partial x} dy$$

so that

$$\Omega_{12} = \frac{\partial^2 g}{\partial x^2} dx \wedge dy = \frac{1}{g}\frac{\partial^2 g}{\partial x^2}\, d\theta_1 \wedge d\theta_2 = -\frac{1}{g}\frac{\partial^2 g}{\partial x^2}\, dA$$

(recall our conventions for volume forms as given in section 1.1). Thus we have that

$$K = -\frac{1}{g}\frac{\partial^2 g}{\partial x^2}$$

We may now use our thus developed machinery to prove the celebrated Gauss-Bonnet theorem.

Theorem 5.1 (Gauss-Bonnet)**.** *Let M be a closed, oriented Riemannian 2-manifold. Then*

$$\int_M \Omega_{12} = \int_M K\, dA = 2\pi \langle e(TM),\, \mu\rangle = 2\pi\chi(M)$$

Proof. The orientation on M gives rise to an orientation of each of the fibers of TM, which we may view as an oriented 2-plane bundle with a Euclidean metric. Before specializing to TM, let's first consider an arbitrary oriented, real 2-plane bundle ξ. ξ has a canonical complex structure J which rotates a vector "counterclockwise" through an angle of $\pi/2$—i.e. in terms of any oriented local orthonormal basis $\{s_i\}$, $Js_1 = s_2$. We may thus canonically identify ξ with a complex line bundle ζ which inherits a connection obeying

$$\nabla s_1 = \omega_{12} \otimes s_2 = \omega_{12} \otimes (is_1) = i\,\omega_{12} \otimes s_1$$

so that

$$\nabla(is_1) = -\omega_{12} \otimes s_1$$

so that the connection one form on ζ is given by

$$\omega_\zeta = i\,\omega_{12}$$

and thus the curvature matrix on ζ by

$$\Omega_\zeta = i\,\Omega_{12}$$

Using the invariant polynomial $\sigma_1 = \mathrm{tr}$, we obtain a cohomology class

$$[\mathrm{tr}\ (i\,\Omega_{12})] = [i\,\Omega_{12}] \in H^2(M;\mathbb{C})$$

This represents a characteristic class for ζ, but we know that there only exists one: the Chern class. Thus

$$i\,\Omega_{12} = \alpha\, c_1(\zeta) = \alpha\, e(\xi)$$

for some $\alpha \in \mathbb{C}$. To evaluate α, it suffices to check a single case.

We thus return to the case of TM. Since $i\,\Omega_{12} = \alpha\, e(TM)$, we must have that

$$i\int_M K\,dA = \int_M i\,\Omega_{12} = \alpha\langle\, e(TM), \mu\,\rangle = \alpha\chi(M)$$

We now evaluate both sides for a sphere, for which $K = 1$, to get

$$4\pi i = 2\alpha$$

hence $\alpha = 2\pi i$. Hence

$$\int_M K\,dA = 2\pi\xi(M)$$

and the theorem is proved. □

6. Describing the Curvature Invariants

We would finally like to show how to express our curvature invariants in terms of the Chern classes. This is done by the following theorem

Theorem 6.1. *Let ζ be a complex vector bundle with connection ω (and corresponding curvature Ω). Then the cohomology class $\sigma_k(\Omega)$ is equal to $(2\pi i)^k c_k(\zeta)$.*

Proof. Our proof of the Gauss-Bonnet theorem from the last section shows us that the theorem holds for all line bundles. To show the general case, we define the invariant polynomial P by

$$P(A) = \det(I + A/2\pi i) = \sum_k \frac{\sigma_k(A)}{(2\pi i)^k}$$

For a complex line bundle λ, we must have that

$$[P(\Omega_\lambda)] = [1 + \sigma_1/2\pi i] = c(\lambda) = 1 + c_1(\lambda)$$

Now consider any bundle which ζ which splits as a Whitney sum of line bundles

$$\zeta = \lambda_1 \oplus \cdots \oplus \lambda_n$$

Choosing connections ω_i on λ_i, we get a unique sum connection ω on ζ. Choosing local sections s_i on each λ_i, the collection $\{s_i\}$ gives a local basis of sections of ζ. The corresponding curvature matrix is diagonal

$$\Omega = \mathrm{diag}\,(\Omega_1, \ldots, \Omega_n)$$

and hence

$$P(\Omega) = P(\Omega_1) \wedge \cdots \wedge P(\Omega_n)$$

by the multiplicative property of the determinant. But we have that

$$[P(\Omega_1) \wedge \cdots \wedge P(\Omega_n)] = c(\lambda_1) \cup \cdots \cup c(\lambda_n) = c(\zeta)$$

by the Whitney sum formula. Hence the result holds for any sum of line bundles.

However, we know that for *any* ζ, there exists a splitting map f that takes ζ to a sum of line bundles, and hence the result holds for $f^*\zeta$, but since f^* is injective, it must hold for any ζ. □

Thus we see that, given any complex vector bundle and any connection on it, we can construct its Chern classes using the invariant polynomials $\det(I + \Omega/2\pi i)$.

We have finally shown that given just the geometrical data of a connection on a complex n-plane bundle ζ, we may find all its characteristic classes. This gives us a bridge from the rigid geometry of a bundle back to its underlying topology.

Appendix A. Sums and Products of Vector Bundles

All proofs for the material of the appendices may be found in [2]. In this appendix, we recall the definitions of the Cartesian product and Whitney sum (i.e. direct sum) of vector bundles.

Definition A.1. Let $\{\xi_i = (E_i, \pi_i, B_i)\}_{i=1}^n$ be a finite collection of vector bundles. The **Cartesian product** of the bundles ξ_i defined as the bundle

$$\xi_1 \times \cdots \times \xi_n = (E_1 \times \cdots \times E_n, \pi_1 \times \cdots \times \pi_n, B_1 \times \cdots \times B_n)$$

with fibers

$$(\pi_1 \times \cdots \times \pi_n)^{-1}(b_1, \ldots, b_n) = \pi_1^{-1}(b_1) \times \cdots \times \pi_n^{-1}(b_n)$$

given the structure of a Cartesian product of vector spaces.

Example A.2. $T(M_1 \times M_2) \simeq TM_1 \times TM_2$ where the isomorphism is taken as an isomorphism of bundles.

Definition A.3. Let ξ_1 and ξ_2 be two bundles over the same base space B. Denote by $d : B \to B \times B$ the diagonal embedding. The bundle

$$\xi_1 \oplus \xi_2 \coloneqq d^*(\xi_1 \times \xi_2)$$

is called the **Whitney sum** of ξ_1 and ξ_2 with each fiber $\pi_\oplus^{-1}(b)$ canonically isomorphic to the direct sum $\pi_1^{-1}(b) \oplus \pi_2^{-1}(b)$.

Example A.4. The exterior algebra of differential forms over a manifold M lives in the bundle

$$\Lambda(T^*M) = \oplus_{k=0}^n \left\{\Lambda^k(T^*M)\right\}$$

Appendix B. Connections and Curvatures of Complex Vector Bundles

Consider a vector bundle $E \xrightarrow{\pi} M$. One of the shortcomings of the vector bundle structure is that we are a priori provided with no way to compare the values of a section $s \in \Gamma(E)$ at two different points (say p and q). The reason for this is simple: $\pi^{-1}(p)$ and $\pi^{-1}(q)$ are two *different* vector spaces, and there is no canonical isomorphism between them.

It is clear that, were we given some way to unambiguously lift a curve γ on M (connecting $p = \pi(v)$ and q) to a curve on E starting at v (connecting $v \in \pi^{-1}(p)$ to $C(v, \gamma) \in \pi^{-1}(q)$), we could obtain a (curve-dependent) identification of the fibers $\pi^{-1}(p)$ and $\pi^{-1}(q)$ (in particular, we would want this implicit map $C(\,\cdot\,,\,\cdot\,)$ to be linear and injective in the first "slot" and smooth in the second). Such a prescription for a curve-dependent identification of fibers is called a **connection**.

It is also clear that being provided with a connection should give us an unambiguous was to take directional derivatives of sections along tangent vectors in M: we simply lift the curve along which we wish to take this derivative (i.e. whose tangent at parameter $t = 0$ is the tangent vector $\dot\gamma$ in question), and consider the limit

$$\nabla_{\dot\gamma} s = \lim_{t \to 0} \frac{s(\gamma(t)) - C\left(s(\gamma(t)), \gamma^{-1}(t)\right)}{t}$$

Similarly, any prescription for "covariantly" (i.e. independent of choice of coordinates or trivialization) taking derivatives defines a connection, since we may identify a vector $s(p)$ with its "parallel transports" along γ, $s(q)$ defined by

$$\nabla_{\dot\gamma} s = 0$$

Thus we will sometimes abuse terminology and call a covariant derivative operator a connection and vice-versa.

Any connection on a principal fiber bundle can be described by a lie algebra valued one form ω, which descends to a one form on each of its associated bundles. Thus all connections on our vector bundles can be specified by an $\mathrm{End}(E)$ valued one form ω, where E is the total space.

For our purposes, let $\zeta = E \xrightarrow{\pi} M$ be a smooth, complex n-plane bundle with smooth (m dimensional) base space M. We denote by $(T^*M)_{\mathbb{C}} = \mathrm{Hom}_{\mathbb{R}}(TM, \mathbb{C})$ the complexification of the cotangent bundle (i.e. each fiber of $(T^*M)_{\mathbb{C}}$ is the complexification $T_p^*M \otimes_{\mathbb{R}} \mathbb{C}$ of the corresponding cotangent space). It is clear that $(T^*M)_{\mathbb{C}} \otimes_{\mathbb{C}} \zeta$ will be a smooth, complex $m + n$ plane bundle over M.

Definition B.1. A **connection** ∇ on ζ is a $\mathbb{C}$-linear mapping

$$\nabla : \Gamma(\zeta) \to \Gamma((T^*M)^{\mathbb{C}} \otimes \zeta)$$

which satisfies the Leibniz rule:

$$\nabla(fs) = df \otimes s + f\nabla s$$

for all $s \in \Gamma(\zeta)$, $f \in C^\infty(M, \mathbb{C})$. The image ∇s of a section s is called the **covariant derivative** of s.

Proposition B.2. *The mapping $s \mapsto \nabla s$* ***decreases supports*** *(that is ∇s vanishes on any open region in which s vanishes).*

Any local mapping $L : \Gamma(\zeta) \to \Gamma(\eta)$ which decreases supports is called a **local operator** since the value of $L(s)$ at a point p only depends on the values of s in an arbitrarily small neighborhood of p. A theorem of Peetre assert that every local operator is a differential operator, i.e. that it can be expressed locally as a finite linear combination of partial derivatives with coefficients in $\Gamma(\eta)$. This is where the one form ω comes in.

In any local trivialization, we may find a basis of sections $\{s_i\}$. We may always write

$$\nabla(s_i) = \sum \omega_{ij} \otimes s_j$$

for some unique matrix of one forms ω_{ij}. This matrix ω_{ij} is just the representative of the one form ω in this basis.

We have the following results about sections on ζ.

Proposition B.3. *Let ∇_1 and ∇_2 be two connections on ζ and g any smooth complex valued function on M. Then $g\nabla_1 + (1-g)\nabla_2$ is another connection on ζ. In particular, the space of all connections on ζ is contractible.*

Proposition B.4. *Any smooth complex vector bundle with paracompact base space posesses a connection.*

Proposition B.5. *Let $g : M' \to M$ be a smooth map and ζ a smooth complex vector bundle over M with connection ∇. Then there is one and only one connection $\nabla' = g^*\nabla$ on $\zeta' = g^*\zeta$ such that the following diagram is commutative*

$$\begin{array}{ccc} \Gamma(\zeta) & \xrightarrow{\nabla} & \Gamma((T^*M)^{\mathbb{C}} \otimes \zeta) \\ \downarrow & & \downarrow \\ \Gamma(\zeta') & \xrightarrow{\nabla'} & \Gamma((T^*M')^{\mathbb{C}} \otimes \zeta') \end{array}$$

We would like as well a notion of "second derivative" for sections. The motivation comes from the case where ζ is the tensor bundle formed from the tangent bundle of ζ. Here, we see that ∇ gives ζ a "shape" (i.e. determines the "straight lines" in ζ as those sections which satisfy $\nabla_{\dot\gamma}\dot\gamma = 0$). The connection uniquely extends to the tensor bundle and thus we get a notion of second derivative. The failure of covariant derivatives to commute (i.e. its lack of symmetry of this second derivative with respect its two tangent vector inputs) tells us about the curvature of this "shape". We thus generalize this notion.

Proposition B.6. *Given ∇, there is one and only one $\mathbb{C}$-linear mapping*

$$\hat\nabla : \Gamma((T^*M)^{\mathbb{C}} \otimes \zeta) \to \Gamma(\Lambda^2 (T^*M)^{\mathbb{C}} \otimes \zeta)$$

which satisfies the Leibniz formula

$$\hat\nabla(\theta \otimes s) = d\theta \otimes s - \theta \wedge \nabla s$$

We consider the mapping

$$\Omega = \hat{\nabla} \circ \nabla : \Gamma(\zeta) \to \Gamma(\Lambda^2 \left(T^*M\right)^{\mathbb{C}} \otimes \zeta)$$

We can show that this is a local operator, and thus defines a smooth section of the complex vector bundle Hom $(\zeta, \Lambda^2 \left(T^*M\right)^{\mathbb{C}} \otimes \zeta) \simeq \Lambda^2 \left(T^*M\right)^{\mathbb{C}} \otimes \text{Hom}\ (\zeta, \zeta)$. This section is called the **curvature tensor** of ∇.

Proposition B.7. *In any local basis of sections, we have that*

$$\Omega(s_i) = \sum \Omega_{ij} \otimes s_j$$

where

$$\Omega_{ij} = d\omega_{ij} - \sum \omega_{ik} \wedge \omega_{kj}$$

i.e. we have

$$\Omega = d\omega - \omega \wedge \omega$$

where ω is the connection one form.

Appendix C. Complex Grassmann Manifolds

In this appendix we recall some basic properties of the canonical n-plane bundles γ^n over the complex Grassmann manifolds G_n. These bundles are particularly useful in proving general statements about characteristic classes due to their "universality" as discussed below.

Definition C.1. We define the **complex Grasmann manifold** $G_n(\mathbb{C}^{n+k})$ to be the space of all complex n-planes through the origin in $\mathbb{C}^{n+k}$.

We note that $G_n(\mathbb{C}^{n+k})$ has a natural complex nk-manifold structure.

Definition C.2. The **canonical complex n-plane bundle** $\gamma^n(\mathbb{C}^{n+k})$ is the bundle over $G_n(\mathbb{C}^{n+k})$ whose fiber over each point is a vector in the plane that point represents. In other words, the total space E consists of all pairs (X, v) where X is a complex n-plane through the origin of $\mathbb{C}^{n+k}$ and v a vector on X. We denote by γ^n the canonical n-plane bundle $\gamma^n(\mathbb{C}^\infty)$.

The utility of Grassmann manifolds follows from the following two theorems

Theorem C.3. *Every complex n-plane bundle ζ over a paracompact base space possesses a bundle map (unique up to homotopy) f into the canonical complex n-plane bundle γ^n over $G_n \coloneqq G_n(\mathbb{C}^\infty)$. Furthermore, we have that $\zeta \simeq f^*\gamma^n$.*

Theorem C.4. *Two induced bundles $f^*(\gamma^n)$ and $g^*(\gamma^n)$ are isomorphic if and only if f is homotopic to g.*

Thus every complex n-plane bundle ζ (since we assume paracompact base space B from the outset) is isomorphic to the induced bundle $f^*(\gamma^n)$ for some (suitably unique) map $f : B \to G_n$. Furthermore, we can use these maps to check whether bundles are isomorphic by merely looking to their homotopy equivalence (or lack thereof). For these reasons, we call the bundle γ^n the **universal complex n-plane bundle** and G_n the **classifying space** for complex n-plane bundles since the map $f : B \to G_n$ completely classifies ζ. Thus we may always work indirectly with γ^n, pulling our results back to the specific bundle ζ only when necessary.

We thus will find it fruitful to acquaint ourselves with the cohomology of the Grassmann manifolds. It is a remarkable fact that the cohomology of the nth Grassmann manifold is completely determined by its Chern classes. This turns out to be related to the fact that *all* characteristic classes of a complex n plane bundle are polynomials in its Chern classes. Further discussion of this fact is beyond the scope of this paper.

Theorem C.5. *The cohomology ring of the Grassmann manifold $G_n(\mathbb{C}^\infty)$ is the polynomial ring over $\mathbb{Z}$ generated by the Chern classes of γ^n, i.e.*

$$H^*\left(G_n(\mathbb{C}^\infty); \mathbb{Z}\right) = \mathbb{Z}[c_1(\gamma^n), \ldots, c_n(\gamma^n)]$$

with no polynomial relations between the generators.

Thus a knowledge of the Chern classes of the canonical n-plane bundles should serve to illuminate much about the Chern classes in general.

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References

[1] Lopez, Mauricio E. "Characteristic Classes. University of Copenhagen Department of Mathematical Sciences .

[2] Milnor, John, and James Stasheff. *Characteristic Classes.* Princeton Univ. Press, 1974.