Linear G-structures by examples

In differential geometry, the "central objects" are manifolds and the relevant maps are "the smooth maps". Smoothness is actually the key word here- that is precisely what we can talk about when dealing with manifolds. And nothing more; e.g. we cannot talk about "length","volume", "slope", "rational slope", etc etc on a general manifolds. But these are concepts of geometric nature, so one should be able to talk about/study them in differential geometry. That can be done, but not on a bare manifold. What one needs is a manifold endowed with an extra-structure. The extra-structure is dictated by the concept one would like to make sense of/study. For instance: to talk about length (of tangent vectors, and then of curves) on a manifold M, then one needs the extra-structure of M called "Riemannian metric". Hence:

- one has to consider geometric structures on manifolds.
- the geometric structure one considers depends on the concept/aspect one would like to make sense of and study.

But what is the meaning of "geometric structure" on a manifold M? Intuitively it is rather clear; moreover, in each example of a geometric structure, one can handle it directly, following the intuition. The theory of G-structures gives a general framework that allows us to treat (the basic properties of) geometric structures in a conceptual, unified manner.

There are several intuitive ideas behind. To get a first feeling of what is going on, let us mention here some rough principles.

- G denotes here a group (a subgroup of the group $GL_n(\mathbb{R})$ of all invertible matrices with real entries); it plays the role of "the symmetry group" of the geometric structure one models. Concentrating on G corresponds to an old idea (originating in Klein's Erlangen program) of studying geometric objects via their symmetry group.
- the notion of G-structure makes use of the "infinitesimal philosophy" of differential geometry. The key concept here is that of "tangent space T_xM" of a manifold M at an arbitrary point $x \in M$, together with the intuition that T_xM is the "infinitesimal (or linear) approximation of M around x".
- hence we first have to make sense of the notion of "linear G-structure on a vector space V" (to be applied to $V = T_xM$ with x varying inside M). The key remark here is that, given a geometric structure on a vector space V, one can then talk about bases of V (frames) "which are compatible with the structure". For instance, for a metric (inner product) on V, one looks at orthonormal bases. The key idea then is to encode/define geometric

structures via the associated "set of special frames" (hence a G-structure will be precisely the choice of "special frames", satisfying the appropriate axioms).

- according to the "infinitesimal philosophy", a global G(eometric) structure on M is to be defined as a collection of "linear G(eometric) structures", one on each T_xM, varying smoothly with respect to x.

Accordingly, we will first discuss linear G-structures (on vector spaces), and then apply them to tangent spaces of manifolds.

1 Part 1: Geometric structures on vector spaces

In this section, V is assumed to be a *general* n-dimensional real vector space.

1.1 Frames, matrices

Let us start by fixing some terminology and notations.

Matrices: Here we restrict to invertible matrices. Recall the interpretation(s) of matrices

$$A = (A_{i,j}) \in GL_n(\mathbb{R})$$

as linear isomorphisms acting on $\mathbb{R}^n$. Since this can be done in two ways, it is good to fix the notations. We have:

- first of all, one can think as vectors as row matrices (without making a distinction in the notation); with this we obtain an action of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$ from the right:

$$\mathbb{R}^n \times GL_n(\mathbb{R}) \to \mathbb{R}^n, \quad (v, A) \mapsto v \cdot A.$$

In coordinates,

$$\begin{pmatrix} (v \cdot A)_1 & \ldots & (v \cdot A)_n \end{pmatrix} = \begin{pmatrix} v_1 & \ldots & v_n \end{pmatrix} \begin{pmatrix} A_{1,1} & \ldots & A_{1,n} \\ \ldots & \ldots & \ldots \\ A_{n,1} & \ldots & A_{n,n} \end{pmatrix}$$

In this way any $A \in GL_n(\mathbb{R})$ may be interpreted as a liner map

$$\hat{A} : \mathbb{R}^n \to \mathbb{R}^n, \quad \hat{A}(v) = v \cdot A.$$

This is not the "standard" way of interpreting matrices as linear maps (for that reason we use the notation $\hat{A}$ instead of simply A) but, in many cases, it is more "conceptual" (or "natural").

- one can also represent vectors by column matrices; for $v \in \mathbb{R}^n$ we denote by ${}^t v$ the resulting column matrix; with this we obtain an action of $GL_n(\mathbb{R})$ on $\mathbb{R}^n$ from the left:

$$GL_n(\mathbb{R}) \times \mathbb{R}^n \to \mathbb{R}^n, \quad (A, v) \mapsto A \cdot v$$

determined by the matrix condition

$${}^t(A \cdot v) = A\, {}^t(v).$$

In coordinates,

$$\begin{pmatrix} (A\cdot v)_1 \\ \dots \\ (A\cdot v)_n \end{pmatrix} = \begin{pmatrix} A_{1,1} & \dots & A_{1,n} \\ \dots & \dots & \dots \\ A_{n,1} & \dots & A_{n,n} \end{pmatrix} \begin{pmatrix} v_1 \\ \dots \\ v_n \end{pmatrix}$$

In this way any $A \in GL_n(\mathbb{R})$ may be interpreted as a liner map

$$A : \mathbb{R}^n \to \mathbb{R}^n, \quad A(v) = A \cdot v.$$

This is the standard way of viewing a matrix as a linear map. The relation with $\hat{A}$ is simply $\hat{A} =^t A$ (as linear maps).

Frames: First of all, by frame of V we mean any ordered basis of V

$$\phi = (\phi_1, \dots, \phi_n) \quad (\text{hence } \phi_i \in V).$$

We denote by $\mathrm{Fr}(V)$ the set of all frames of V. Any such frame determines a linear isomorphism

$$\hat{\phi} : \mathbb{R}^n \to V$$

by sending the element e_i of the standard basis of $\mathbb{R}^n$ to $\phi_i \in V$. Conversely, any isomorphism arises in this way, and we will actually identify (at least in our mind) ϕ with $\hat{\phi}$. In other words, we think of a frame on V as an isomorphism of V and the "standard model" $\mathbb{R}^n$.

Action of matrices on frames: There is a canonical right action of $GL_n(\mathbb{R})$ on $\mathrm{Fr}(V)$, i.e. an operation

$$\mathrm{Fr}(V) \times GL_n(\mathbb{R}) \to \mathrm{Fr}(V), \quad (\phi, A) \mapsto \phi \cdot A.$$

It is given by the explicit formula:

$$(\phi \cdot A)_i = \sum_j A_{i,j} \phi_j,$$

but it is better to think in terms of the linear maps $\hat{\phi}$ and $\hat{A}$ associated to ϕ and A- and then the operation $\phi \cdot A$ becomes simply composition of maps:

Exercise 1.1. *Show that* $\widehat{\phi \cdot A} = \hat{\phi} \circ \hat{A}$.

"Division" of frames: Another simple remark is that, for any two frames ϕ and ϕ' of V, one can form the linear isomorphism

$$(\hat{\phi'})^{-1} \circ \hat{\phi} : \mathbb{R}^n \to \mathbb{R}^n$$

hence we obtain a matrix, denoted

$$[\phi : \phi'] \in GL_n(\mathbb{R}), \tag{1}$$

uniquely determined by

$$\phi = \phi' \cdot [\phi : \phi'].$$

This is precisely the matrix of coordinate changes from ϕ to ϕ' (given a frame ϕ on V, vectors $v \in V$ are determined by its coordinates with respect to the frame-which are organized in a line matrix denoted $[v]_\phi$; choosing another frame ϕ', the resulting $[v]_{\phi'}$ can then be computed from $[v]_\phi$ by $[v]_{\phi'} = [v]_\phi \cdot [\phi : \phi'])$.

Geometric structures: steps to follow. We now move to geometric structures. Here we start by looking at examples in which the situations is pretty clear. However, to understand the general framework, it is important that in each such example, after we specify the structure:

S1: we specify the corresponding notion of isomorphism.

S2: we specify the standard model (on $\mathbb{R}^n$).

S3: we specify the symmetry group of the standard model.

S4: we specify the "special frames" on V associated to the structure (read: the frames of V that are compatible with the geometric structure; a bit more precise, one may think of the "special frames" as those frames ϕ with the property that the associated isomorphism $\hat{\phi}$ is an isomorphism between the standard model and V preserving the structure).

The very last point is the key point for the notion of G-structure. Indeed, in each example we will look at, the "special frames" will be clear. Passing to the general theory, specifying a G-structure will simply mean specifying which frames are "special". Ok, this gets too vague, so let's look at examples, where everything is pretty clear (however, I do advise you that, after you look at some examples, you do return to the discussion above, and see how it all makes more sense).

1.2 Metrics

Recall that a linear metric (inner product) on the vector space V is a symmetric bilinear map

$$g : V \times V \to \mathbb{R}$$

with the property that $g(v, v) \geq 0$ for all $v \in V$, with equality only for $v = 0$. This is the first geometric structure we look at. For this:

<u>S1</u>: Given two vector spaces endowed with inner products, (V, g), (V', g'), it is clear what an isomorphism between them should be: any linear isomorphism $A : V \to V'$ with the property that

$$g'(A(u), A(v)) = g(u, v) \quad \forall\ u, v\ \in V.$$

One also says that A is an isometry between (V, g) and (V', g').

<u>S2</u>: The standard model is $\mathbb{R}^n$ with the standard inner product g_{can}:

$$g_{\text{can}}(u, v) = \langle u, v \rangle = \sum_i u_i v_i,$$

or, in the matrix notation (see the previous subsection):

$$g_{\text{can}}(u, v) = u^{\,t} v. \tag{2}$$

Exercise 1.2. *Show that, indeed,any vector space endowed with an inner product, (V, g), is isomorphic to $(\mathbb{R}^n, g_{can})$.*

<u>**S3**</u>**:** The symmetry group of the standard model becomes

$$O(n) = \{A \in GL_n(\mathbb{R}) : g_{\text{can}}(A(u), A(v)) = g_{\text{can}}(u, v) \text{ for all } u, v \in \mathbb{R}^n\} =$$
$$= \{A \in GL_n(\mathbb{R}) : A^t A = I\} \subset GL_n(\mathbb{R}).$$

<u>**S4**</u>**:** We find that "the special frames" of (V, g) are those frames $\phi \in \text{Fr}(V)$ with the property that the induced linear isomorphism $\hat{\phi} : \mathbb{R}^n \to V$ is an isomorphism between $(\mathbb{R}^n, g_{\text{can}})$ and (V, g).

Exercise 1.3. *Check that this is the case if and only of the frame $\phi = (\phi_1, \ldots, \phi_n)$ is orthonormal with respect to g, i.e.*

$$g(\phi_i, \phi_j) = \delta_{i,j} \quad \forall\ i, j. \tag{3}$$

Hence one ends up with the set of orthonormal frames of (V, g), denoted

$$\mathcal{S}_g \subset \text{Fr}(V).$$

The following Proposition makes precise the fact that (and explains how) the "inner product g" is encoded by the associated set of frames $\mathcal{S}_g$.

Proposition 1.4. *Given the vector space V, show that*

1. *if g and g' are inner products on V such that $\mathcal{S}_g = \mathcal{S}_{g'}$, then $g = g'$.*
2. *a subset $\mathcal{S} \subset Fr(V)$ is of type $\mathcal{S}_g$ for some inner product g if and only if:*

   A1: $\mathcal{S}$ is $O(n)$-invariant, i.e.: $\phi \in \mathcal{S},\ A \in O(n) \Longrightarrow \phi \cdot A \in \mathcal{S}$.
   A2: if $\phi, \phi' \in \mathcal{S}$ then $[\phi : \phi'] \in O(n)$.

Proof. The key (but obvious) remark is that, if we know an orthonormal frame ϕ for a metric g, then we know g. This clearly implies 1. Also the reverse implication in 2, i.e. starting with $\mathcal{S}$ satisfying the two axioms, one just picks up an arbitrary $\phi \in \mathcal{S}$ and then considers the associated g (given by (3). The axioms imply that g does not depend on the choice of ϕ in $\mathcal{S}$ (check this!) and that $\mathcal{S} = \mathcal{S}_g$. The direct implication is a simple check. □

Corollary 1.5. *Given a vector space V, there is a 1-1 correspondence between:*

1. *inner products g on V.*
2. *subsets $\mathcal{S} \subset Fr(V)$ satisfying (A1) and (A2) above.*

This corollary indicates the way we will proceed in general, when discussing G-structures for an arbitrary $G \subset GL_n(\mathbb{R})$. There is a slightly different way of encoding inner products (which can also be used in order to deal with arbitrary G-structures), but which is less intuitive; however, we mention it here:

Exercise 1.6. *Given a vector space V, show that there is a 1-1 correspondence between:*

1. *inner products g on V.*
2. *elements of $Fr(V)/O(n)$ (quotient modulo the right action of $O(n)$).*

The correspondence associates to an inner product g the equivalence class $[\phi] \in Fr(V)/O(n)$ of a(ny) frame ϕ orthonormal with respect to g.

1.3 Orientations

The next example of "structure" that we consider is that of "orientation". This is very well suited to our discussion because the notion of orientation is itself defined (right from the start) using frames. More precisely, given our n-dimensional vector space V, one says that two frames $\phi, \phi' \in \mathrm{Fr}(V)$ induce the same orientation if the matrix $[\phi : \phi']$ of change of coordinates has positive determinant; this defines an equivalence relation $\sim$ on $\mathrm{Fr}(V)$ and an orientation of V is the choice of an equivalence class:

$$\mathcal{O} \in \mathrm{Orient}(V) := \mathrm{Fr}(V)/\sim .$$

Exercise 1.7. ***S1:*** *Given two oriented vector spaces $(V, \mathcal{O})$, $(V', \mathcal{O}')$, what are the isomorphisms between them?*

S2, S3: The local model is $\mathbb{R}^n$ with the orientation $\mathcal{O}_{\mathrm{can}}$ induced by the standard basis (frame) and computing the induced symmetry group (isomorphisms from the local model to itself) we find the subgroup of GL_n of matrices with positive determinant:

$$GL_n^+ = \{A \in GL_n : \det(A) > 0\}.$$

S4: The special frames of $(V, \mathcal{O})$ are, of course, the frames which induce the given orientation; they correspond to oriented isomorphism $(\mathbb{R}^n, \mathcal{O}_{\mathrm{can}}) \to (V, \mathcal{O})$. Denote the set of such frames by

$$\mathcal{S}_{\mathcal{O}} \subset \mathrm{Fr}(V).$$

Exercise 1.8. *State and prove the analogue of Proposition 1.4 in this context.*

Note also that the analogue of Exercise 1.6 in this context, i.e. a 1-1 correspondence between orientations on V and elements of $\mathrm{Fr}(V)/GL_n^+$, holds by the very definition of orientations, since the equivalence relation $\sim$ discussed above is precisely the one induced by the action of GL_n^+ on $\mathrm{Fr}(V)$.

1.4 Volume elements

Recall that a volume element on V is a non-zero element $\mu \in \Lambda^n V^*$ (where n is the dimension of V) or, equivalently, a non-zero skew-symmetric multilinear map

$$\mu : \underbrace{V \times \ldots \times V}_{n\,\text{times}} \to \mathbb{R}.$$

S1: An isomorphism between (V, μ) and $V', \mu')$ is any linear isomorphism $A : V \to V'$ with the property that

$$\mu'(A(v_1), \ldots, A(v_n)) = \mu(v_1, \ldots, v_n)$$

for all v_i's. Equivalently, a linear map A induces $A^* : V'^* \to V^*$ and then $A^* : \Lambda^n(V')^* \to \Lambda^n V^*$, and we are talking about the condition $A^*\mu' = \mu$.

S2: The standard model is, again, $\mathbb{R}^n$ with the canonical volume element given μ_{can} given by (or uniquely determined by):

$$\mu_{\mathrm{can}}(e_1, \ldots, e_n) = 1.$$

<u>S3</u>: The associated symmetry group becomes (after the computation)

$$SL_n(\mathbb{R}) = \{A \in GL_n(\mathbb{R}) : \det(A) = 1\}.$$

<u>S4</u>: Given (V, μ), its special frames will be those $\phi \in \mathrm{Fr}(V)$ satisfying:

$$\mu(\phi_1, \ldots, \phi_n) = 1.$$

Exercise 1.9. *Again, state + prove the analogue of Prop. 1.4 in this context.*

Remark 1.10. Note that a volume element on V determines, in particular, an orientation $\mathcal{O}_\mu$ on V: the one induced by frames ϕ with the property that $\mu(\phi) > 0$. This is basically due to the inclusion $SL_n(\mathbb{R}) \subset GL_n^+$ (try to make this more precise and to generalize it).

Geometrically, a volume element allows us to talk about the oriented volume of any "body" (simplex) induced by n-vectors $v_1, \ldots, v_n$.

Remark 1.11. It is useful to remember a few things about tensor calculus. $\Lambda^p V^*$ are defined as above for any $p \geq 0$, with the convention $\Lambda^0 V^* = \mathbb{R}$. All these spaces are finite dimensional (see also below), and they vanish for $p > n$. Hence $\Lambda^n V^*$ is the last one of them which does not vanish (for that reason it is also called the top exterior power of V); it is actually one-dimensional. These spaces interact with each other via the wedge products, i.e. the operations

$$\Lambda^p V^* \times \Lambda^q V^* \to \Lambda^{p+q} V^*, \quad (\omega, \eta) \mapsto \omega \wedge \eta$$

$$(\omega \wedge \eta)(v_1, \ldots, v_{p+q}) = \sum_{\sigma} \mathrm{sign}(\sigma) \omega(v_{\sigma(1)}, \ldots, v_{\sigma(p)}) \eta(v_{\sigma(p+1)}, \ldots, v_{\sigma(p+q)})$$

where the sum is over all (p, q)-shuffles, i.e. all permutations σ with

$$\sigma(1) < \ldots < \sigma(p), \quad \sigma(p+1) < \ldots < \sigma(p+q).$$

The basic properties of the wedge operation are: it is bilinear, associative:

$$(\omega \wedge \eta) \wedge \theta = \omega \wedge (\eta \wedge \theta) \quad \text{for all } \omega, \eta, \theta$$

and grade antisymmetric:

$$\eta \wedge \omega = (-1)^{pq} \omega \wedge \eta \quad \text{for all } \omega \in \Lambda^p V^*, \eta \in \Lambda^q V^*.$$

Note that, in particular (but also immediate from the definition),

$$\theta \wedge \theta = 0 \quad \text{for all } \theta \in \Lambda^1 V^*.$$

Using the wedge operation, one can express all elements in the exterior powers using (sums of wedges of) elements of $\Lambda^1 V^* = V^*$. Even better, fixing a frame $\phi = (\phi_1, \ldots, \phi_n)$ of V, one has an associated (dual) frame

$$\phi^* = (\phi^1, \ldots, \phi^n)$$

of V^* (determined by $\phi^i(\phi_j) = \delta_{i,j}$) and then

$$\{\phi_{i_1} \wedge \ldots \wedge \phi_{i_p} : 1 \leq i_1 < \ldots < i_p \leq n\}$$

forms a basis of $\Lambda^p V^*$. In particular, $\Lambda^n V^*$ is one dimensional. For $V = \mathbb{R}^n$, the canonical volume element is simply the element induced by the standard basis:

$$\mu_{\mathrm{can}} = e^1 \wedge \ldots \wedge e^n.$$

1.5 p-directions

Another interesting "structure" one can have on a vector space V (and becomes more interesting when passing to manifolds) is that of "p-directions" in V, where p is any positive integer less or equal to the dimension n of V. By that we simply mean a p-dimensional vector subspace $W \subset V$. The S1-S4 steps are quite clear: Given (V, W) and (V', W'), and isomorphism between them is any linear isomorphism $A : V \to V'$ satisfying $A(W) \subset W'$. The local model is $(\mathbb{R}^n, \mathbb{R}^p)$, where we view $\mathbb{R}^p$ sitting inside $\mathbb{R}^n$ via the inclusion on the first components

$$\mathbb{R}^p \hookrightarrow \mathbb{R}^n = \mathbb{R}^p \times \mathbb{R}^{n-p}, \quad u \mapsto (u, 0).$$

For the symmetry group we find

$$GL(p, n-p) = \{ \begin{pmatrix} A & 0 \\ B & C \end{pmatrix} \in GL_n(\mathbb{R}) : A \in GL_p(\mathbb{R}), C \in GL_{n-p}(\mathbb{R}) \}.$$

(B is an $(n-p) \times p$ matrix). The special frames of (V, W) will be those of type

$$\phi = (\phi_1, \ldots, \phi_n) \ \text{ with } \phi_1, \ldots, \phi_p \in W$$

and they define

$$\mathcal{S}_W \subset \mathrm{Fr}(V).$$

1.6 Integral affine structures

By an (linear) integral affine structure on a vector space V we mean a lattice

$$\Lambda \subset V.$$

That means that Λ is a discrete subgroup of $(V, +)$. Equivalently, Λ is of type

$$\Lambda = \mathbb{Z}\phi_1 + \ldots + \mathbb{Z}\phi_n$$

for some frame $\phi = (\phi_1, \ldots, \phi_n)$. Frames of this type are called integral frames of (V, Λ) and they play the role of "special frames". The local model is $(\mathbb{R}^n, \mathbb{Z}^n)$, while the symmetry group becomes

$$GL_n(\mathbb{Z}) \subset GL_n(\mathbb{R}).$$

Note that an integral affine structure Λ on an oriented vector space $(V, \mathcal{O})$ induces a volume element μ_Λ on V

$$\mu_\Lambda = \phi^1 \wedge \ldots \phi^n$$

where ϕ is a(ny) positively oriented integral basis (and we use the dual basis in the last equation). This comes from the fact that any matrix $A \in GL_n(\mathbb{Z})$ has determinant ± 1.

1.7 Complex structures

A (linear) complex structure on a vector space V is a linear map

$$J : V \to V$$

satisfying

$$J^2 = -\mathrm{Id}.$$

Note that such structures can exist only on even dimensional vector spaces (see the next remark and exercise). An isomorphism between (V, J) and (V', J') is any linear isomorphism $A : V \to V'$ satisfying

$$J' \circ A = A \circ J.$$

Remark 1.12. Such a complex structure allows us to promote the (real) vector space V to a complex vector space by defining

$$(r + is) \cdot v := rv + sJ(v).$$

When we view the vector space V as a complex vector space in this way, we will denote it by V_J. Note that the notion of isomorphism corresponds to the fact that A is an isomorphism between the complex vector spaces V_J and $V'_{J'}$.

Exercise 1.13. *Show that, given (V, J), one can find a frame of V of type*

$$\phi = (\phi_1, \ldots, \phi_k, J(\phi_1), \ldots, J(\phi_k)).$$

($k = \frac{1}{2} dim(V)$).

The frames of this type, also called "complex frames" (why?), will be the special frames of (V, J); they define

$$\mathcal{S}_J \subset \mathrm{Fr}(V).$$

The local model is $\mathbb{R}^{2k} = \mathbb{R}^k \oplus \mathbb{R}^k$ with the complex structure

$$J_{\mathrm{can}}(u, v) = (-v, u)$$

(think $i \cdot (u + iv) = -v + iu$). It is sometimes useful to use the matrix notations. Using the previous decomposition of $\mathbb{R}^{2k}$, we see that a linear automorphism of $\mathbb{R}^{2k}$ can be represented as a matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where such a matrix encodes the map

$$\mathbb{R}^{2k} \ni (u, v) \mapsto (Au + Bv, Cu + Dv) \in \mathbb{R}^{2k}.$$

With this, one has:

$$J_{\mathrm{can}} = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}. \tag{4}$$

The associated symmetry group consists of those $M \in GL_{2k}(\mathbb{R})$ satisfying $J_{\mathrm{can}} M = M J_{\mathrm{can}}$. In the matricial notation, working out this condition, we find out that we are looking at Ms of type

$$M = \begin{pmatrix} A & B \\ -B & A \end{pmatrix}.$$

We find that the symmetry group is a copy of $GL_k(\mathbb{C})$ embedded in $GL_{2k}(\mathbb{R})$ via

$$GL_k(\mathbb{C}) \ni A + iB \mapsto \begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in GL_{2k}(\mathbb{R}).$$

We hope that the rest of the story (and the analogue of Proposition 1.4) is clear.

Remark 1.14. A complex structure J on V induces an orientation $\mathcal{O}_J$ on V: one just considers the orientation induced by the complex frames. The fact that the orientation does not depend on the frame (which is the main point of this remark) comes from the fact that, via the previous inclusion of $GL_k(\mathbb{C})$, one ends up in GL_{2k}^+.

Exercise 1.15. *Prove the last statement. More precisely, denoting by incl the inclusion of $GL_k(\mathbb{C})$ into $GL_{2k}(\mathbb{R})$, show that*

$$det(incl(Z)) = |det(Z)|^2 \quad \forall\, Z \in GL_k(\mathbb{C}).$$

(Hint: there exists a matrix $X \in GL_k(\mathbb{R})$ such that

$$\begin{pmatrix} A & B \\ -B & A \end{pmatrix} = \begin{pmatrix} X & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} I & X^{-1} \\ -I & X \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix})$$

Remark 1.16 (another way of encoding complex structures)**.** There is another way of encoding complex structures: via (sub)spaces instead of maps. This will be useful when passing to manifolds. This is based on some yoga with complex vector spaces. Start with a real vector space V (no J yet). Then we introduce the complexification of V as the complex vector space

$$V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C} = \{u + iv : u, v \in V\}$$

with the obvious complex structure. Note that it also comes with an obvious conjugation map

$$u + iv \mapsto \overline{u + iv} := u - iv.$$

Now, if J is a complex structure on V, just view it as an $\mathbb{R}$-linear map on V and extend it to a $\mathbb{C}$-linear map on $V_{\mathbb{C}}$:

$$J(u + iv) = J(u) + iJ(v).$$

Using now that $J^2 = -\mathrm{Id}$ and that J acts now on a complex vector space (namely on $V_{\mathbb{C}}$), $V_{\mathbb{C}}$ can be split into the i and $-i$ eigenspaces of J:

$$V = V^{1,0} \oplus V^{0,1},$$

where the two summands are the complex vector spaces given by:

$$V^{1,0} = \{w \in V_{\mathbb{C}} : J(w) = iw\} = \ldots = \{u - iJ(u) : u \in V\},$$

$$V^{0,1} = \{w \in V_{\mathbb{C}} : J(w) = -iw\} = \ldots = \{u + iJ(u) : u \in V\}.$$

Of course, the two are related by conjugation:

$$V^{0,1} = \overline{V^{1,0}}.$$

The key remark here is that the complex structure J on V not only gives rise to, but it is actually determined by the resulting complex subspace of $V_{\mathbb{C}}$

$$V^{1,0} \subset V_{\mathbb{C}}.$$

The key properties of this subspace is that

$$V_{\mathbb{C}} = V^{1,0} \oplus \overline{V^{1,0}}.$$

and that the first projection $V^{1,0} \to V$ is an isomorphism. Note that these properties are, indeed, formulated independently of J and they do determine J.

Note also the connection between the complex vector spaces that arise from the two points of view. On one hand, J gave rise to the complex vector space V_J (V viewed as a complex vector space using J). On the other hand, we had the complex subspace $V^{1,0}$ of $V_{\mathbb{C}}$. Note that the two are isomorphic as complex vector spaces, by the obvious map

$$V_J \xrightarrow{\sim} V^{1,0}, \quad u \mapsto u - iJ(u).$$

Indeed, denoting by I this map, one has:

$$i \cdot I(u) = iu + J(u) = J(u) - iJ(J(u)) = I(J(u)).$$

Similarly, the obvious map $V_J \to V^{0,1}$ is a conjugate-linear isomorphism. One often identifies $V^{1,0}$ with V_J using I and then $V^{0,1}$ with the conjugate of V_J.

1.8 Linear symplectic forms

A linear symplectic form on a vector space V is a non-degenerate antisymmetric 2-forms

$$\omega : V \times V \to \mathbb{R}.$$

One can think of linear symplectic forms as some antisymmetric versions of inner products. Note however that, in contrast with inner products, they can only exist on even dimensional vector spaces (this will be proven below).

S1: An isomorphism between (V, ω) and (V', ω') is any linear isomorphism $A : V \to V'$ satisfying

$$\omega'(A(u), A(v)) = \omega(u, v) \quad \forall\ u, v \in V.$$

S2: The local model is not so obvious before one thinks a bit about symplectic structures. Postponing a bit the "thinking", let us just describe the resulting local model from several points of view. First of all, it is $(\mathbb{R}^{2k}, \omega_{\text{can}})$ with

$$\omega_{\text{can}}((x, y), (x', y')) = \langle x', y\rangle - \langle x, y'\rangle, \tag{5}$$

where:

- we use $\mathbb{R}^{2k} = \mathbb{R}^k \times \mathbb{R}^k$ to represent the elements of $\mathbb{R}^{2k}$ as pairs

$$(x, y) = (x_1, \ldots, x_k, y_1, \ldots, y_k).$$

- $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^k$.

More compactly, writing the standard frame of $\mathbb{R}^{2k}$ as

$$(e_1, \ldots, e_k, f_1, \ldots, f_k),$$

and using the associated dual frame and the wedge-products (Remark 1.11),

$$\omega_{\text{can}} = f^1 \wedge e^1 + \ldots + f^k \wedge e^k.$$

Even more compactly, one can use the canonical inner product g_{can} and the canonical complex structure J_{can} on $\mathbb{R}^{2k}$ (see the previous subsections) to write

$$\omega_{\text{can}}(u,v) = g_{\text{can}}(u, J_{\text{can}}(v)), \quad \forall\, u,v \in \mathbb{R}^{2k}.$$

This is a fundamental relation between metrics, complex structure and symplectic strutures that will be further discussed in the next section. One can go further and use the matrix expressions (2) and (4) and write:

$$\omega_{can}(u,v) = u\, J_{\text{can}}\, {}^{t}(v).$$

<u>**S3**</u>**:** To describe the symmetry group of $(\mathbb{R}^{2k}, \omega_{\text{can}})$ one can proceed in various ways, depending on which of the previous formulas for ω_{can} one uses. The most elegant way is to use the last formula; hence we are looking at those matrices $M \in GL_{2k}(\mathbb{R})$ for which

$$u\, J_{\text{can}}\, {}^{t}(v) = \omega_{\text{can}}(u,v) = \omega_{\text{can}}(M(u), M(v)) = {}^{t}(M\, {}^{t}(u))\, J_{\text{can}}\, M\, {}^{t}(v) = u\, {}^{t}(M) J_{\text{can}}\, M\, {}^{t}(v)$$

for all $u, v \in \mathbb{R}^{2k}$. Hence the symmetry group of interest is

$$Sp_k(\mathbb{R}) := \{M \in GL_{2k}(\mathbb{R}) :\, {}^{t}(M) J_{can} M = J_{can}\}$$

and is called the symplectic group. Note that, writing

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

the previous equations for M become

$${}^{t}(A)C = {}^{t}(C)A,\ {}^{t}(D)B = {}^{t}(B)D,\ {}^{t}(D)A - {}^{t}(B)C = I_k,$$

equations at which one can arrive directly using the formula (5).

<u>**S3**</u>**:** The "special frames" for a pair (V, ω) (called also the symplectic frames) are somehow similar to the orthonormal frames for metrics. However, unlike there, it is not so clear "right away" how to define the notion of "symplectic frame". Of course, since we have already described the local model, we could just say that $\phi \in \text{Fr}(V)$ is a symplectic frame for (V, ω) if the induced linear isomorphism $\hat{\phi}$ is an isomorphism between $(\mathbb{R}^{2k}, \omega_{\text{can}})$ and (V, ω). Explicitely, that means frames of type

$$(\phi_1, \ldots, \phi_k, \psi_1, \ldots, \psi_k) \quad \text{(in particular } n = 2k)$$

with the property that

$$\omega(\phi_i, \psi_i) = -1,\ \omega(\psi_i, \phi_i) = 1,\ \omega(\phi_i, \psi_j) = 0 \text{ for } i \neq j,\ \omega(\phi_i, \phi_j) = 0 = \omega(\psi_i, \psi_j).$$

Ok, we take this as the definition of symplectic frames, but one should keep in mind that there is still some cheating involved since, on the first place, the local model was not that obvious.

Let's now pass to a more careful analysis of symplectic vector spaces (V, ω) which should clarify the previous discussion. So, we fix our (V, ω). The key notion will be that of Lagrangian subspace. First of all, one calls a subspace $L \subset V$ isotropic if ω vanishes on it:

$$\omega(u, v) = 0 \quad \forall\, u, v \in L.$$

Those which are maximal (of highest possible dimension) among all isotropic subspaces are called Lagrangian.

Note that for any isotropic L one has a well-defined map

$$\Phi_L : L \to (V/L)^*, \quad \Phi_L(u)(v) = \omega(u, v)$$

(where one thinks of $(V/L)^*$ as consisting of linear maps from V to $\mathbb{R}$ that vanish on L).

Lemma 1.17. *An isotropic subspace $L \subset V$ is Lagrangian if and only if Φ_L is an isomorphism.*

Note that this implies that all Lagrangian subspaces have the same dimension, namely

$$\dim(L) = \frac{1}{2}\dim(V).$$

(in particular, the dimension of V is even).

Proof. Note that the map Φ_L is automatically injective, because ω is nondegenerate. In particular, the dimension of the domain is less than of the codomain, hence

$$\dim(L) \leq \frac{1}{2}\dim(V)$$

(... for all isotropic L). If Φ_L is an isomorphism then this becomes an equality hence L is clearly maximal among the isotropic subspaces. Assume now that L is Lagrangian. To check that Φ_L is also surjective, let $\xi \in (V/L)^*$, i.e. $\xi : V \to \mathbb{R}$ linear satisfying $\xi|_L = 0$. Using again the nondegeneracy of ω,

$$V \to V^*, u \mapsto \omega(u, \cdot) \tag{6}$$

is an isomorphism, hence we find $u_0 \in V$ such that $\xi(v) = \omega(u_0, v)$ for all $v \in V$. We have to show that $u_0 \in L$. Assume this is not the case. But then

$$L' := L + \mathbb{R}u_0$$

will be larger than L and still isotropic: indeed, since $\xi|_L = 0$, we also have $\omega(u_0, u) = 0$ for all $u \in L$. Hence the desired contradiction. $\square$

Proposition 1.18. *Given (V, ω) and $L \subset V$ Lagrangian:*

1. *L admits a Lagrangian complement, i.e. $L' \subset V$ Lagrangian with*

$$V = L \oplus L'.$$

2. *for any Lagrangian complement L', one has a linear isomorhism*

$$\Phi : L' \to L^*, \quad \Phi(l)(l') = \omega(l, l').$$

3. *The resulting linear isomorhism $V \cong L \oplus L^*$ is an isomorphism between (V, ω) and $(L \oplus L^*, \omega_L)$, where*

$$\omega_L((l, \xi), (l', \xi')) = \xi(l') - \xi'(l)).$$

(for $(l, \xi), (l', \xi') \in V_L$)

Note that the very last part tells us that (V, ω) is determined, up to isomorphism, by the bare vector space L (indeed, no extra-structure is needed to define ω_L). This provides "the local model". It is now easy to see that, for $L = \mathbb{R}^k$ one recovers, after the identifications

$$\mathbb{R}^k \oplus (\mathbb{R}^k)^* \cong \mathbb{R}^k \oplus \mathbb{R}^k \cong \mathbb{R}^{2k},$$

the local model $(\mathbb{R}^{2k}, \omega_{\text{can}})$.

Proof. Start with any complement C of L in V (Lagrangian or not), with corresponding projection $p_C : V \to C$. Then, for any $c \in C$, the map

$$v \mapsto \frac{1}{2}\omega(c, \text{pr}_C(v))$$

vanishes on L (because pr_C does) hence, using that Φ_L is an isomorphism, this map is of type $\omega(l_c, \cdot)$ for some $l_c \in L$. In particular,

$$\omega(l_c, c') = \frac{1}{2}\omega(c, c') \quad \forall\ c' \in C.$$

Replace now C by

$$L' := \{c - l_c : c \in C\}.$$

We claim that L' is the Lagrangian complement we were looking for. Indeed, using the last equation and the fact that $l_c \in L$, which is Lagrangian, we get for any two elements $c - l_c, c' - l_{c'} \in L'$

$$\begin{aligned} \omega(c - l_c, c' - l_{c'}) &= \omega(c, c') - \omega(l_c, c') + \omega(l_{c'}, c) + \omega(l_c, l_{c'}) \\ &= \omega(c, c') - \frac{1}{2}\omega(c, c') + \frac{1}{2}\omega(c', c) \\ &= 0 \end{aligned}$$

Moreover, $V = L \oplus C$ implies $V = L \oplus L'$. This proves the first part.

Assume now that L' is an arbitrary Lagrangian complement. Note that Φ is precisely $\Phi_{L'}$ after the identification $V/L' \cong L$. We deal with the resulting sequence of isomorhisms:

$$V = L \oplus L' \cong L \oplus (V/L')^* \cong L \oplus L^*.$$

For $v \in V$, denote by (l_v, ξ_v) the resulting element in $L \oplus L^*$. Going through the maps involved, we find the characterization:

$$\xi_v \circ \text{pr}_L(\cdot) = \omega(v - l_v, \cdot).$$

For $v, w \in L$, compute now

$$\omega(v, w) - \omega_L((l_v, \xi_v), (l_w, \xi_w))$$

(we have to prove it is zero). Applying the definition of ω_L and the fact that $\xi_v(l_w) = \omega(v, l_w)$ we find

$$\omega(v, w) - \omega(v - l_v, l_w) + \omega(w - l_w, l_v).$$

For the middle term, using $\omega(l_v, l_w) = 0$ (L is isotropic), we find

$$\omega(v, w) - \omega(v, l_w) + \omega(w - l_w, l_v)$$

hence we find

$$\omega(v, w - l_w) + \omega(w - l_w, l_v) = \omega(v - l_v, w - l_w)$$

where we have also used the antisymmetry of ω. Since $v - l_v \in L'$ for all v and L' is isotropic, the last expression is indeed zero. $\square$

Exercise 1.19. *As before, state and prove the analogue of Proposition 1.4 in this context. Try to do it coordinate/formula free (e.g. do not us the previous explicit matrix description of $Sp_k(\mathbb{R})$, but the direct definition- as linear isomorphism preserving ω_{can}.*

Remark 1.20 (from symplectic forms to volume elements)**.** Any (linear) symplectic form ω on V induces (canonical- i.e. without any further choices) a volume element

$$\mu_\omega := \omega^k = \underbrace{\omega \wedge \ldots \wedge \omega}_{k\text{times}} \in \Lambda^n V^*.$$

This is related to the fact that

$$Sp_k(\mathbb{R}) \subset SL_{2k}(\mathbb{R}).$$

1.9 Hermitian structures

Let's now briefly mention Hermitian structures (which can be thought of as analogues of metrics on complex vector spaces). First of all, given a complex vector space W (we reserve the letter V for real spaces), a Hermitian metric on W is an $\mathbb{R}$-bilinear map

$$h : W \times W \to \mathbb{C}$$

which is $\mathbb{C}$-linear in the first argument (and then, because of the next axiom, $\mathbb{C}$-antilinear in the second), satisfies the conjugates symmetry

$$h(w_2, w_1) = \overline{h(w_1, w_2)} \quad \forall\, w_1, w_2 \in W$$

ans $h(w, w) \geq 0$ for all $w \in W$ with equality only for $w = 0$.

Given a real vector space V, a hermitian structure on V is a pair (J, h) consisting of a complex structure J on V and a Hermitian metric on the resulting complex vector space V_J,

$$h : V \times V \to \mathbb{C}.$$

The main remark we want to make here is that such pairs can be unravelled and thought of in several equivalent ways. Here is the summary:

Lemma 1.21. *Given a real vector space V, there is a 1-1 correspondence between*

1. *Hermitian structures (h, J) on V.*

2. *pairs (J, g) consisting of a complex structure J and a metric g on V satisfying*
$$g(Ju, Jv) = g(u, v) \quad \forall\ u, v \in V,$$

3. *pairs (J, ω) consisting of a complex structure J and a symplectic structure ω on V, satisfying*
$$\omega(Ju, Jv) = \omega(u, v) \quad \forall\ u, v \in V,$$
and such that
$$\omega(u, Ju) > 0 \quad \forall\ u \in V \setminus \{0\}.$$

The defining relations between them are:

$$h(u, v) = g(u, v) - i\omega(u, v), \omega(u, v) = g(Ju, v), g(u, v) = -\omega(Ju, v).$$

Note that, using $J^2 = -\mathrm{Id}$, the equations in the statement can be written in the equivalent forms

$$g(Ju, v) = -g(u, Jv) \quad \forall\ u, v \in V,$$
$$\omega(u, Jv) = -\omega(Ju, v) \quad \forall\ u, v \in V.$$

Proof. The key remark is that, writing

$$h(u, v) = g(u, v) - i\omega(u, v)$$

(where, priory, g and ω are just $\mathbb{R}$-bilinear):

- the conjugated symmetry for h is equivalent to the fact that g is symmetric and ω is antisymmetric,
- the $\mathbb{C}$-linearity of h in the first argument of h is equivalent to
$$h(Ju, v) = ih(u, v)$$
and then to the two very last equations in the statement.
- the non-degeneracy of h is equivalent to that of g as a real bilinear form.

The rest is simple manipulations with these identities. □

Finally, there is some terminology that comes out of this lemma:

Definition 1.22. *Given a vector space V, a complex structure J on V and a symplectic structure ω on V, one says that J is ω-compatible if 3. of the previous proposition is satisfied.*

This terminology reflects the key idea for relating symplectic and complex structures- actually, on how to use complex structures in the study of symplectic ones. It is interesting to know that, for any symplectic ω, one can choose a ω-compatible J.

Exercise 1.23. *Let (V, ω) be a symplectic vector space and fix a Lagrangian sub-space $L \subset V$. Show that, fixing a metric g on V, there is a canonical way of constructing an ω-compatible complex structure J on V ("canonical" means that no extra-choices are necessary).*

(Hint: first look at the proof of Prop 1.18, where you use as complement of L the orthogonal with respect to g. Then identify V with $L \oplus L$ where you use again the metric. On the later use $J(u, v) = (v, -u)$).

1.10 General linear G-structures

It is now clear that all the previous examples fit in a general framework.

Definition 1.24. *Let G be a subgroup of $GL_n(\mathbb{R})$.*

A linear G-structure on an n-dimensional vector space V is a subset

$$\mathcal{S} \subset Fr(V)$$

satisfying the axioms:

A1: $\mathcal{S}$ is G-invariant, i.e.: $\phi \in \mathcal{S},\ A \in G \Longrightarrow \phi \cdot A \in \mathcal{S}$.

A2: if $\phi, \phi' \in \mathcal{S}$ then $[\phi : \phi'] \in G$ (for notation, see subsection 1.1).

Definition 1.25. *Let $\mathcal{S}$ and $\mathcal{S}'$ be G-structures on V and V', respectively. An isomorphism between them is a linear map isomorphism $A : V \to V'$ such that, for any frame ϕ of V, one has:*

$$A(\phi) \in \mathcal{S}' \Longleftrightarrow \phi \in \mathcal{S}.$$

(where, for $\phi = (\phi_1, \ldots, \phi_n)$, $A(\phi) := (A(\phi_1), \ldots, A(\phi_n))$).

Note that, for any $G \subset GL_n$, one has the "standard G-structure"

$$\mathcal{S}_G^{\text{can}},$$

on $\mathbb{R}^n$, consisting of those frames $\phi = (\phi_1, \ldots, \phi_n)$ of $\mathbb{R}^n$ which, when interpreted as a matrix (in which each row contains the components of ϕ_i with respect to the standard basis), belong to G.

Exercise 1.26. *For a linear G-structure $\mathcal{S}$ on V check that a frame $\phi \in Fr(V)$ belongs to $\mathcal{S}$ if and only if $\hat{\phi}$ is an isomorphism from $(\mathbb{R}^n, \mathcal{S}_G^{can})$ to $(V, \mathcal{S})$.*

Note that one can generalize also Exercise 1.6 to this context; actually, due to the generality of our discussion, the proof becomes almost tautological (so, better think about the proof yourself than reading it).

Proposition 1.27. *Given an n-dimensional vector space V and $G \subset GL_n(\mathbb{R})$, there is a 1-1 correspondence between*

- *G-structures on V.*
- *elements of $Fr(V)/G$.*

Proof. Given $\mathcal{S} \subset \mathrm{Fr}(V)$ a G-structure, axiom (A2) implies that any two elements $\phi, \phi' \in \mathcal{S}$ induces the same element in $\mathrm{Fr}(V)/G$; hence one obtains a well-defined element

$$u_{\mathcal{S}} \in \mathrm{Fr}(V)/G.$$

Conversely, given an element u, one just defines

$$\mathcal{S}_u := \{\phi \in \mathrm{Fr}(V) : u = \phi \text{ modulo } G\}.$$

□

Note in particular that, when applied to $V = \mathbb{R}^n$, due to the identification of $\mathrm{Fr}(\mathbb{R}^n)$ with $GL_n(\mathbb{R})$, we find that the set of G-structures on $\mathbb{R}^n$ is (in natural bijection with) $GL_n(\mathbb{R})/G$, the quotient modulo the action of G on $GL_n(\mathbb{R}_n)$ by right multiplication of matrices.

Example 1.28. The examples that we discussed show that choosing an $O(n)$-structure on V is the same thing as choosing an inner product on V, an GL_n^+-structure is an orientation, an $SL_n(\mathbb{R})$-structure is a volume form, etc etc.

Example 1.29. *[e-structures]* One case which looks pretty trivial in this linear discussion but which becomes important when we pass to manifolds is the case when G is the trivial group

$$G = \{I\} \subset GL_n(\mathbb{R}).$$

In this case one talks about e-structures ("e" refers to the fact that G is trivial- fact that is often written as $G = \{e\}$ with e denoting the unit). Note that the subset

$$\mathcal{S} \subset \mathrm{Fr}(V)$$

encoding an e-structure has one element only. Hence an e-structure on V is the same thing as the a frame ϕ of V.

Example 1.30 (G-structures associated to tensors)**.** At the other extreme, many of the previous examples fit into a general type of structure: associated to various tensors. More precisely:

- for inner products, we deal with elements $g \in S^2V^*$.
- for volume elements, we deal with $\mu \in \Lambda^n V^*$.
- for symplectic forms, we deal with $\omega \in \Lambda^2 V^*$.
- for complex structures, we deal with $J \in V^* \otimes V$.

The fact that these elements were of the type we were interested in (positive definite for g, non-degenerate for ω, non-zero for μ, satisfying $J^2 = -\mathrm{Id}$) came from the fact that the standard models had these properties (actually, were determined by them, up to isomorphisms). The above examples can be generalized. In principle we can start with any

$$t_0 := \text{any canonical "tensor" on } \mathbb{R}^n.$$

To such a tensor we can associate a group $G(t_0)$ of linear isomorphisms, which preserve t_0, i.e.

$$G(t_0) := \{A \in GL_n(\mathbb{R}^n) : A^* t_0 = t_0\}.$$

Then a $G(t_0)$-structure on a vector space V corresponds to a tensor t on V, of the same type as the original t_0, with the property that (V, t) is isomorphic to $(\mathbb{R}^n, t_0)$.

Example 1.31. As a general construction that puts together Remark 1.10 and 1.20 note that for subgroups

$$H \subset G \subset GL_n(\mathbb{R}),$$

any H-structure $\mathcal{S}$ on a vector space V induces a G-structure $\mathcal{S}G$ on V. Here:

$$\mathcal{S}G := \{\phi \cdot A : \phi \in \mathcal{S}, A \in G\}.$$

Equivalently, one can use the point of view given by Proposition 1.27 and use the canonical projections

$$\mathrm{Fr}(V)/H \to \mathrm{Fr}(V)/G.$$

Particular cases of this are:

1. Remark 1.10: a volume element on V induces an orientation on V; this comes from $SL_n(\mathbb{R}) \subset GL_n^+$.
2. Remark 1.20: a symplectic form induces a volume element; this comes from $Sp_k(\mathbb{R}) \subset SL_{2k}(\mathbb{R})$.
3. Remark Remark 1.14: a complex structure induces an orientation; this comes from $GL_k(\mathbb{C}) \subset GL_{2k}^+$.

2 G-structures on manifolds by examples

2.1 G-structures on manifolds

To discuss G-structures on manifolds it is rather natural to proceed like in the previous section and start with the examples. However, the exposition/terminology is simpler/clearer if we start with the general theory first. However, one should always have an eye on the examples.

So, let us fix a closed subgroup

$$G \subset GL_n(\mathbb{R})$$

and we will discuss G-structures on n-dimensional manifolds M. The idea is to consider G-structures on all the tangent space T_xM for $x \in M$, so that "they vary smoothly with respect to x". More precisely:

Definition 2.1. *A G-structure on an n-dimensional manifold M consists of a collection*

$$\mathcal{S} = \{\mathcal{S}_x : x \in M\}$$

of linear G-structures, $\mathcal{S}_x$ on T_xM (one for each $x \in M$), satisfying the following smoothness condition: for any $x_0 \in M$ and any

$$\phi = (\phi_1, \ldots, \phi_n) \in \mathcal{S}_{x_0},$$

there exist vector fields $X_1, \ldots, X_n$ defined on some open neighborhood U of x_0 such that

$$X_i(x_0) = \phi_i, \quad (X_1(x), \ldots, X_n(x)) \in \mathcal{S}_x \quad \forall \quad x \in U.$$

Remark 2.2 (the frame bundle)**.** As for the tangent (and cotangent) bundle, one can define the frame bundle $\mathrm{Fr}(M)$ of a manifold M by considering the disjoint union of all the frame bundles of the tangent spaces T_xM:

$$\mathrm{Fr}(M) := \{(x, \phi_x) : x \in M, \phi_x \in \mathrm{Fr}(T_xM)\}.$$

It comes with an obvious projection

$$p : \mathrm{Fr}(M) \to M, \ (x, \phi_x) \mapsto x.$$

Moreover (and still as for the tangent bundle), $\mathrm{Fr}(M)$ can be naturally made into a smooth manifold. Passing to G-structures, a G-structure can be viewed as a sub-space:

$$\mathcal{S} := \cup_x \mathcal{S}_x \subset \mathrm{Fr}(T_xM) = \mathrm{Fr}(M).$$

Moreover, the smoothness of $\mathcal{S}$ from the previous definition is equivalent to the smoothness of $\mathcal{S}$ as a smooth submanifold of $\mathrm{Fr}(M)$ (this will become clearer when we discuss principal bundles).

Remark 2.3 (global/local frames)**.** One can talk about global frames on a manifold M: they are maps ϕ:

$$M \ni x \mapsto \phi(x) = (\phi_1(x), \ldots, \phi_n(x)) \in \mathrm{Fr}(T_xM)$$

with the property that any of the components ϕ_i are smooth vector fields. Hence a frame on M is a (ordered) collection

$$(\phi_1, \ldots, \phi_n)$$

of vector fields on M with the property that

$$\phi_1(x), \ldots, \phi_n(x)$$

is a basis of T_xM for all $x \in M$. Equivalently, a global frame is a (smooth) section of the frame bundle $\mathrm{Fr}(M)$, i.e. a map

$$M \ni x \mapsto \phi_x \in \mathrm{Fr}(T_xM)$$

which is smooth as a map from M to TM (see the previous remark).

Global frames rarely exist on interesting manifolds. However, one can talk about local frames- i.e. defined over some open $U \subset M$ (we call them local frames of M, defined over U). Given a G-structure $\mathcal{S}$ on M, we say that a local frame ϕ of M (defined over some open $U \subset M$) is a *local frame adapted to the G-structure* if

$$\phi(x) \in \mathcal{S}_x \quad \forall\, x \in U.$$

Important: When looking at geometric structures, one should think that giving a G-structure as in the definition is the same thing as specifying which frames are adapted to the structure (after all, if we know the adapted local frames, then we also know the G-structure). With this, the smoothness condition on $\mathcal{S}$ says that, for any pointwise frame $\phi_x \in \mathcal{S}_x$, there exist an adapted local frame ϕ, defined on some open containing x, such that $\phi(x) = \phi_x$.

Inspired by the previous discussion on adapted frames, we have:

Definition 2.4. *Given a G-structure $\mathcal{S}$ on M, a coordinate chart (U, χ) of M is called adapted to the G-structure if the induced frame is adapted, i.e.:*

$$(\frac{\partial}{\partial \chi_1}(x), \ldots, \frac{\partial}{\partial \chi_n}(x)) \in \mathcal{S}_x \quad \forall\, x \in U.$$

A G-structure is called integrable if around any point $x \in M$ one can find an adapted coordinate chart.

The notion of integrability is very much related to isomorphisms of G-structures and the so-called equivalence problem (when are two G structures locally equivalent?) applied to the original G-structure and the local model. Here are the details. First of all, the notion of isomorphism of G-structure is the natural one: any diffeomorphism $f : M \to M'$ induces linear isomorphism

$$(df)_x : T_xM \to T_{f(x)}M'$$

and then, at the level of frames,

$$f_* : \mathrm{Fr}(T_xM) \to \mathrm{Fr}(T_{f(x)}M')$$

(or, in terms of bundles: a bundle map $f_* : \mathrm{Fr}(M) \to \mathrm{Fr}(M')$ covering f).

Definition 2.5. *Given two G-structures, $\mathcal{S}$ on M and $\mathcal{S}'$ on M', an isomorphism between them is any diffeomorphism $f : M \to M'$ with the property that*

$$f_*(\mathcal{S}_x) = \mathcal{S}_{f(x)} \quad \forall\, x \in M.$$

Next, given G, there is a "standard G-structure" $\mathcal{S}_G^{\text{can}}$ on $\mathbb{R}^n$. Indeed, each of the spaces $T_x\mathbb{R}^n$ is canonically isomorphic to $\mathbb{R}^n$ (using the standard basis $\partial/\partial x_i$) and then one just uses the standard linear G-structure on $\mathbb{R}^n$. Equivalently, the frames of $\mathbb{R}^n$ that are adapted to $\mathcal{S}_G^{\text{can}}$ are precisely those of type $\phi = (\phi_1, \ldots, \phi_n)$,

$$\phi_i(x) = \sum g_{i,j}(x) \frac{\partial}{\partial x_j}(x)$$

where all the matrices $(g_{i,j}(x))_{i,j}$ belong to G.

Proposition 2.6. *A G-structure $\mathcal{S}$ on M is integrable if and only if $(M, \mathcal{S})$ is locally isomorphic to $(\mathbb{R}^n, \mathcal{S}_G^{can})$.*

Proof. tautological. □

2.2 $G = \{e\}$: frames and coframes

Let us briefly look at the extreme case when G is the trivial group. In that case we talk about e-structures (see also Example 1.29). We see that an e-structure on a manifold M is simply a global frame

$$\phi = (\phi_1, \ldots, \phi_n)$$

on M. The local model is, of course, $\mathbb{R}^n$ with the standard global frame $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_1}$. An isomorphism between two manifolds endowed with global frames, (M, ϕ) and (M, ϕ'), is any diffeomorphism $f : M \to M'$ with the property that

$$(df)_x(\phi_i(x)) = \phi_i^{'}(f(x)) \ \ \forall \ c \in M.$$

The integrability of (M, ϕ) (when interpreted as an e-structure) means that: around any point $x \in M$ one can find a coordinate chart (U, χ) such that

$$\phi_i = \frac{\partial}{\partial \chi_i}$$

for all i. The integrability of a coframe can be characterized in terms of the "structure functions" of the coframes, which are the smooth functions $c_{i,j}^k$ that arise as the coefficients with respect to the frame ϕ of the Lie brackets of the vectors of ϕ:

$$[\phi_i, \phi_j] = \sum_k c_{i,j}^k \phi_k.$$

Theorem 2.7. *A frame ϕ is integrable if and only if its structure functions vanish identically.*

Proof. The condition is $[\phi_i, \phi_j] = 0$ for all i and j; this is clearly a local condition and is satisfied by the local model, so it is necessary in order to have integrability. For the sufficiency, we use the flows Φ_i^t of the vector fields ϕ_i. The condition we have implies that each two of them commute. Let $x_0 \in M$ be arbitrary and consider

$$F : \mathbb{R}^n \to M, \quad F(t_1, \ldots, t_n) = \Phi_1^{t_1} \circ \ldots \Phi_n^{t_n}(x_0);$$

strictly speaking, this is defined on an open neighborhood of $0 \in \mathbb{R}^n$ (corresponding to the domains of the flows). Clearly $F(0) = x_0$. Computing the partial derivatives of F at an arbitrary point $x \in M$ we find the vectors $\phi_i(F(x))$. In particular, by the inverse function theorem, F is a local diffeomorphism from an open containing 0 to an open containing x_0; this defines the desired local chart. □

There is one more comment we would like to make here: very often one prefers to work with differential forms instead of vector fields. In particular, one prefers to work with coframes instead of frames, i.e. with a family of 1-forms:

$$\theta = (\theta^1, \ldots, \theta^n)$$

which, at each $x \in M$, induces a basis of T_x^*M. Of course, the two points of view are equivalent, as one has a 1-1 correspondence $\phi \longleftrightarrow \theta$ between frames and coframes given by the usual duality

$$\theta^i(\phi_j) = \delta_{i,j}.$$

However, when one goes deeper into the theory, the parallelism between the two points of view becomes a bit less obvious. One thing to keep in mind is that the role played by the Lie derivatives of vector fields, in the dual picture of forms, is taken by the DeRham differential. For instance, for a coframe θ, the resulting structure functions are characterized by

$$d\theta^k = \sum_{i<j} c_{i,j}^k \theta^i \wedge \theta^j.$$

Exercise 2.8. Prove that, indeed, given a frame ϕ and the induced coframe θ, the structure functions of ϕ satisfies the previous equation.

Of course, in the dual picture, the local model is $(\mathbb{R}^n, dx_1, \ldots, dx_n)$, the integrability of (M, θ) is about writing (locally) $\theta^i = d\chi_i$ and the previous proposition say that this is possible if and only if all the 1-forms are closed.

Exercise 2.9. Give now another proof of the previous proposition, using the fact that, locally, all closed 1-forms are exact.

For the curious reader. As we have already mentioned, it is vary rarely that interesting manifolds admit e-structure (global frames). Manifolds that do are called parallelizable. The situation is interesting even for the spheres. You should first do the following:

Exercise 2.10. *Show that S^1 is parallelizable. Then do the same for S^3 (try to use the quaternionic numbers for this one).*

Let us discuss the spheres in a bit more detail. Note that, the question of whether S^n is parallelizable can be rewritten in very down to earth terms: one is looking for $n+1$ functions

$$F^1, \ldots, F^n \to \mathbb{R}^{n+1}$$

with the property that, for each $x \in S^n \subset \mathbb{R}^{n+1}$, vectors $F^1(x), \ldots, F^n(x)$ are linearly independent and take value in the hyperplane P_x orthogonal to x:

$$F^i(x) \in P_x := \{v \in \mathbb{R}^{n+1} : \langle v, x \rangle = 0\}.$$

This comes from the standard identification of the tangent spaces T_xS^n with $P_x \subset \mathbb{R}^{n+1}$. In the case $n = 1$ one can take

$$F^1(x_1, x_2) = (x_2, -x_1)$$

or, interpreting $\mathbb{R}^2$ as $\mathbb{C}$ ($(x_1, x_2) \mapsto x_1 + ix_2$),

$$F^1(z) = iz.$$

It works quite similarly for S^3, but using quaternions (previous exercise). The 2-sphere however is not parallelizable. Actually, on S^2 any vector field must vanish at at least one point. This is popularly known as the "hairy ball theorem", and stated as: "you can't comb a hairy ball flat without creating a cowlick"(see also the "Cyclone consequences" on the wikipedia). The mathematical reason that S^2 is not parallelizable (and actually that it does not admit any no-where vanishing vector field) is that its "Euler characteristic" does not vanish. Actually, a nice (but non-trivial) theorem in differential topology says that the Euler characteristic of a compact manifold vanishes if and only if it admits a no-where vanishing vector field. By the way, for the same reason, all the even dimensional spheres S^{2n} are not parallelizable.

Are the other (odd dimensional) spheres parallelizable? Yes: there is also S^7. This uses octonions (also called the Cayley algebra) instead of quaternions (S^3) and complex numbers (S^1). How general is this? In what dimensions does it work? Looking at the proofs (e.g. for S^1 and S^3) we see that what we need is a "normed division algebra" structure on $\mathbb{R}^{n+1}$ (in order to handle the n-sphere S^n). By that we mean a multiplication "$\cdot$" on $\mathbb{R}^{n+1}$, with unit element $1 := (1, 0, \ldots, 0)$ (but not necessarily associative- as in the case of the octonions) and satisfying the norm condition

$$|x \cdot y| = |x||y| \quad \forall\ x, y \in \mathbb{R}^{n+1},$$

where $|\cdot|$ is the standard norm. The term "division algebra" comes from the fact that the norm condition implies that the product has the "division property": $xy = 0$ implies that either $x = 0$ or $y = 0$ (if we keep this condition, but we give up on the norm condition, we talk about division algebras). Indeed, any such operation induces

$$F^i(x) = e_i \cdot x$$

proving that S^n is parallelizable. But on which $\mathbb{R}^{n+1}$ do there exist such operations? Well: only on $\mathbb{R}$, $\mathbb{R}^2$ (complex), $\mathbb{R}^4$ (quaternions) and $\mathbb{R}^8$ (octonions)! This was known since the 19th century. It is also interesting to know why was there interest on such operations already in the 19th century: number theory and the question of which numbers can be written as sums of two, three, etc squares. For sum of two squares, the central formula which shows that a product of two numbers that can be written as a sum of two squares can itself be written as a sum of two squares is:

$$(x^2 + y^2)(a^2 + b^2) = (xa - yb)^2 + (xb + ya)^2.$$

Or, in terms of the complex numbers $z_1 = x + iy, z_2 = a_i b$:

$$|z_1 z_2| = |z_1||z_2|.$$

The search for similar "magic formulas" for sum of three squares never worked, but it did for four:

$$(x^2 + y^2 + z^2 + t^2)(a^2 + b^2 + c^2 + d^2) =$$

$$(xa + yb + zc + td)^2 + (xb - ya - zd + tc)^2 +$$

$$+(xc + yd - za - tb)^2 + (xd - yc + zb - ta)^2.$$

This is governed by the quaternions and its norm equation.

Any way, returning to the spheres, we see that the trick with the multiplication can only work for S^1, S^3 and S^7. And, indeed, one can prove that there are the only parallelizable spheres! Well, S^0 as well if you insist. The proof is highly non-trivial and makes use of the machinery of Algebraic Topology.

There are a few more things that are worth mentioning here. One is that probably the largest class of manifolds that are automatically parallelizable (because of their structure) are the Lie groups (see also later); this applies in particular to the closed subgroups of GL_n. This is one of the reasons that S^1 and S^3 are parallelizable. The circle is clearly a Lie group (with complex multiplication), which can be identified with the rotation group

$$\left\{ \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix} \right\} \subset GL_2(\mathbb{R}).$$

The fact that S^3 can be made into a Lie group follows by characterizing S^3 as the set of unit quaternionic vectors (i.e. similar to S^1, but by using quaternions instead of complex numbers). The similar argument for S^7 (but using octonions instead of quaternions) does not work because the multiplication of octonions is not associative (to ensure that S^7 is parallelizable, the division property is enough, but for making it into a Lie group one would need associativity). Can other spheres be made into Lie groups (besides S^1 and S^3)? No!

What else? Note that the negative answer in the case of even dimensional spheres - due to the fact that such spheres do not even admit nowhere vanishing vector fields- comes with a very natural question: ok, but, looking at S^n, which is the largest number of linearly independent vector fields that it admits? Again, this is a simple but very non-trivial problem, whose solution (half way in the 20th century) requires again the machinery of Algebraic Topology. But here is the answer: write

$n+1 = 2^{4a+r}m$ with m odd, $a \geq 0$ integer, $r \in \{0,1,2,3\}$. Then the maximal number of linearly independent vector fields on S^n is

$$r(n) = 8a + 2^r - 1.$$

Note that $r(n) = 0$ if n is even (no nowhere vanishing vector fields on the even dimensional spheres), while the parallelizability of S^n is equivalent to $r(n) = n$, i.e. $8a + 2r = 2^{4a+r}m$ which is easily seen to have the only solutions

$$m = 1, a = 0, r \in \{0,1,2,3\}$$

giving $n = 0, 2, 4, 8$ hence again the spheres S^0, S^1, S^3 and S^7.

Exercise 2.11. *Show that $S^2 \times S^1$ is parallelizable.*

2.3 $G = GL_n^+$: orientations

Here we briefly discuss the case $G = GL_n^+$ (orientations). An orientation on a manifold M consists of a collection

$$\mathcal{O} = \{\mathcal{O}_x : x \in M\}$$

of orientations on the tangent spaces T_xM which vary smoothly with respect to x in the following sense: for any $x_0 \in M$ one can find a local frame ϕ of M defined over some open $U \subset M$ containing x_0, such that ϕ_x induces the orientation $\mathcal{O}_x$ for all $x \in M$. In other words, an orientation on M is an GL_n^+-structure on M. One says that a manifold is orientable if it admits an orientation.

Remark 2.12. We see that, from the point of view of G-structures, an isomorphism between two oriented manifolds $(M, \mathcal{O})$ and $(M', \mathcal{O}')$ is a diffeomorphism $f : M \to M'$ with the property that $(df)_x : T_xM \to T_{f(x)}M'$ sends oriented frames of M to oriented frames of M'.

Proposition 2.13. *Any orientation $\mathcal{O}$ on M, when interpreted as an GL_n^+-structure, is integrable. In other words, for any point $x_0 \in M$, one can find a coordinate chart (U, χ) around x_0 such that, for any $x \in U$, the orientation $\mathcal{O}_x$ on T_xM is induced by the frame*

$$\frac{\partial}{\partial \chi_1}(x), \ldots, \frac{\partial}{\partial \chi_n}(x).$$

Proof. We know we find a frame ϕ defined over some open U containing x_0 such that ϕ_x induces $\mathcal{O}_x$ for all $x \in U$. Choose a coordinate chart (U_0, χ) defined over some open $U_0 \subset U$ containing x_0 which we may assume to be connected, such that the corresponding frame at x_0 induces the orientation $\mathcal{O}_{x_0}$. For $x \in U_0$, the matrix $A(x)$ of coordinate changes from the frame ϕ_x to the frame induced by (U_0, χ) at x, has the entries $A_{i,j}(x)$ smooth with respect to $x \in U_0$. Hence also det(A) is smooth; the determinant is also non-zero, and we know it is strictly positive at x_0. Hence it is positive on the (connected) U_0. This implies that the frame induced by (U_0, χ) at any $x \in U_0$ induces the orientation $\mathcal{O}_x$. □

Note that the previous proposition shows that an orientation on M is determined by the choice of an atlas $\mathcal{A}$ for the smooth structure of M with the property that, for any (U, χ), (U', χ') in M, the change of coordinates

$$c_{\chi,\chi'} := \chi' \circ \chi^{-1}$$

(function between opens in $\mathbb{R}^n$) has positive Jacobian at all points. Such atlases are called oriented. Two oriented atlases $\mathcal{A}$ and $\mathcal{A}'$ are called oriented equivalent if $\mathcal{A} \cup \mathcal{A}'$ is an oriented atlas. We see that:

- a manifold M is orientable if and only if it admits an oriented atlas.
- the choice of an orientation is equivalent to the choice of an equivalence class of an oriented atlas.

Exercise 2.14. Do the same for arbitrary subgroups $G \subset GL_n(\mathbb{R})$. More precisely:

- define the notion of G-atlas and equivalence of G-atlases.
- explain how a G-atlas induces a G-structure and show that two different G-atlases induce the same G-structure if and only if they are equivalent.
- show that a G-structure comes from a G-atlas if and only if it is integrable.

Conclusion: the choice of an integrable G-structure is equivalent to the choice of an equivalence class of G-atlases (or to the choice of a maximal G-atlas).

Exercise 2.15. *Exhibit an oriented atlas for the sphere S^n.*

Of course, exhibiting oriented atlases is not quite the (practical) way to obtain orientations. Here is a general procedure (that works e.g. for S^n):

Exercise 2.16. A unit normal vector field on a hypersurface (or codimension one submanifold) S in $\mathbb{R}^n$ is a smooth map $\mathbf{n} : S \to \mathbb{R}^n$ with the property that $\mathbf{n}(x) \perp T_x S$ and $\|\mathbf{n}(x)\| = 1$ for all $x \in S$.

1. Prove that S is orientable if and only if S admits a unit normal vector field.
2. Let U be open in $\mathbb{R}^n$ and $\Phi : U \to \mathbb{R}$ be a smooth function. Let c be a regular value of Φ. Then the manifold $\Phi^{-1}(c)$ has a unit normal vector field and is therefore orientable.

For the curious reader. Do all manifolds admit an orientation? The answer is: no. However, most of them do.

It should be clear that any parallelizable manifold is also orientable. As pointed out in the previous exercises, the spheres are orientable. Actually, so are all the simply connected manifolds (i.e. for which the first fundamental group vanishes). What would be the non-orientable examples then? Well, probably the simplest/nicest are the even-dimensional (real) projective spaces $\mathbb{P}^{2n}$. The complex projective spaces however (as any complex manifold- see also below) are orientable.

And, there is a simple trick to replace a non-orientable manifold M by one which is orientable, denoted $\tilde{M}$, and which covers M via a projection

$$p : \tilde{M} \to M$$

which is a 2-cover (each fiber consists of two points). Explicitly,

$$\tilde{M} = \{(x, \mathcal{O}_x) : x \in M, \mathcal{O}_x - \text{orientation on } T_x M\},\ p(x, \mathcal{O}_x) = x.$$

(I let you guess the smooth structure on $\tilde{M}$). For instance: which are the oriented covers of the (non-orientable) projective spaces $\mathbb{P}^{2n}$? (yes, it is the spheres S^{2n}).

Remark 2.17 (integration). One of the main uses of orientations comes from the fact that, once an orientation is fixed on a manifold M, one can integrate n-forms with compact supports, i.e. there is an associated integration map

$$\int_M : \Omega^n_{\text{cpt}}(M) \to \mathbb{R},$$

where, as above, n is the dimension of M and subscript c denotes "compact supports" (i.e. forms that vanish outside a compact). Here are some details. To understand why n-forms (and not functions) and how the orientation is relevant, let us start with an n-form ω which is supported inside the domain of a coordinate chart

$$\chi : U \xrightarrow{\sim} \Omega \subset \mathbb{R}^n.$$

Using the basis induced by the coordinate chart for the tangent spaces T_xM, and then for $\Lambda^n T_x^* M$, we see we can write, on U:

$$\omega = f_\chi \circ \chi (d\chi_1) \wedge \ldots \wedge (d\chi_n)$$

for some smooth function $f_\chi : \Omega \to \mathbb{R}$ (with compact supports). Of course, we would like to use the standard integration of functions on (opens in) $\mathbb{R}^n$ and define

$$\int_M \omega = \int_U \omega := \int_\Omega f_\chi(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$

The question is: doesn't this depend on the choice of the coordinate chart on U? Of course, this should be related to the change of variable formulas for the standard integration: if $h : \Omega' \to \Omega$ is a diffeomorphism between two opens in $\mathbb{R}^n$ then, for any $f \in C_c^\infty(\Omega)$,

$$\int_{\Omega'} |\mathrm{Jac}(h)| f \circ h = \int_\Omega f,$$

where $|\mathrm{Jac}(h)|$ is the absolute value of the Jacobian of h.

So, let's assume that

$$\chi' : U \to \Omega'$$

is another coordinate chart. Then we obtain another function $f_{\chi'}$ on Ω' and the question is whether its integral coincides with that of χ. The relationship between the two functions can be expressed in terms of the coordinate change

$$h = \chi \circ (\chi')^{-1} : \Omega' \to \Omega.$$

More precisely, writing

$$(d\chi_i)_x = \sum_j \frac{\partial h_i}{\partial x_j}(\chi'(x))(d\chi_j')_x,$$

we find that $f_{\chi'} = \mathrm{Jac}(h) f_\chi \circ h$. Hence we are almost there: if the Jacobian of h was everywhere positive then, by the standard change of variable formula, $\int_\Omega f_\chi = \int_{\Omega'} f_{\chi'}$ hence $\int_M \omega$ will be defined unambiguously. This is where the fixed orientation comes in: it allows us to talk about, and work only with, positive charts; and, for two such charts, the change of coordinates h will have positive Jacobian!

This explains how $\int_M \omega$ is defined if ω is supported inside a coordinate chart. Now, for an arbitrary $\omega \in \Omega^n_{\mathrm{cpt}}(M)$ one uses a partition of unity to decompose ω as a finite sum

$$\omega = \omega_1 + \ldots + \omega_k$$

where each ω_i is supported inside some coordinate chart (with compact support there). Define then

$$\int_M \omega = \int_M \omega_1 + \ldots + \int_M \omega_k.$$

Again, this does not depend on the way we decompose ω as a sum as before; this follows basically from the additivity of the usual integral.

Note also the reason that we work with forms (and, e.g., not with functions): locally they are represented by functions f_χ (that we can integrate) and the way these functions change when we change the coordinates is compatible with the change of variables formula for the standard integration. We only had the "small problem" that the Jacobians had to be positive- which was fixed by the orientation. One can wonder here: ok, but can't one work with something else instead of forms, so that even the "small problem" disappears, so that the integration is defined even on non-orientable manifolds? The answer is yes: use "densities". We do not give further details here, but let us only mention that they are very similar to top forms: they are represented locally by functions f_χ and the formula when changing the coordinates is precisely the one coming from the change of variable formula (so that the integration can be defined without further complications). Of course, the choice of an orientation on M induces an identification between the space of densities and the one of n-forms, and then the resulting integration is the one we discussed.

Remark 2.18 (relationship with cohomology)**.** Let $(M, \mathcal{O})$ be a compact n-dimensional oriented manifold, so that the integration

$$\int_M : \Omega^n(M) \to \mathbb{R}$$

is well-defined. Then the Stokes formula implies that the integral vanishes on all exact forms (forms of type $d\theta$). Interesting enough (and not completely trivial) is the fact that also the converse is true if M is connected: if the integral of an n-form vanishes, then it must be exact.

This discussion fits very well with DeRham cohomology. Since $d\omega = 0$ for all n-forms ω (just because any form of degree strictly larger than the dimension of the manifold vanishes), the Stokes argument tells us that the integral descends to a linear map

$$\int_M : H^n(M) \to \mathbb{R},$$

while the previous comment tells us that, if M is also connected, then the last map is injective (hence $H^n(M)$ is either zero or isomorphic to $\mathbb{R}$). This is an indication of the following characterization for orientability: a compact connected manifold is orientable if and only if $H^n(M)$ is isomorphic to $\mathbb{R}$ (see also the similar discussion on volume forms).

This discussion may be viewed as the starting point of an important tool of Algebraic/Differential Topology: Poincare duality. Given the compact, connected, oriented M then for any $k \in \{0, 1, \ldots, n\}$, using the wedge product of forms, passing to cohomology and then using the integration gives a pairing

$$H^k(M) \times H^{n-k}(M) \to \mathbb{R}, \ ([\omega], [\eta]) \mapsto \int_M \omega \wedge \eta.$$

One version of the Poincare duality says that this pairing is perfect ("non-degenerate"). From this it follows for instance that, for odd dimensional M, its Euler characteristic is zero. The previous pairing is an important tool in the study of manifolds. For instance, for 4-dimensional manifolds, choosing $k = 2$, one obtains a non-degenerate bilinear form on $H^2(M)$- an algebraic invariant that tells us a lot about M. But, again, we are moving to other territories here ...

2.4 Volume forms

We are again very brief. A volume form on a manifold M consists of a collection

$$\mu = \{\mu_x : x \in M\}$$

of linear volume elements on the tangent spaces T_xM varying smoothly with respect to x, in the sense that for any vector fields $X_1, \ldots, X_n$ on M,

$$\mu(X_1, \ldots, X_n) : M \to \mathbb{R}$$

is smooth. Equivalently

$$\mu \in \Omega^n(M)$$

is a differential form of top degree which does not vanish at any point $x \in M$. In other words, a volume element on M is an $SL_n(\mathbb{R})$-structure on M. In particular, the local model is $\mathbb{R}^n$ together with

$$\mu_{\text{can}} = (dx_1) \wedge \ldots \wedge (dx_n).$$

Remark 2.19. We see that, from the point of view of G-structures, an isomorphism between two manifolds endowed with volume forms, (M, μ) and (M', μ') is a diffeomorphism $f : M \to M'$ with the property that $f^*(\mu) = \mu'$. These are called volume preserving diffeomorphisms.

Proposition 2.20. *Any volume form μ on a manifold M, when interpreted as a $SL_n(\mathbb{R})$-structure, is integrable. In other words, around any $x \in M$, one can find a coordinate chart (U, χ) such that, on U,*

$$\mu = d\chi_1 \wedge \ldots \wedge d\chi_n.$$

Proof. Start with an arbitrary chart (U, χ) around x. Writing μ in the resulting coordinates, it looks like

$$\mu = f d\chi_1 \wedge \ldots \wedge d\chi_n$$

for some smooth function f on U. Then just choose any function χ_1' near x such that $\partial\chi_1'/\partial\chi_1 = f$ and replace the old coordinates by $(\chi_1', \chi_2, \ldots, \chi_n)$. □

Note that the linear story implies that the choice of a volume form on M determines the choice of an orientation on M. Actually more is true when it comes to the existence question:

Exercise 2.21. Show that a manifold M admits a volume form if and only if it is orientable. (Hint: time to exercise a bit with partitions of unity!)

For the curious reader. By the previous exercise, the question of existence of volume forms brings nothing new (see however below).

One of the main uses of volume forms comes from the fact that, once we fix a volume form μ on a manifold M, one can integrate compactly supported smooth functions, i.e. there is an induced integration map:

$$\int_M : C^\infty_{\text{cpt}}(M) \to \mathbb{R}.$$

This goes via the integration of n-forms (see the previous subsection) and the fact that μ induces an orientation on M and an isomorphism

$$C^\infty(M) \xrightarrow{\sim} \Omega^n(M), \ f \mapsto f\mu.$$

In other words, the integral of a function M is defined as the integral of the n-form $f\mu$ (associated to the orientation induced by μ).

In particular, the choice of a volume form allows us to talk about the volume of M:

$$\text{Vol}_\mu(M) := \int_M 1$$

(where 1 is the constant function equal to 1). It is remarkable that this is invariant characterizes μ uniquely up to isomorphism:

Theorem 2.22. *[Moser] If μ and μ' are two volume elements on a compact manifold M then the following are equivalent:*

1. *there exists a diffeomorphism $h : M \to M$ such that $\mu' = h^*\mu$.*
2. *$Vol_\mu(M) = Vol_{\mu'}(M)$.*

Proof. For the direct implication, note that the integration is invariant under oriented diffeomorphisms (...). Let's concentrate on the other, more difficult, implication. If you did the previous exercise, you noticed that an affine combinations of volume forms is a volume form. Hence for any $t \in [0, 1]$,

$$\mu_t := t\mu + (1 - t)\mu'$$

is a volume form, and it is not difficult to see that it induces the same orientation as μ. Since $\int_M \mu = \int_M \mu'$ it follows (see the relationship of integration with DeRham cohomology) that $\mu - \mu'$ is exact. We conclude that there exists a form $\eta \in \Omega^{n-1}(M)$ such that

$$\frac{d\mu_t}{dt} = \mu - \mu' = d\eta$$

for all t. Moreover, since μ_t is a volume form, it is not difficult to see that the operation

$$\mathcal{X}(M) \to \Omega^{n-1}(M), \quad X \mapsto i_X(\eta_t)$$

is an isomorphism (think first what happens for vector spaces) hence we find a vector field X_t such that

$$i_{X_t}(\mu_t) = -\eta.$$

All together, $\{X_t : t \in [0, 1]\}$ form a smooth family of vector fields on M (time dependent vector field). Finally, consider the flow of this family, which can be seen as a family of diffeomorphisms

$$\varphi^t : M \to M.$$

Using the basic properties of flows, one computes:

$$\frac{d}{dt}\varphi_t^*\mu_t = \varphi_t^*(L_{X_t}(\mu_t) + \dot{\mu}_t) = \ldots = 0,$$

hence $\varphi_t^*\mu_t$ is constant with respect to t. In particular $\varphi_1^*\mu_1 = \mu_0$, i.e. $\varphi_1^*\mu = \mu'$. □

Remark 2.23 (more on the relationship with integration)**.** Note that, since the integral of a volume form is always strictly positive, the resulting integration map

$$\int_M : H^n(M) \to \mathbb{R}$$

is non-zero. Combining with the similar discussion from the orientability subsection, we deduce that : for an n-dimensional compact connected manifold, the following are equivalent:

- M is orientable.
- M admits a volume form.
- $H^n(M) \neq 0$.
- $H^n(M)$ is isomorphic to $\mathbb{R}$.

2.5 Foliations

The story gets more interesting now. A p-dimensional distribution of a manifold M is a collection

$$\mathcal{F} = \{\mathcal{F}_x : x \in M\}$$

of p-dimensional subspaces $\mathcal{F}_x$ of T_xM which vary smoothly with respect to x in the sense that, for any $x_0 \in M$ one can find a local frame ϕ defined over some neighborhood of x_0 such that $\{\phi_1(x), \ldots, \phi_p(x)\}$ is a frame for $\mathcal{F}_x$ for all $x \in U$. In terms of vector bundles (to be done in more detail later), we are talking abouts p-dimensional (smooth) vector sub-bundles

$$\mathcal{F} \subset TM.$$

Again, it should be clear that we are now talking about $GL(p, n-p)$-structures on M (see also subsection 1.5). What about integrability? In what follows, we will denote by $\Gamma(\mathcal{F})$ the space of (smooth) sections of $\mathcal{F}$, i.e. of vector fields X on M with the property that $X_x \in \mathcal{F}_x$ for all $x \in M$.

Definition 2.24. *A distribution $\mathcal{F} \subset TM$ is called involutive if:*

$$[X, Y] \in \Gamma(\mathcal{F}) \quad \forall \; X, Y \in \Gamma(\mathcal{F}).$$

A (p-dimensional) foliation on M is an (p-dimensional) distribution that is involutive.

Note that the local model is involutive. It is $\mathbb{R}^n$ with the distribution $\mathcal{F}_{\text{can}}$ given by

$$\mathcal{F}_{\text{can},x} = \text{Span}\{\frac{\partial}{\partial x_1}(x), \ldots, \frac{\partial}{\partial x_p}(x)\}.$$

Theorem 2.25 (Frobenius)**.** *For a p-dimensional distribution $\mathcal{F}$ on a manifold M, the following are equivalent:*

1. *$\mathcal{F}$ is involutive.*
2. *$\mathcal{F}$, interpreted as an $GL(p, n-p)$-structures, is integrable. Equivalently, for any $x_0 \in M$ one finds a coordinate chart*

   $$(U, \chi) = (U, \chi_1, \ldots, \chi_p, \chi_{p+1}, \ldots, \chi_n)$$

   around x_0 such that

   $$\mathcal{F}_x = \textit{Span}\{\frac{\partial}{\partial \chi_1}(x), \ldots, \frac{\partial}{\partial \chi_p}(x)\} \quad \forall \; x \in U.$$

Proof. (rather sketchy, but with all the main ingredients) The reverse direction should be clear since the bracket of two vector fields of type $\partial/\partial\chi_i$ is zero. Let us prove the direct implication. Fix $x_0 \in M$ and fix any coordinate chart (U, χ) around x_0. After eventually renumbering the coordinates, we may assume:

$$T_xM = \mathcal{F}_x \oplus \text{Span}_{\mathbb{R}}\{\frac{\partial}{\partial \chi_{p+1}}(x), \ldots, \frac{\partial}{\partial \chi_n}(x)\}$$

for all x in a neighborhood $W \subset U$ of x_0 (note: for dimensional reasons the sum is any way direct; then, by assuming the previous equation to hold at x_0, it follows by a continuity argument that it holds in a neighborhood W of x_0). We may assume that W is some small ball. Consider the projection on the first p coordinates:

$$\pi : W \to \mathbb{R}^p.$$

It follows that its differential restricted to $\mathcal{F}$ induces isomorphisms

$$(d\pi)_x : \mathcal{F}_x \xrightarrow{\sim} T_{\pi(x)}\mathbb{R}^p \quad \forall\, x \in W.$$

Hence we find

$$V_x^1, \ldots, V_x^p \in \mathcal{F}_x \quad (\text{for } x \in W)$$

which are projectable (via $(d\pi)$) to

$$\frac{\partial}{\partial x_1}(\pi(x)), \ldots, \frac{\partial}{\partial x_p}(\pi(x)) \in T_{\pi(x)}\mathbb{R}^p.$$

Moreover, each V^i is smooth w.r.t. $x \in W$ (show this using the smoothness of $\mathcal{F}$!). Hence we obtain the vector fields $V^1, \ldots, V^p$ defined on W, spanning $\mathcal{F}$ and π-projectable to $\partial/\partial x_1$, ..., $\partial/\partial x_p$. In general, the property of being π-projectable is compatible with the Lie bracket (show that!) hence the Lie brackets $[V^i, V^j]$ are π-projectable to

$$[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = 0.$$

Since they also belong to $\mathcal{F}$ (the involutivity condition!), from the choice of W, it follows that $[V^i, V^j] = 0$ for all i and j. Then, as in the proof of the similar result for e-structures (Theorem 2.7), it follows that the flows of the vector fields V^i combine to give F which is a diffeomorphism from a neighborhood of $0 \in \mathbb{R}^n$ to W; this provides the desired coordinates. □

Remark 2.26 (more about foliations)**.** A p-dimensional distribution $\mathcal{F}$ on M specifies certain directions. It is natural to look at curves that follow the given directions (as for flows of vector fields). Even better, one can look for integrals of $\mathcal{F}$, by which we mean submanifolds $L \subset M$ with the property that

$$T_xL = \mathcal{F}_x \quad \forall\, x \in L.$$

When is it true that through each point of M there passes an integral of $\mathcal{F}$? Well ... precisely in the involutive (integrable) case. In one direction, it is clear that the existence of integrals through each point implies involutivity, because for any two vector fields X and Y tangent to a submanifold L, $[X, Y]$ remains tangent to L. The Frobenius theorem implies the converse: around each x_0, choosing the coordinate chart as in the statement, we may assume that $\chi(x_0) = 0$ and then

$$\{x \in M : \chi^{p+1}(x) = \ldots = \chi^n(x) = 0\}$$

is an integral of $\mathcal{F}$ through x.

However, the story does not stop here: like for flows of vector fields, when one looks for *maximal* integral curves, one looks here for maximal integrals of $\mathcal{F}$. There are also called leaves of $\mathcal{F}$. These are (immersed) submanifolds $L \subset M$ which are integrals of $\mathcal{F}$ and which are maximal with this property. Starting from Frobenius theorem and proceeding as for vector fields, we see that leaves exist through each point, an the collection of all leaves of $\mathcal{F}$ gives a partition of M. The Frobenius theorem says that this partition looks locally like the partition

$$\mathbb{R}^n = \bigcup_{y \in \mathbb{R}^{n-p}} \mathbb{R}^p \times \{y\}.$$

Actually, that is precisely one should think of (and picture) a foliation: such a partition which, locally, looks trivially (the bundle $\mathcal{F}$ is handy to encode and work with the foliation). Foliation Theory study the intricate geometry of such partitions.

Exercise 2.27. *Let $\lambda \in \mathbb{R}$ and consider the foliation on the torus $M = S^1 \to S^1$ given by*

$$\{\frac{\partial}{\partial \theta_1} + \lambda \frac{\partial}{\partial \theta_2}\}.$$

How do the leaves of $\mathcal{F}$ look like?

For the curious reader. The question of existence of foliations of a given codimension (where the codimension is the dimension of the ambient manifold minus the dimension of the foliation), on a given compact manifold, is a very interesting (and hard) question. Actually, the very birth of "Foliation Theory" is identified with the PhD thesis of George Reeb (1943), where he answered the question of existence of codimension one foliations on S^3. The answer was positive, and the examples he found is well-known under the name of "the Reeb foliation". It arises by first realizing S^3 as obtained from two copies of the solid torus $S^1 \times D^2$ (think on the picture) by gluing them along their boundary torus $S^1 \times S^1$ (indicated on picture **??**). into two solid tori.

In formulas, one thinks of the 3-sphere as

$$S^3 = \{(u, v) : u, v \in \mathbb{C} : |u|^2 + |v|^2 = 1\},$$

one consider the copy of the 2-torus inside S^3 given by

$$A = \{(u, v) \in S^3 : |v| = \frac{\sqrt{2}}{2}\}$$

and the two connected components X_1 and X_2 of $S^3 \setminus A$ (described similarly to A above, but with "$\leq$" and "$\geq$" instead of "$=$"). It is not difficult to see that X_1 and X_2 are indeed solid tori.

The idea is now to foliate (partition) each solid torus with a codimension one foliation which has, as one of the leaves, the boundary torus. This is done by wrapping around"planes" inside the solid torus, as indicated in pictures **??** and **??**.

After gluing, one gets a foliation on S^3. It is interesting to note that, although it is pretty clear that the result is smooth, the result can never be done analytically (the geometric phenomena of wrapping around is quite non-analytic). Actually, an old theorem of Haefliger says that S^3 does not admit any analytic codimension foliation. The theorem actually proves that compact manifolds that admit analytic codimension one foliation have infinite fundamental group.

Back to the existence of codimension one foliations on manifolds, it is not so difficult to see that the even dimensional spheres cannot carry such; actually, the existence of such a foliation on a compact simply connected manifold would imply the vanishing of the Euler characteristic. So one is left with the odd dimensional spheres. The case of S^5 was answered by Lawson in 1971 and then, after the work of several people, it was proved that all odd-dimensional spheres do admit codimension one foliations. This story (about codimension one foliations on compact manifold) came to an end around 1976 with the work of Thurston who showed that, on a compact orientable manifold M such a foliation exists if and only if its Euler characteristic vanishes. The story continues with the existence of foliations of other interesting dimensions/codimensions (e.g. there is a 2-dimensional one on S^7) etc etc ... and with Foliation Theory.

2.6 Complex structures

Another very interesting case is that of (almost) complex structures. An almost complex structure on a manifold M is, by definition, a family

$$J = \{J_x : x \in M\}$$

of linear complex structures on the tangent spaces T_xM, varying smoothly with respect to x, in the sense that for any vector field X on M, $J(X)$ is smooth (in terms of vector bundles, we are talking about vector bundle morphisms $J : TM \to TM$ satisfying $J^2 = -\text{Id}$). As in the linear story (see Subsection 1.7), the dimension of M must be even:

$$n = 2k.$$

It should be clear now that almost complex structures on M correspond to $GL_k(\mathbb{C})$-structures on M (where we use the embedding $GL_k(\mathbb{C}) \subset GL_{2k}(\mathbb{R})$ as in Subsection 1.7.

Remark 2.28 (from almost complex to orientations)**.** It should be clear from the linear story that an almost complex structure on M induces an orientation on M.

Again, the local model is $\mathbb{C}^k$, with the almost complex structure induced by the multiplication by i, after identifying the tangent spaces of $\mathbb{C}^k$ with $\mathbb{C}^k$. In other words, identifying

$$\begin{array}{ccc} \mathbb{C}^n & \stackrel{\sim}{\to} & \mathbb{R}^{2n} \\ (z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n) & \mapsto & (x_1, \ldots, x_n, y_1, \ldots, y_n), \end{array}$$

we are talking about $\mathbb{R}^{2n}$ with the almost complex structure is given by

$$J_{\text{can}}(\frac{\partial}{\partial x_k}) = \frac{\partial}{\partial y_k}, J_{\text{can}}(\frac{\partial}{\partial y_k}) = -\frac{\partial}{\partial x_k}.$$

Definition 2.29. *The Nijenhuis tensor of an almost complex structure J is the map*

$$\mathcal{N}_J : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M),$$
$$\mathcal{N}_J(X, Y) = [X, Y] + J([JX, Y] + [X, JY]) - [JX, JY].$$

Exercise 2.30. Show that $\mathcal{N}_J$ comes indeed from a tensor, i.e. it is $C^\infty(M)$-linear in its entries:

$$\mathcal{N}_J(fX, Y) = \mathcal{N}_J(X, fY) = f\mathcal{N}_J(X, Y)$$

for any two vector fields X, Y oi M and any smooth function f on M.

With these, one has the following:

Theorem 2.31 (Newlander-Nirenberg)**.** *For an almost complex structure J on M the following are equivalent:*

1. *the Nijenhuis tensor of J vanishes.*
2. *J, interpreted as a $GL_k(\mathbb{C})$-structure, is integrable. Equivalently, for any $x_0 \in M$, one finds a coordinate chart*

$$(U, \chi) = (U, x_1, \ldots, x_k, y_1, \ldots, y_k)$$

around x_0 such that, on U,

$$J(\frac{\partial}{\partial x_k}) = \frac{\partial}{\partial y_k}, \frac{\partial}{\partial y_k} = -\frac{\partial}{\partial x_k}.$$

Proof. too difficult to give here. □

Remark 2.32 (complex manifolds)**.** This implies that we are talking about complex manifolds- which are defined exactly in the same way as smooth (real) manifolds, but with charts taking values in opens in $\mathbb{C}^k$ and with transition function holomorphic: these are the charts that arise from the theorem. Indeed, if one considers a transition function corresponding to two such charts:

$$f = (f_1, \ldots, f_k, g_1, \ldots, g_k) : V \to W \quad (V, W \subset \mathbb{R}^{2k} \text{ opens})$$

we see that

$$(df) \circ J_{\text{can}} = J_{\text{can}} \circ (df),$$

or, equivalently,

$$\frac{\partial f_k}{\partial x_j} = \frac{\partial g_k}{\partial y_j}, \frac{\partial f_k}{\partial y_j} = -\frac{\partial g_k}{\partial x_j},$$

which are precisely the Cauchy-Riemann equations that characterize the holomorphicity of $f_k + ig_k$. Hence F is holomorphic.

Note also that, for a complex manifold M, one can talk about its complex tangent space and, as a real vector space, it can be canonically identified with the standard tangent space of M, viewed as a smooth (real) manifold. This gives an intrinsic description of the almost complex structure on the smooth manifold underlying a complex manifold. It is also useful to think in coordinates. Then complex charts $(U, z_1, \ldots, z_k)$ induce then a basis over $\mathbb{C}$ of the tangent space

$$\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_k}.$$

The identification of the complex tangent space (as a real vector space) with the standard tangent space of M comes from splitting th complex coordinates (of complex charts) as $z_k = x_k + iy_k$; then

$$\frac{\partial}{\partial z_k} = \frac{1}{2}(\frac{\partial}{\partial x_k} - i\frac{\partial}{\partial y_k}).$$

It is customary to consider also

$$\frac{\partial}{\partial \overline{z}_k} = \frac{1}{2}(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k})$$

so that, for a smooth function f, the Cauchy-Riemann equations ensuring the holomorphicity of f take the form $\frac{\partial f}{\partial \overline{z}_k} = 0$.)

Remark 2.33 (complex foliations)**.** Almost complex structures can be interpreted (formally) like "complex foliations". This comes from the reinterpretation of a linear complex structure J on a (real) vector space V as a certain complex subspace $V^{1,0}$ of the complexification $V_{\mathbb{C}}$ of V. See Remark 1.16. That discussion applied to each tangent space T_xM tells us that an almost complex structure J on a manifold M is encoded in the resulting $T^{1,0}M \subset T_{\mathbb{C}}M$ which, in terms of J is

$$T^{1,0}M = \{X - iJ(X) : X \in \mathcal{X}(M)\}$$

It is now easy to see that the condition that the Nijenhuis tensor of J vanishes is equivalent to the Frobenius-type condition:

$$[X, Y] \in \Gamma(T^{1,0}M) \quad \forall\ X, Y \in \Gamma(T^{1,0}M).$$

(where $[\cdot, \cdot]$ is just the Lie bracket of vector fields, extended by $\mathbb{C}$-linearity to the entire $\Gamma(T_{\mathbb{C}}M)$).

For the curious reader. Also in the case of complex structures, the question of existence of a complex structure on a manifold M is a rather hard one. For instance, it seems to be an open problem the question of whether S^6 admits a complex structure. On the other hand, from all the spheres, only S^2 and S^6 admit an almost complex structure; for S^2 it comes from an honest complex structure (by viewing S^2 as a complex manifold, e.g. as $\mathbb{CP}^1$); for S^6 it follows by using, again, the octonions. Actually, there is a simple trick that allows us to relate the existence of almost complex structures to parallelizability and then, using the fact that S^1, S^3 and S^7 are the only parallelizable sphere, to conclude that S^2 and S^6 are the only spheres which admit an almost complex structure. Here is the trick:

Exercise 2.34. *Assume that $S^n \subset \mathbb{R}^{n+1}$ admits an almost complex structure J. Interpret it as a family of linear endomorphisms*

$$J_y : P_y \to P_y \quad \text{for } y \in S^n$$

of the hyperplane $P_y = \{v \in \mathbb{R}^{n+1} : \langle v, y\rangle = 0\} \subset \mathbb{R}^{n+1}$ orthogonal to y.

Pass now to $\mathbb{R}^{n+2}$; denote by $e_0, \ldots, e_{n+1}$ its standard basis and interpret $\mathbb{R}^{n+1}$ (and all its subspaces, e.g. the P_ys) as the subspace of $\mathbb{R}^{n+2}$ via the standard inclusion

$$(y_0, \ldots, y_n) \mapsto (y_0, \ldots, y_n, 0).$$

For $x = (x_0, \ldots, x_{n+1}) \in S^{n+1} \subset \mathbb{R}^{n+2}$ different from $\pm e_{n+1}$, we denote by $p(x)$ its orthogonal projection on $\mathbb{R}^{n+1}$,

$$p(x) = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$$

and the induced element on the sphere

$$s(x) = \frac{1}{||p(x)||} p(x) \in S^n.$$

Consider the orthogonal projection onto $P_{s(x)}$,

$$pr^{\perp}_{s(x)} : \mathbb{R}^{n+1} \to P_{s(x)},\ \ pr^{\perp}_{s(x)}(\lambda) = \lambda - \langle \lambda, s(x)\rangle s(x)$$

and define:

$$F_i(x) := x_{n+1}e_i - x_i e_{n+1} + ||p(x)|| J_{s(x)}(pr^{\perp}_{s(x)}(e_i))$$

for all $x \in S^{n+1}$, where the very last term is set to be zero when it does not make sense (i.e. for $x = \pm e_{n+1}$). Show that $\{F^0, \ldots, F^n\}$ is a frame of S^{n+1}. Hence S^{n+1} is parallelizable.

Interesting enough, the product of any two odd dimensional spheres admits a complex structure.

2.7 Riemannian metrics I: existence

A Riemannian metric on a manifold M is a collection

$$g = \{g_x : x \in M\}$$

of metrics (inner products) g_x on the tangent spaces T_xM, which "vary smoothly with respect to x", where smoothness is given by one of the following equivalent descriptions:

Exercise 2.35. *Given the family g, show that the following are equivalent:*

1. *For any two vector fields X and Y on M,*

$$g(X,Y) : M \to \mathbb{R}, \quad x \mapsto g_x(X(x), Y(x))$$

is smooth.

2. *For any coordinate chart (U, χ) of M, the coefficients of g given by*

$$g_{i,j}(x) := g_x\left(\frac{\partial}{\partial \chi_i}(x), \frac{\partial}{\partial \chi_j}(x)\right)$$

are smooth (as functions of $x \in U$).

We talk about $O(n)$-structures.
The first basic result about Riemannian metrics is:

Proposition 2.36. *Any manifold admits a Riemannian metric.*

Proof. (sketch) Locally it is clear (just transport the standard inner products via coordinate charts). Then use a partition of unity. □

Exercise 2.37. Using Riemannian metrics prove again that a manifold admits a volume form if and only if it is orientable.

Moving to the point of view of G-structures, one has:

Definition 2.38. *An isomorphism (or isometry) between two Riemannian manifolds (M, g) and (M', g') is any diffeomorphism $f : M \to M'$ with the property that $(df)_x : T_xM \to T_{f(x)}M'$ is an isomorphism between (T_xM, g_x) and $(T_{f(x)}M', g_{f(x)})$.*

The local flat model of Riemannian geometry is $\mathbb{R}^n$, endowed with the standard metric given by

$$g^{\mathrm{can}}\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \delta_{i,j}.$$

2.8 (Almost) symplectic structures

Next, we move to Symplectic Geometry.

Definition 2.39. *An almost symplectic structure on a manifold M is a 2-form*

$$\omega \in \Omega^2(M)$$

with the property that each ω_x is non-degenerate.

Hence each ω_x is a linear symplectic form on T_xM. From the linear story, we know that the dimension n of M must be even. Here we will use the notation:

$$n = 2k.$$

Interpreting each ω_x as a linear $Sp_k(\mathbb{R})$-structure $\mathcal{S}_x$ on T_xM (see the linear story of Subsection 1.8), the smoothness of $\omega = \{\omega_x : x \in M\}$ as a 2-form is easily seen to be equivalent to the smoothness of $\mathcal{S} = \{\mathcal{S}_x : x \in M\}$ as a

$Sp_k(\mathbb{R})$-structure on M. Hence an almost symplectic structure on M is the same thing as an $Sp_k(\mathbb{R})$-structure on M.

Note that the local model is $\mathbb{R}^{2k}$, with

$$\omega_{\text{can}} = dy_1 \wedge dx_1 + \ldots + dy_k \wedge dx_k,$$

where $(x_1, \ldots, x_k, y_1, \ldots, y_k)$ denote the coordinates of $\mathbb{R}^{2k}$. What about integrability?

Definition 2.40. *A symplectic structure on a manifold M is an almost symplectic structure $\omega \in \Omega^2(M)$ which is closed.*

Isomorphisms between symplectic manifolds (M, ω) and (M', ω') correspond to diffeomorphisms $f : M \to M'$ satisfying $\omega = f^*\omega'$. They are also called symplectomorphisms.

Theorem 2.41 (Darboux)**.** *For an almost symplectic manifolds (M, ω), the following are equivalent:*

1. *ω is symplectic.*
2. *ω, interpreted as a $Sp_k(\mathbb{R})$-structure, is integrable. Equivalently, for any $x \in M$, one finds a coordinate chart*

$$(U, \chi) = (U, x_1, \ldots, x_k, y_1, \ldots, y_k)$$

around x such that

$$\omega = dy_1 \wedge dx_1 + \ldots + dy_k \wedge dx_k.$$

Proof. This is very similar to the proof of Theorem 2.22. Let's write it in a slightly different order. The reverse implication is clear since the standard symplectic form is closed. Assume now that ω is symplectic. We may assume we are on $\mathbb{R}^{2n}$ with coordinates denoted (x, y) and we work around the origin. We may also assume that $\omega = \omega_{\text{can}}$ at the origin (why???). The idea is to realize the desired diffeomorphism between open neighborhoods of the origin taking ω to ω_{can} as the flow at time $t = 1$ of a vector field. Just using vector fields does not work, but it will if we allow time-dependent vector fields. Let X_t be the time-dependent vector field we are looking for and let φ be its flow. The next idea is to connect ω to ω_{can} by a smooth family of forms, $\{\omega_t : t \in [0, 1]\}$, satisfying

$$\omega_0 = \omega_{\text{can}}, \ \omega_1 = \omega$$

and make sure that $\varphi_t^*\omega_t$ stays constant or, equivalently, $\frac{d}{dt}\varphi_t^*\omega_t = 0$. Using the properties of the flow (see Exercise 2.42 below) we can write this condition as follows:

$$L_{X_t}(\omega_t) + \frac{d}{dt}\omega_t = 0,$$

or, if the ω_t's are all closed and using *Cartan's magic formula* $L_X = di_X + i_X d$:

$$d(i_{X_t}\omega_t) + \frac{d}{dt}\omega_t = 0.$$

That is all we have to do. Here is how one does it. First of all, one considers the most obvious family:

$$\omega_t = t\omega + (1-t)\omega_{\text{can}},$$

so that the equation we have to solve (to find X_t) becomes:

$$d(i_{X_t}\omega_t) + (\omega - \omega_{\text{can}}) = 0.$$

Note that, since $\omega - \omega_{\text{can}}$ is closed, by working in a ball around the origin, we may assume it is exact (Poincare lemma), say of type $d\eta$ for some 1-form η. Then it is enough to solve the (simpler) equation

$$i_{X_t}(\omega_t) + \eta = 0.$$

Here we deal with the operation from vector fields to 1-forms which sends a vector V to the 1-form $i_V(\omega_t)$. This we know to be a bijection (in particular surjective) if ω_t was non-degenerate. This is certainly so for $t \in \{0, 1\}$, so we just have to make sure that, after eventually shrinking the neighborhood of 0 that we are working on, all the ω_t are symplectic. Here is where the assumption that ω and ω_{can} coincide at 0 comes in. They also coincide with ω_t at 0. Since ω_t is non-degenerate at 0, it will be non-degenerate in a neighborhood W_t of 0. Since we are interested in $t \in [0, 1]$ and since ω_t is a continuous family (even smooth) and $[0, 1]$ is compact, it follows that one can choose the same W_t for all t; call it W. If you followed the argument, you see we are done. Otherwise, start from here and read the proof backwards. □

Exercise 2.42. Let α_t be a smooth family of d-forms and X_t a time-dependent vector field. Let φ_t be the flow generated by X_t. Prove that

$$\frac{d}{dt}\varphi_t^*\alpha_t = \varphi_t^*\left(\frac{d}{dt}\alpha_t + L_{X_t}\alpha_t\right).$$

(Hint: if you cannot figure it out, look at the appendix from Geiges' book.)

Remark 2.43 (from almost symplectic to volume forms and orientations)**.** Again, it should be clear by looking at the local picture that an almost symplectic structure ω on M induces a volume forms (hence also an orientation). It is customary to use a certain scaling: given ω one defines the Liouville (volume) form associated to ω as:

$$\mu_\omega := \frac{1}{k!}\omega^k = \frac{1}{k!}\underbrace{\omega \wedge \ldots \omega}_{k \text{ times}}.$$

Exercise 2.44. Using Riemannian metrics prove that if M admits an almost complex structure then it also admits and almost symplectic one. More precisely, show that if J is an almost complex structure and g is a Riemannian metric, then

$$\omega(X, Y) := g(X, JY) - g(JX, Y)$$

defines an almost symplectic structure on M.

Remark 2.45 (almost symplectic versus almost complex)**.** It is not true that an almost symplectic structure induces ("naturally") an almost complex one, or the other way. What is true however is that a manifold M admits an almost symplectic structure if and only if it admits an almost complex one. The reverse implication is rather easy to show once we have Riemannian metrics (Exercise 2.44). The direct implication is slightly more difficult, but it should be easy if you did already Exercise 1.23.

Exercise 2.46. Use Exercise 1.23 to show that if M admits an almost symplectic structure ω then it admits an almost complex structure J.

For the curious reader.
While the existence of an almost complex structure is equivalent to the existence of an almost symplectic structure, anything else can happen:

- There are compact manifolds which admit almost complex structures, but do not admit complex or symplectic structures. This was proven for the connected sum of three copies of $\mathbb{CP}^2$ by Taubes in 1995.
- there are compact manifolds that admit complex structures but do not admit symplectic ones. One example is $S^3 \times S^1$.
- There are compact manifolds that admit symplectic structures but not complex ones.

The key idea for relating (almost) symplectic with (almost) complex structure comes from the notion of ω-compatibility - see subsection 1.9 for the linear story; on manifolds, we require that condition at each point. With this, one can show that, given any almost symplectic structure ω one can find an ω-compatible almost complex structure. After that, one can proceed an use J as it was an actual complex structure, exploiting ideas from complex geometry, to derive information about ω. This is the basic idea behind the theory of "pseudo-holomorphic curves" in Symplectic Geometry.

Regarding existence results, due to the difference between symplectic and almost symplectic structures, there are really two questions to answer here. The question of which (compact) manifolds admit symplectic structure is a very hard one, especially in the case of a negative answer; this is due to the lack of (known) invariants/obstructions for symplectic structures. What one can say right away, for compact manifolds M, is that if they admit a symplectic structure, then they must be even dimensional and orientable. And a bit more: the second DeRham cohomology group $H^2(M)$ is non-zero (so this excludes all the sphere S^n with $n \neq 2$). This last remark follows using integration: if ω is a symplectic form, then one has an induced volume form and orientation and the integral of $[\omega]^k \in H^n(M)$ is non-zero, hence *omega* itself will represent a non-zero cohomology class. Here are some examples/results:

- in dimension 2, a symplectic structure is the same thing as a volume form. Hence orientable surfaces are symplectic.
- from all the spheres, only S^2 and S^6 admit an almost symplectic structure, and only S^2 admits a symplectic structure.
- the connected sum of three copies of $\mathbb{CP}^2$ admits an almost symplectic structure but neither a complex nor a symplectic one (Taubes, 1995).
- For any finitely presented group Γ one can find a compact 4-manifold M, which is symplectic, with fundamental group isomorphic to Γ (Gompf, 2000).
- The existence of an almost symplectic structure is equivalent to the existence of an almost complex structure (see also subsection 2.6).
- On open manifolds (i.e. manifolds which are not compact), there is quite some flexibility- due to the so called h-principle of Gromov ("h" stands for "homotopy"). For symplectic structures it says that: starting with any almost symplectic structure on an open manifold M, one can smoothly deform it (through almost symplectic structures) into a symplectic structure.

Remark 2.47 (Hamiltonian dynamics)**.** The motion of a mass 1 particle in $\mathbb{R}^k$ in the presence of a potential force $\Phi(x) = -\frac{\partial V}{\partial x}$ is governed by Newton's second law, $\ddot{q} = \Phi(q)$. If we introduce the auxiliary variable $p = \dot{q}$, the total energy of the particle is given by

$$H = \frac{1}{2}p^2 + V(q)$$

and Newton's equation transforms into a system of first order ODE's (or, equivalently, a differential equation in the $2k$-dimensional space $\mathbb{R}^{2k}$ with coordinates (x, y)), known as a *Hamiltonian system*:

$$\frac{dy}{dt} = -\frac{\partial H}{\partial x}, \qquad \frac{dx}{dt} = \frac{\partial H}{\partial y}$$

The corresponding Hamiltonian flow φ_t sends (x_0, y_0) to the solution of the Hamiltonian system satisfying the initial condition $(x(0), y(0)) = (x_0, y_0)$. In classical mechanics, these diffeomorphisms were sometime referred to as *mechanical motions* and they have the property that they preserve the volume form

$$\mu = dy_1 \wedge dx_1 \wedge \ldots \wedge dy_k \wedge dx_k$$

The crucial observation here is that, in fact, mechanical motions preserve the canonical symplectic form on $\mathbb{R}^{2n}$, i.e. they are symplectomorphisms. Obviously this second property implies the first one, in view of

$$\mu = \mu_\omega = \frac{\omega^k}{k!}$$

and this should be more evidence that the similarity between the proofs of the Darboux theorem for symplectic forms and the Moser theorem for volume forms is not accidental (of course, the two coincide in dimension two). The volume preserving property of Hamiltonian flows already attractted a lot of attention more than a century ago. The fact that these mechanical motions preserve the symplectic form was first pointed out explicitly by Arnol'd in the 1960's and the question of whether volume-preserving diffeomorphisms were the same as symplectomorphisms is one that kept symplectic topologists busy for quite some time. The (negative) answer was provided by Gromov in the 1980's, with his famous proof of the non-squeezing theorem, and makes use of very sophisticated techniques (*J-holomorphic curves*).

So the punchline is: Hamiltonian mechanics makes sense and can be studied on any manifold endowed with a symplectic form. For any such manifold (M, ω), the symplectic form ω sets up a one-to-one correspondence between vector fields and one-forms on M (in other words, an isomorphism between the tangent and the cotangent bundle). So given a smooth function $H : M \to \mathbb{R}$ (the Hamiltonian function) there is a uniquely defined vector field X_H corresponding to the differential of H: it is called the Hamiltonian vector field and it is defined by $i_{X_H}\omega = -dH$. We are interested in integral curves of this vector field, i.e. curves γ on M that satisfy the equation:

$$\dot{\gamma}(t) = X_H(\gamma(t)),$$

which should be interpreted as the most general form of Hamilton's equations.

Exercise 2.48. *Check that if $M = \mathbb{R}^{2n}$ with coordinates (x, y) and standard symplectic form ω_{can}, the Hamiltonian vector field is given by $J_0\nabla H = (\partial_y H, -\partial_x H)$ and we recover the classical system of Hamiltonian equations. Can you guess what the form of the Hamiltonian vector field should be for an arbitrary (M, ω)?*

2.9 Integral affine structures

An almost integral affine structure on a manifold M is, by definition, a family

$$\Lambda = \{\Lambda_x\}_{x \in M}$$

of lattices Λ_x on the tangent spaces T_xM, which vary smoothly with respect to x in the sense that, for any $X_0 \in M$, there exists a local frame ϕ defined on a neighborhood U of x_0 such that

$$\Lambda_x = \mathrm{Span}_{\mathbb{Z}}\{\phi^1(x), \ldots, \phi_n(x)\} \;\; \forall\, x \in U.$$

As before, Λ can be interpreted as a subspace

$$\Lambda \subset TM.$$

We also see the we are talking about $GL_n(\mathbb{Z})$-structures on M (see also subsection 1.6). The adapted frames in this case are the ones whose components (vector fields) take values in Λ. Vector fields with this property are called *integral vector fields of* (M, Λ).

Definition 2.49. *We say that Λ is an integral affine structure if, when interpreted as an $GL_n(\mathbb{Z})$-structure, it is integrable.*

Note that the local model is $\mathbb{R}^n$ with the lattice

$$\Lambda_{\text{can}} = \mathbb{Z}\frac{\partial}{\partial x_1} + \ldots + \mathbb{Z}\frac{\partial}{\partial x_n}.$$

Hence the integrability of Λ means that, around any point we can find a coordinate chart (U, χ) such that

$$\Lambda_x = \mathbb{Z}\frac{\partial}{\partial \chi_1}(x) + \ldots + \mathbb{Z}\frac{\partial}{\partial \chi_n}(x)$$

for all $x \in U$. The atlas consisting of such charts have very special transition functions $F : V \to W$ (between opens $V, W \subset \mathbb{R}^n$): they are integral affine, i.e. of type

$$F(v) = v_0 + A(v) \quad v_0 \in \mathbb{R}^n,\ A \in GL_n(\mathbb{Z}). \tag{7}$$

Such an atlas is called an integral affine atlas. As before, the existence of such an atlas is equivalent to the existence of an integral affine structure.

Ok, but can one characterize the integrability of Λ more directly (as we did for symplectic or complex structures?). The answer can be given in several different ways. One of them makes use of the dual

$$\Lambda^{\vee} \subset T^*M$$

of Λ defined by

$$\Lambda^{\vee}_x = \{\xi \in T^*_x M : \xi(\Lambda_x) \subset \mathbb{Z}\}.$$

Theorem 2.50. *For an almost integral affine structure Λ on a manifold M the following are equivalent:*

1. *Λ is integrable (hence an integral affine structure).*
2. *the Lie bracket of any two local vector fields which are integral (i.e. take values in Λ) vanishes.*
3. *$\Lambda^{\vee}$ is locally spanned by closed 1-forms, i.e. any point in M admits a neighborhood U and closed 1-forms $\theta_1, \ldots, \theta_n$ on U such that*

$$\Lambda^{\vee}_x = Span_{\mathbb{Z}}\{\theta_1(x), \ldots, \theta_n(x)\}.$$

Proof. That 1 implies 2 comes from the fact that the condition in 2 can be checked locally and is valid for the local model. Assume now that 2 holds. Locally, we choose any frame $\{\phi_1, \ldots, \phi_n\}$ which spans Λ (as in the description of smoothness above). The condition in 2 tells us that ϕ is integrable in the sense of e-structures (see subsection 2.2), hence we find charts (U, χ) such that $\phi_i = d\chi_i$, and this implies 1. The equivalence with 3 is similar, using the dual point of view (see again subsection 2.2). ☐

Exercise 2.51. *Show that if a manifold M is orientable and admits an integral affine structure, then it admits a volume form.*

2.10 Affine structures

In the previous subsection we have discussed *integral affine structures* (where "'integral" reflects the role of $\mathbb{Z}$ in the story). What about affine structures? At least when it comes to integral affine atlases, it is clear that the role of $\mathbb{Z}$ is not important, and one can talk about affine atlases as those with the property that the transitions functions are of type (7) but with $A \in GL_n(\mathbb{R})$ (i.e. just affine transformations). But the rest of the story is more problematic: affine structure is a type of geometric structures which is interesting (and worth looking at) but which does not fit in the general theory of G-structures, at least not in an obvious way, and not in the way we discussed it so far. Nevertheless, let's us discuss it a bit.

The central notion here is that of (affine) connection.

Definition 2.52. *An affine connection on a manifold M is a bi-linear map*

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M),\ (X, Y) \mapsto \nabla_X(Y), \tag{8}$$

satisfying the following equations:

$$\nabla_{fX}(y) = f\nabla_X(Y),\ \nabla_X(fY) = f\nabla_X(Y) + L_X(f)Y$$

for all $X, Y \in \mathcal{X}(M)$, $f \in C^\infty(M)$.

One should think of a connection as a rule that allows to take derivatives along a vector field Y along another vector field X. Note that, for functions f, one can talk about such derivatives- they are precisely the Lie derivatives $L_X(f)$. The analogue of this with f replaced by a vector field Y would be

$$L_X(Y) := [X, Y],$$

but this encodes "the variation of Y along X" (rather than the "derivative"); in particular, this last operation does not satisfy the first of the equations from the definition (but it does satisfy the second). By the way, $\nabla_X(Y)$ is also called the covariant derivative of Y along X, with respect to ∇. Any way, the point is that, although we would sometimes like to be able to talk about such "derivatives" of vector fields along vector fields, they do not come for free- we have to make a choice- and that is what an affine connection is. Using a partition of unity argument, one can show that any manifold admits an affine connection.

Remark 2.53 (More general connections)**.** In the previous definition, $\mathcal{X}(M)$ (or, even better: TM) plays a double role. One role comes from the first appearance of $\mathcal{X}(M)$ in (8) (i.e. the role of X when one write $\nabla_X(Y)$): it tells us "along what" we take derivatives: *along vector fields*. This is the most important role. The other one comes from the second and third appearance of $\mathcal{X}(M)$ in (8) and it tells us "of what" we take derivatives: *of vector fields* (with the result being the same kind of object). With this in mind, one sees that one can talk about different type of connections: keep the first $\mathcal{X}(M)$ but replace the other two. For instance, one can look at operations

$$\nabla : \mathcal{X}(M) \times \Omega^1(M) \to \Omega^1(M),\ (X, \theta) \mapsto \nabla_X(\theta)$$

satisfying exactly the same conditions as above, but with Y replaced by θ. This are connections on T^*M". More generally, one can talk about connections on any vector bundle E over M, completely similarly, as operations:

$$\nabla : \mathcal{X}(M) \times \Gamma(E) \to \Gamma(E), \ (X, s) \mapsto \nabla_X(s)$$

where one uses the space $\Gamma(E)$ of sections of E.

Exercise 2.54. Given an affine connection on M, show that for any open $U \subset M$ there is an induced connection ∇^U on U (sometimes also denoted $\nabla|_U$- the restriction of ∇ to U) uniquely characterized by the condition that, for any two vector fields X, Y on U,

$$\nabla_X(Y)|_U = \nabla^U_{X|_U}(Y|_U).$$

Although we are not talking about G-structures, note that there is a natural notion of "special" (or adapted) frame. More precisely, given ∇, the interesting vector fields are those $V \in \mathcal{X}(M)$ with the property that

$$\nabla_X(V) = 0 \quad \forall \ X \in \mathcal{X}(M).$$

These are called *flat vector fields* (flat with respect to ∇). With this, one can talk about flat frames (and also local flat frames- cg. the previous exercise) of (M, ∇)- and these are the adapted frames. Of course, the local model is $\mathbb{R}^n$ with ∇^{can} uniquely determined by the condition that the standard vector fields $\frac{\partial}{\partial x_i}$ are flat.

Definition 2.55. *An affine structure on a manifold M is an affine connection ∇ with the property that (M, ∇) is locally isomorphic to $(\mathbb{R}^n, \nabla^{can})$ i.e. around any point in M one can find a local chart (U, χ) such that the vector fields $\frac{\partial}{\partial \chi_i}$ are flat.*

As before, this gives rise to an atlas for which the change of coordinates are affine transformations between opens in $\mathbb{R}^n$ (hence of type $v \mapsto v_0 + A(v)$ with $v_0 \in \mathbb{R}^n$, $A \in GL_n(\mathbb{R})$). Conversely, any such atlas gives rise to an integrable ∇. Hence, with the obvious notion of equivalence of affine atlases, an integrable affine structure on M is the same things as an (equivalence class of an) affine atlas.

But can one characterize the integrability of ∇? This brings us to the notion of curvature and torsion:

Definition 2.56. *Given an affine connection ∇ on M, $X, Y \in \mathcal{X}(M)$, one considers:*

- *the torsion expressions*

$$T_\nabla(X, Y) = \nabla_X(Y) - \nabla_Y(X) - [X, Y].$$

- *the curvature expressions*

$$K_\nabla(X, Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]} : \mathcal{X}(M) \to \mathcal{X}(M).$$

Exercise 2.57. *Check that these expressions are $C^\infty(M)$-linear in all its entries:*

$$T_\nabla(fX, Y) = T_\nabla(X, fY) = fT_\nabla(X, Y),$$

$$K_\nabla(fX, Y)(Z) = K_\nabla(X, fY)(Z) = K_\nabla(X, Y)(fZ) = fK_\nabla(X, Y)(Z).$$

The exercise ensures that we deal with tensors, called the torsion tensor T_∇ of ∇ and the curvature tensor K_∇. Abstractly, they belong to:

$$T_\nabla \in \Omega^2(M, TM) = \Gamma(\Lambda^2 T^*M \otimes TM),$$

$$K_\nabla \in \Omega^2(M, \mathrm{End}(TM)) = \Gamma(\Lambda^2 T^*M \otimes \mathrm{End}(TM)).$$

It is clear that, for the local model, these tensors must vanish.

Theorem 2.58. *An affine connection is integrable if and only if its torsion and curvature vanish.*

Proof. Will not be given in these lectures. □

2.11 Riemannian metrics II: integrability

As we have seen in section 2.7, Riemannian metrics exist on any smooth manifold. The integrability of Riemannian metrics (interpreted as $O(n)$-structures) is interesting, but a bit more involved then in the symplectic or complex case, and certainly not so important as in those cases. The key concept here is that of Levi-Civita connection.

Theorem 2.59. *Given a Riemannian metric g on M, there exists and is unique an affine connection ∇ on M (called the Levi-Civita connection associated to ∇) with the following properties:*

- *it is torsion free: $T_\nabla = 0$.*
- *it is compatible with g in the sense that, for all $X, Y, Z \in \mathcal{X}(M)$,*

$$L_X(g(Y, Z)) = g(\nabla_X(Y), Z) + g(Y, \nabla_X(Z)).$$

Proof. This proof here (which is the standard one) is not very enlightening (hopefully we will see a much more transparent/geometric argument a bit later in the course). One just plays with the compatibility equations applied to (X, Y, Z) and their cyclic permutations, combined in such a way that most of the appearances of the ∇ disappear by using the torsion free condition. One ends up with the following identity (which, once written down, can also be checked directly):

$$2g(\nabla_X(Y), Z) = L_X(g(Y, Z)) + L_Y(g(X, Z)) - L_Z(g(X, Y)) +$$

$$+ g([X, Y], Z) + g([Z, X], Y) + g([Z, Y], X).$$

Hence, fixing X and Y, we see that for the expression we are looking for, $\nabla_X(Y)$, the product with respect to any Z is predetermined. Since g is a metric, this forces the definition of $\nabla_X(Y)$ pointwise. It remains to check all kinds of identities for ∇, identities that say that some expression E vanishes. For that one proves that $g(E, V) = 0$ for all vector fields V. In that way, one can use the above (defining) equation for ∇ and the check is straightforward. □

Theorem 2.60. *A Riemannian manifold (M, g) is integrable as a $O(n)$-structure if and only if its Levi-Civita connection is flat, i.e. $K_\nabla = 0$.*

For this reason, such manifolds are also called flat Riemannian manifolds.

Proof. Follows immediately from the similar theorem for affine connections. □

2.12 Kahler geometry

This is an example of interesting geometric structure for which "integrability" is less interesting (it is too restrictive). However, the more general equivalence problem is relevant. More precisely, one does not have *one* local model but several (depending on the choice of a certain function) and the interesting question is when a Hermitian structure is equivalent to one of the models. And that happens precisely in the Kahler case. This can be explained nicely/easily.

2.13 Contact structures

Shall we say something about them as well? Or maybe make a section in which we enumerate a few more: contact structures, symplectic foliations (≅ regular Poisson structures), etc.
****

Remark 2.61. *Mention and emphasize that a lot of the theory of G-structures is about understanding the conditions under which integrability holds. Sometimes, this is carried out as part of a more general program of the "equivalence problem".*

AT THIS POINT: WE SAY THAT, AT THIS POINT, WE STOP FOR A WHILE WITH THE STUDY OF G-STRUCTURES, AND WE CONCENTRATE ON THE GENERAL FRAMEWORK (e.g. OF PRINCIPAL BUNDLES) THAT ALLOWS US TO TREAT G-STRUCTURES MORE CONCEPTUALLY. OF COURSE, THE STUDY OF G-STRUCTURES IS OUR MOTIVATION.

3 Closed subgroups of $GL_n(\mathbb{R})$; Lie groups

3.1 Matrices and the exponential map

Recall that we denote by $M(n, \mathbb{R})$ the vector space of $n \times n$ matrices with real entries and by $GL(n, \mathbb{R})$ the group of invertible $n \times n$ matrices, i.e.

$$GL(n, \mathbb{R}) = \{A \in M(n, \mathbb{R}) \,:\, \det A \neq 0\}.$$

If we identify (in the obvious way) $M(n, \mathbb{R})$ with $\mathbb{R}^{n^2}$, we see that $GL(n, \mathbb{R})$, as an open submanifold, is equipped with a smooth structure and that the group operation (matrix multiplication), is smooth with respect to this structure, since it is just polynomial in the entries (same holds for the inverse matrix). Moreover, if we define left translation on $GL(n, \mathbb{R})$ by an element A as the map $L_A : B \mapsto AB$, this gives a way to identify the tangent space at any matrix A with the tangent space at the identity element I. In particular, the tangent bundle to

$GL(n,\mathbb{R})$ is trivial and isomorphic to $GL(n,\mathbb{R})\times T_I GL(n,\mathbb{R})$. In the language of the previous section, $GL(n,\mathbb{R})$ (as a manifold) is parallelizable. You are asked to work out the details of the proof in the exercise below. We could summarize these observations by saying that $GL(n,\mathbb{R})$ has a *Lie group structure*, but we will only come back to this concept at the end of the lecture.

Exercise 3.1. Prove that $GL(n,\mathbb{R})$ is parallelizable.

On the space of $n \times n$ matrices there is a norm, namely

$$\|X\|^2 = \sum_{1\le i,j\le n} |a_{ij}|^2 = \mathrm{trace}({}^t A\, A),$$

which, under the identification with $\mathbb{R}^{n^2}$, is just the usual Euclidean norm.

We would like to discuss here in some details the exponential map for matrices.

Lemma 3.2. *For each matrix $X \in M(n,\mathbb{R})$, the exponential sequence*

$$I + X + \frac{X^2}{2!} + \frac{X^3}{3!} + \cdots$$

converges.

Exercise 3.3. Prove Lemma 3.2. (Hint: every Cauchy sequence converges in $\mathbb{R}^{n^2}$.)

In view of the above lemma, the following definition is meaningful.

Definition 3.4. *For $X \in M(n,\mathbb{R})$ we define the exponential of X as the limit of the exponential series, i.e.*

$$e^X = I + \sum_{k=1}^{\infty} \frac{X^k}{k!}.$$

Exercise 3.5. Prove the following properties of the matrix exponential:

(i) if $X = 0$, then $e^0 = I$;

(ii) if X is diagonal, say $X = \mathrm{diag}(\lambda_1,\dots,\lambda_n)$, then $e^X = \mathrm{diag}(e^{\lambda_1},\dots,e^{\lambda_n})$;

(iii) if X and Y commute, then $e^X e^Y = e^{X+Y} = e^Y e^X$.

The third property has the following important consequence:

Corollary 3.6. *For all $X \in M(n,\mathbb{R})$, e^X is invertible with $(e^X)^{-1} = e^{-X}$, hence $e^X \in GL(n,\mathbb{R})$.*

Thus the exponential establishes a map

$$\exp : M(n,\mathbb{R}) \to GL(n,\mathbb{R}), \quad X \mapsto e^X. \tag{9}$$

Proposition 3.7. *If the eigenvalues of X are $\lambda_1,\dots,\lambda_n$, then the eigenvalues of e^X are $e^{\lambda_1},\dots,e^{\lambda_n}$.*

Using this proposition one can obtain a simple relation between the determinant of the matrix exponential and the usual exponential, namely:

Corollary 3.8.
$$\det e^X = e^{\operatorname{trace} X}$$

Exercise 3.9. Prove Proposition 3.7 and Corollary 3.8.

We have seen that exp maps the zero matrix to the identity matrix. In fact it even sets up a diffeomorphism between suitable neighborhoods of these elements.

Proposition 3.10. *There exists an open neighborhood $\mathcal{U}$ of $0 \in M(n, \mathbb{R})$ which is mapped diffeomorphically by* exp *onto an open neighborhood $\mathcal{V}$ of $I \in GL(n, \mathbb{R})$.*

Proof. The idea is very easy, namely: as in the scalar case, the inverse of the exponential map should be the logarithm.

Suppose $X \in M(n, \mathbb{R})$ satisfies $\|X\| < 1$. Then the matrix $T = I + X$ is invertible, i.e. an element of $GL(n, \mathbb{R})$, with inverse

$$(I + X)^{-1} = I - X + X^2 - X^3 + \dots$$

Moreover, the logarithmic series

$$\log T = X - \frac{X^2}{2} + \frac{X^3}{3} - \dots$$

is convergent and, if we denote its limit by $\log T$, the following identity holds:

$$e^{\log T} = T.$$

Now let $\mathcal{V} = \{T \in GL(n, \mathbb{R}) : \|T - I\| < 1\}$ and let $\mathcal{U} = \log \mathcal{V} = \{\log T : T \in \mathcal{V}\}$. Then the maps $\mathcal{V} \to \mathcal{U}$, $T \mapsto \log T$, and $\mathcal{U} \to \mathcal{V}$, $X \mapsto e^X$, are mutually inverse. □

Notice that one can identify (in a canonical way) $M(n, \mathbb{R})$ with the tangent space to $GL(n, \mathbb{R})$ at I. This feature is very important in view of the general theory of Lie groups. Recall that the smooth structure on $GL(n, \mathbb{R})$ is that of an open submanifold of $M(n, \mathbb{R})$ and hence that, as a manifold, it also has dimension n^2. Let $X \in M(n, \mathbb{R})$: then the curve $g : \mathbb{R} \to GL(n, \mathbb{R}), t \mapsto e^{tX}$ starts at the identity at time $t = 0$ and $g'(0) = X$. In fact, from the definition of the exponential map we have

$$e^{tX} = I + tX + o(t)$$

and therefore

$$\left.\frac{d}{dt}\right|_{t=0} e^{tX} = X.$$

This shows that we can view $M(n, \mathbb{R})$ as a subspace of $T_I GL(n, \mathbb{R})$ and then, by a dimension argument, we conclude that we can in fact identify these two linear spaces.

Remark 3.11. We can actually conclude something more: the tangent vector $g'(t)$ at each time t is the image of $g'(0) = X$ under the differential of $L_{g(t)}$, left translation by $g(t)$.

3.2 Linear groups and their Lie algebra

Recall that on $M(n,\mathbb{R})$ we have the commutator operation given by the Lie bracket:

$$[X,Y] = XY - YX.$$

This fits into the following general framework:

Definition 3.12. *A Lie algebra $\mathfrak{l}$ over $\mathbb{R}$ is a real vector space together with a bracket $[\ ,\ ]$, i.e., a skew-symmetric bilinear map, satisfying Jacobi's identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0.$$

Notice that if $\mathfrak{l}$, as a vector space, has a basis $\{e_1,\dots,e_m\}$, to define the Lie bracket it's enough to give the $m(m-1)/2$ elements $[e_i,e_j]$, $i<j$. Each of these brackets, in fact, can be written as

$$[e_i,e_j] = \sum_{k=1}^{m} c_{ij}^k e_k$$

and the scalars c_{ij}^k are called the *structure constants* of the Lie algebra.

We would now like to extend the discussion about $GL(n,\mathbb{R})$, $M(n,\mathbb{R})$ and the exponential map to general closed subgroups of $GL(n,\mathbb{R})$ (closed in the topology induced by the norm). We have already encountered a few such subgroups in the previous part of the course: for instance, $O(n)\subset GL(n,\mathbb{R})$, $GL_n^+\subset GL(n,\mathbb{R})$, $SL_n(\mathbb{R})\subset GL(n,\mathbb{R})$, $GL(k,\mathbb{C})\subset GL(2k,\mathbb{R})$, $Sp(k,\mathbb{R})\subset GL(2k,\mathbb{R})$.

Definition 3.13. *A linear group is a closed subgroup of $GL(n,\mathbb{R})$.*

Definition 3.14. *For $G\subset GL(n,\mathbb{R})$ a linear group, we set*

$$\mathfrak{g} := \{X\in M(n,\mathbb{R}) : e^{tX}\in G \quad \forall \quad t\in\mathbb{R}\}.$$

So far we have established that $M(n,\mathbb{R}) = \mathfrak{gl}_n$ is the Lie algebra of $GL(n,\mathbb{R})$. What we want to do next, is to show that for any linear group G, the corresponding space $\mathfrak{g}$, with the restriction of the matrix Lie bracket, is a Lie algebra (which will be called the *Lie algebra of G*). In fact we will show that it is a subalgebra of $\mathfrak{gl}$.

Proposition 3.15. *Let G be a linear group.Then*

- (i) *$\mathfrak{g}$ is a vector subspace of $\mathfrak{gl}_n$;*
- (ii) *$\mathfrak{g}$ is closed with respect to the Lie bracket.*

For the proof of this proposition we will need the following lemma (stated and proved in Sternberg as Lemma 4.2, Chapter V):

Lemma 3.16. *Let $X_i\to X$ be a sequence of elements in $\mathfrak{gl}_n$ and $t_i\to 0$ a sequence of non-zero real numbers. Suppose $e^{t_iX_i}\in G$ for all i: then $e^{tX}\in G$ for all t, i.e., $X\in\mathfrak{g}$.*

Remark 3.17. The fact that G is closed as a subspace of $GL(n,\mathbb{R})$ is used in the proof of this Lemma

Proof of Proposition 3.15. (*i*) $\mathbb{R}$-linearity follows immediately from the definition. We still need to prove that if $X, Y \in \mathfrak{g}$, then $X + Y \in \mathfrak{g}$. If $X, Y \in \mathfrak{g}$, then $e^{tX}e^{tY} \in G$ for all t. We have:

$$e^{tX}e^{tY} = e^{t(X+Y)} + o(t),$$

so we can write

$$e^{tX}e^{tY} = e^{f(t)}, \quad \text{with} \quad \lim_{t\to 0} \frac{f(t)}{t} = X + Y.$$

Taking t_i a sequence of real numbers converging to 0 and $X_i = \frac{f(t_i)}{t_i}$, Lemma 3.16 implies that $X + Y \in \mathfrak{g}$.

(*ii*) We need to prove that if $X, Y \in \mathfrak{g}$, then $[X, Y] \in \mathfrak{g}$. If $X, Y \in \mathfrak{g}$, then $e^{tX}e^{tY}e^{-tX}e^{-tY} \in G$. But as in the first part of the proposition, we can write

$$e^{tX}e^{tY}e^{-tX}e^{-tY} = e^{f(t)}, \quad \text{with} \quad \lim_{t\to 0} \frac{f(t^2)}{t^2} = [X, Y],$$

so with $t_i \to 0$, $s_i = t_i^2$ and $X_i = \frac{f(s_i)}{s_i}$ the lemma applies and we conclude that $[X, Y] \in \mathfrak{g}$.

□

Proposition 3.18. *Any linear group G is an embedded submanifold of $GL(n, \mathbb{R})$.*

Proof. We already know that the Lie algebra $\mathfrak{g}$ of G is a linear subspace of $\mathfrak{gl}_n$. What remains to prove is that $\exp(\mathfrak{g})$ is a neighborhood of I in G, then left translation will provide us with charts around other points of G.

Choose a complementary subspace $\mathfrak{g}'$ s.t. $\mathfrak{gl}_n = \mathfrak{g} \oplus \mathfrak{g}'$. Then the first claim is that there exists a neighborhood V' of 0 in $\mathfrak{g}'$ such that $e^{X'} \notin G$ for any element $X' \neq 0$ of V'. In fact, if we suppose the claim does not hold, we can construct a sequence $Y_i \to 0$ in $\mathfrak{g}'$ such that $e^{Y_i} \in G$. Let $A = \{X \in \mathfrak{g}' \,|\, 1 \leq \|X\| \leq 2\}$ and choose integers n_i such that $X_i = n_i Y_i \in A$. Since A is compact, the X_i's converge to some nonzero element X in $\mathfrak{g}'$. But $\frac{1}{n_i} \to 0$ and $e^{\frac{1}{n_i}X_i} \in G$ for all i, so by Lemma 3.16 we would also have $X \in \mathfrak{g}$, which is a contradiction.

Now let W be a neighborhood of 0 in $\mathfrak{gl}_n$ such that

$$\phi : \mathfrak{gl}_n = \mathfrak{g} \oplus \mathfrak{g}' \to GL(n, \mathbb{R})\,, \quad X + X' \mapsto e^X e^{X'}$$

is also a diffeomorphism. We may assume $W = V \times V'$. Let $U = \phi(W)$: if $T \in U$, then $T = e^X e^{X'}$ and if $T \in U \cap G$, then $X' = 0$ and hence $U \cap G = \exp \mathfrak{g}$. □

Example 3.19. *If $G = SL(n, \mathbb{R})$, then $\mathfrak{g} = \{X \in \mathfrak{gl}_n \,:\, \operatorname{trace} X = 0\}$. In fact, $X \in \mathfrak{g}$ implies $\det(e^{tX}) = 1$ for all t. To the first order in t, $e^{tX} = I + tX + o(t)$ and hence*

$$\det(e^{tX}) = 1 + t \operatorname{trace} X + o(t)$$

(on expanding the determinant, the off-diagonal terms in tX will only appear in second or higher order terms). Hence $\operatorname{trace} X$ must be zero. Conversely, if $\operatorname{trace} X = 0$, then $\det(e^{tX}) = e^{t \operatorname{trace} X} = 1$ and hence $X \in \mathfrak{g}$.

Exercise 3.20.

- Let G be the orthogonal group $O(n) = \{T \in GL(n, \mathbb{R}) : {}^tTT = I\}$. Prove that its Lie algebra is

$$\mathfrak{g} = \{X \in \mathfrak{gl}_n : {}^tX + X = 0\}.$$

- Determine the Lie algebra of $GL(k, \mathbb{C})$, $U(k)$ and $Sp(n)$.

Remark 3.21 (The adjoint representation)**.** Every element $T \in G = GL(n, \mathbb{R})$ defines a linear transformation Ad_G of $\mathfrak{gl}_n$: an element $X \in \mathfrak{gl}_n$ is sent to the velocity at $t = 0$ of the parametric curve $c_T \circ e^{tX}$, where c_T denotes conjugation automorphism $A \mapsto TAT^{-1}$. This curve has velocity TXT^{-1} at $t = 0$, so $\mathrm{Ad}_G(T)$ is smooth (T-conjugation on $\mathfrak{gl}_n$). It therefore makes sense to consider its derivative at the identity matrix. This is related to the Lie bracket by the formula

$$[X, Y] = \frac{d}{dt}\bigg|_{t=0} (\mathrm{Ad}_G(e^{tX})(Y)).$$

3.3 General Lie groups

The above discussion can be generalized to general Lie groups, i.e., smooth manifolds with a group structure such that the group operations $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ are smooth. One can associate to these groups a Lie algebra and this Lie algebra can be identified with the tangent space to the group at the identity element. Closed subgroups of a Lie group are smooth submanifolds (hence Lie subgroups). And so on. One of many good references for general Lie group theory is F.W. Warner, *Foundations of Differentiable manifolds and Lie groups.*

Exercise 3.22. *In the definition of G-structures, show that the smoothness condition is equivalent to the fact that B_G is a smooth submanifold of $Fr(M)$.*

4 Vector bundles and principal bundles

4.1 Vector bundles

In this subsection we recall some basics on vector bundles.

Definition 4.1. *Let M be a manifold, $r \geq 0$ integer. A (real) vector bundle of rank r over M consists of*

- *a manifold E,*
- *a surjective map $\pi : E \to M$; for $x \in M$, the fiber $\pi^{-1}(x)$ is denoted E_x*
- *for each $x \in M$, a structure of r-dimensional (real) vector space on E_x*

satisfying the following local triviality condition: for each $x_0 \in M$, there exists an open neighborhood U of x_0 and a diffeomorphism

$$h : E|_U := \pi^{-1}(U) \xrightarrow{\sim} U \times \mathbb{R}^r$$

with the property that it sends each fiber E_x into the $\{x\} \times \mathbb{R}^r$ and

$$h_x := h|_{E_x} : E_x \xrightarrow{\sim} \{x\} \times \mathbb{R}^r \cong \mathbb{R}^r$$

is a linear isomorphism for all $x \in U$. Complex vector bundles are defined similarly, replacing $\mathbb{R}$ by $\mathbb{C}$.

As terminology: although we should say that $\pi : E \to M$ is a (real) vector bundle over M, we often just mention E; in such cases we usually denote by π_E the associated map into M and we refer to it as the projection of the vector bundle E.

One should think about E as the collection

$$\{E_x\}_{x \in M}$$

of vector spaces (of rank r), "smoothly parametrized by $x \in M$".

Note that, as hinted by the notation $E|_U$, a vector bundle E over M can be restricted to an arbitrary open $U \subset M$. More precisely,

$$E|_U := \pi^{-1}(U)$$

together with the restriction of π gives a vector bundle $\pi_U : E|_U \to U$ over U.

Here are some basic concepts/constructions regarding vector bundles.

4.2. Morphisms: Given two vector bundles E and F over M, a morphism from E to F (of vector bundles over M) is a smooth map $u : E \to F$ with the property that, for each $x \in M$, u sends E_x to F_x and

$$u_x := u|_{E_x} : E_x \to F_x$$

is linear. We say that u is an isomorphism (of vector bundles over M) if each u_x is an isomorphism (or, equivalently, if u is also a diffeomorphism).

Again, one should think of a morphism f as the collection $\{u_x\}_{x\in M}$ of linear maps f_x, "smooth in x".

4.3. Trivial vector bundles; trivializations: The trivial vector bundle of rank r over M is the product $M \times \mathbb{R}^r$ together with the first projection

$$\mathrm{pr}_1 : M \times \mathbb{R}^r \to M$$

and the usual vector space structure on each fiber $\{x\} \times \mathbb{R}^r$.

We say that a vector bundle E (of rank r) over M is trivializable if E is isomorphic to $M \times \mathbb{R}^r$. A trivialization of E is the choice of such an isomorphism.

With these in mind, we see that the local triviality condition from the definition of vector bundles says that E is locally trivializable, i.e. each point in M admits an open neighborhood U such that the restriction $E|_U$ is trivializable.

4.4. Sections: The main objects associated to vector bundles are their (local) sections. Given a vector bundle $\pi : E \to M$, a section of E is a smooth map $s : M \to E$ satisfying $p \circ s = \mathrm{Id}$, i.e. with the property that

$$s(x) \in E_x \quad \forall\, x \in M.$$

We denote by

$$\Gamma(M, E) = \Gamma(E)$$

the space of all smooth sections. For $U \subset M$ open, the space of local sections of E defined over U is

$$\Gamma(U, E) := \Gamma(E|_U).$$

Sections can be added pointwise:

$$(s + s')(x) := s(x) + s'(x)$$

and, similarly, can be multiplied by scalars $\lambda \in \mathbb{R}$:

$$(\lambda s)(x) := \lambda s(x).$$

With these, $\Gamma(E)$ becomes a vector space. Furthermore, any section $s \in \Gamma(E)$ can be multiplied pointwise by any real-valued smooth function $f \in C^\infty(M)$ giving rise to another section $fs \in \Gamma(E)$:

$$(fs)(x) := f(x)s(x).$$

The resulting operation

$$C^\infty(M) \times \Gamma(E) \to \Gamma(E), \quad (f, s) \mapsto fs$$

makes $\Gamma(E)$ into a module over the algebra $C^\infty(M)$. Actually the entire vector bundle E is fully encoded in the space of sections $\Gamma(E)$ together with this module structure; a precise formulation of this is Swan's theorem which says that the construction $E \mapsto \Gamma(E)$ gives a 1-1 correspondence between vector bundles over M and finitely generated projective modules over $C^\infty(M)$. A simpler illustration of this is the following:

Lemma 4.5. *Let E and F be two vector bundles over M. Then there is a bijection between':*

- *morphisms $u : E \to F$ of vector bundles over M.*
- *morphisms $u_* : \Gamma(E) \to \Gamma(F)$ of $C^\infty(M)$-modules.*

Explicitely, given u, the associated u_* is given by

$$u_*(s)(x) = u_x(s(x)).$$

4.6. Frames: Let $\pi : E \to M$ be a vector bundle over M. A frame of E is a collection

$$s = (s^1, \ldots, s^r)$$

consisting of sections s^i of E with the property that, for each $x \in M$,

$$(s^1(x), \ldots, s^r(x))$$

is a frame of E_x. A local frame of E is a frame s of $E|_U$ for some open $U \subset M$; we then say that the local frame s is defined over U.

Remark 4.7. Choosing a frame s of E is equivalent to choosing a trivialization $u : M \times \mathbb{R}^r \to E$ of E. Hence, the local triviality condition from the definition of vector bundles can be phrased in terms of local frames as follows: around any point of M one can find a local frame of E.

4.8. Remark on the construction of vector bundles: Often the vector bundles that one encounters do not arise right away as in the original definition of vector bundles. Instead, one has just a collection $E = \{E_x\}_{x \in M}$ of vector spaces indexed by $x \in M$ and certain "smooth sections". Let us formalize this a bit. We will use the name "discrete vector bundle over M (of rank r)" for any collection $\{E_x\}_{x \in M}$ of (r-dimensional) vector spaces indexed by $x \in M$. We will identify such a collection with the resulting disjoint union and the associated projection

$$E := \{(x, v_x) : x \in M, v_x \in E_x\}, \ p : E \to M, \quad (x, v_x) \mapsto x$$

(so that each E_x is identified with the fiber $p^{-1}(x)$).

For such a discrete vector bundle E we can talk about discrete sections, which are simply functions s as above,

$$M \ni x \mapsto s(x) \in E_x$$

(but without any smoothness condition). Denote by $\Gamma_{\mathrm{discr}}(E)$ the set of such sections. Similarly we can talk about discrete local sections, frames and local frames. As in the case of charts of manifolds, there is a natural notion of "smooth compatibility" of local frames. To be more precise, we assume that

$$s = (s^1, \ldots, s^r), \ \tilde{s} = (\tilde{s}^1, \ldots, \tilde{s}^r)$$

are two local frames defined over U and $\tilde{U}$, respectively. Then, over $U \cap \tilde{U}$, one can write

$$\tilde{s}^i(x) = \sum_{j=1}^{r} g^i_j(x) s^j(x),$$

giving rise to functions

$$g^i_j : U \cap \tilde{U} \to \mathbb{R} \ \ (1 \le i, j \le r).$$

We say that s and $\tilde{s}$ are smoothly compatible if all the functions g^i_j are smooth. The following is an instructive exerse.

Exercise 4.9. Let $E = \{E_x\}_{x \in M}$ be a discrete vector bundle over M of rank r. Assume that we are given an open cover $\mathcal{U}$ of M and, for each open $U \in \mathcal{U}$, a discrete local frame s_U of E over U. Assume that, for any $U, V \in \mathcal{U}$, s_U and s_V are smoothly compatible. Then E admits a unique smooth structure which makes it into a vector bundle over M with the property that all the s_U become (smooth) local frames.

Moreover, the (smooth) sections of E can be recognized as those discrete sections s with the property that they are smooth with respect to the given data $\{s_U\}_{U \in \mathcal{U}}$ in the following sense: for any $U \in \mathcal{U}$, writing

$$u(x) = f_1(x) s^1_U(x) + \ldots + f_r(x) s^r_U(x) \quad (x \in U),$$

all the functions f_i are smooth on E.

Example 4.10. For a manifold M one consider all the tangent spaces $TM = \{T_x M\}_x \in M$ and view it as a discrete vector bundle. Given a chart $\chi : U_\chi \to \mathbb{R}^n$ for M, then the associated tangent vectors

$$(\frac{\partial}{\partial \chi_1}(x), \ldots, \frac{\partial}{\partial \chi_n}(x))$$

can be seen as a discrete local frame of TM over U_χ. Starting with an atlas $\mathcal{A}$ of M, we obtain in this way precisely the data that we need in order to apply the previous lemma; this makes TM into a vector bundle over M in the sense of the original definition.

4.11. Operations with vector bundles: The principle is very simple: natural operations with vector spaces, applied fiberwise, extend to vector bundles.

Direct sums: Let us start with the direct sum operation. Given two vector spaces V and W we consider their direct sum vector space $V \oplus W$. Assume now that $p_E : E \to M$ and $p_F : F \to M$ are vector bundles over M. Then the direct sum $E \oplus F$ is another vector bundle over M, with fibers

$$(E \oplus F)_x := E_x \oplus F_x. \tag{10}$$

These equations force the definition of the total space $E \oplus F$ and of the projection into M. To exhibit the smooth structure of $E \oplus F$ one can e.g. use Exercise 4.9. Indeed, choosing opens $U \subset M$ over which we can find (smooth) local frames $e = (e_1, \ldots, e_p)$ of E and $f = (f_1, \ldots, f_q)$ of F, one can form the direct sum local frame

$$e \oplus f = (e_1, \ldots, e_p, f_1, \ldots, f_q)$$

and we consider the smooth structure on $E \oplus F$ which makes all the local frames of type $e \oplus f$ smooth.

This procedure of extending operations between vector spaces to operations between vector bundles is rather general. In some cases however, one can further take advantage of the actual operation one deals with and obtain "more concrete" descriptions. This is the case also with the direct sum operation. Indeed, recall that for any two vector spaces V and W, their direct sum $V \oplus W$ can be described as the set-theoretical product $V \times W$ with the vector space operations

$$(v, w) + (v', w') = (v + v', w + w'), \lambda \cdot (v, w) = (\lambda \cdot v, \lambda \cdot w)$$

(the passing to the notation $V \oplus W$ indicates that we identify the elements $v \in V$ with $(v, 0) \in V \times W$, $w \in W$ with $(0, w) \in V \times W$, so that an arbitrary element (v, w) can be written uniquely as $v + w$ with $v \in V$, $w \in W$). Hence one can just define $E \oplus F$ as the submanifold of $E \times F$

$$E \times_M F := \{(e, f) \in E \times F : p_E(e) = p_F(f)\}.$$

The condition (10) is clearly satisfied (and specify the vector space structure on the fibers) and is not difficult to see that the resulting $E \oplus F$ is a vector bundle over M. Note that the space of sections of $E \oplus F$ coincides with the direct sum $\Gamma(E) \oplus \Gamma(F)$.

Duals: Let us now look at the operation that associates to a vector space V its dual V^*. Starting with a vector bundle E over M, its dual E^* is another vector bundle over M with the property that

$$(E^*)_x = (E_x)^*$$

for all $x \in M$. Again, this determines E^* as a set and its projection into M. Moreover, using dual basis, we see that any smooth local frame $e = (e_1, \ldots, e_r)$ of E induces a local frame e^* for E^* and we can invoke again Exercise 4.9 to obtain the smooth structure of E^*.

Hom-bundles: Next we look at the operation that associates to two vector spaces V and W the vector space $\mathrm{Hom}(V, W)$ consisting of all linear maps from V to W. Given now two vecctor bundles E and F over M, we form the new vector bundle $\mathrm{Hom}(E, F)$ over M with fibers

$$\mathrm{Hom}(E, F)_x = \mathrm{Hom}(E_x, F_x).$$

And, again, we see that local frames of E and F induce a local frame of $\mathrm{Hom}(E, F)$, and then we obtain a canonical smooth structure on the hom-vector bundle. Note that a section of $\mathrm{Hom}(E, F)$ is the same thing as a morphism $u : E \to F$ of vector bundles over M. Hence Lemma 4.5 identifies sections of $\mathrm{Hom}(E, F)$ with $C^\infty(M)$-linear maps $\Gamma(E) \to \Gamma(F)$.

Of course, when F is the trivial vector vector bundle of rank 1 ($F = M \times \mathbb{R}$), we recover the dual of E:

$$E^* = \mathrm{Hom}(E, M \times \mathbb{R}).$$

Hence Lemma 4.5 identifies the sections of E^* with the dual of $\Gamma(E)$ as an $C^\infty(M)$-module.

Tensor products: One proceeds similarly for the tensor product operation on vector spaces. Since we work with finite dimensional vector spaces, this operation can be expressed using duals and homs:

$$V \otimes W = \mathrm{Hom}(V^*, W).$$

(where for $v \in V$, $w \in W$, the tensor $v \otimes w$ is identified with (or stands for) the linear map

$$V^* \to W, \quad \xi \mapsto \xi(v)w.)$$

Other operations: Similar to taking the dual of a vector space one can consider operations of type

$$V \mapsto S^k(V^*), \ (\text{or: } V \mapsto \Lambda^k(V^*))$$

which associate to a vector space V the space of all k-multilinear symmetric (or: anti-symmetric) maps $V \times \ldots \times V \to \mathbb{R}$. Again, one has to remember/notice that any frame of V induces a frame of S^kV^* (or: Λ^kV^*). Slightly more generally, one can consider operations of type

$$(V, W) \to S^k(V^*) \otimes W \ (\text{or: } (V, W) \mapsto \Lambda^k(V^*) \otimes W)$$

which associate to a pair (V, W) of vector spaces the space of k-multilinear symmetric (or: anti-symmetric) maps on V with values in W and then one obtains similar operations on vector bundles. Note the following generalization of Exercise 4.9:

Exercise 4.12. Show that, for any two vector bundles E and F over M and $k \geq 1$ integer, there is a 1-1 correspondence between:

- sections u of $S^kE^* \otimes F$.
- symmetric maps

  $$u_* : \underbrace{\Gamma(E) \times \ldots \times \Gamma(E)}_{k-\text{times}} \to \Gamma(F)$$

  which is $C^\infty(M)$-linear in each argument.

Similarly for sections of $\Lambda^kE^* \otimes F$ and antisymmetric maps as above.

Pull-backs Another important operation with vector bundles, but which does not fit in the previous framework, is the operation of taking pull-backs. More precisely, given a smooth map

$$f : M \to N,$$

starting with any vector bundle E over N, one can pull-it back via f to a vector bundle f^*E over M. Fiberwise,

$$(f^*E)_x) = E_{f(x)}$$

for all $x \in M$. One can use again Exercise 4.9 to make f^*E into a vector bundle; the key remark is that any section s of E induces a section f^*s of F^*E by

$$(f^*s)(x) := s(f(x))$$

and similalrly for local sections and local frames.

Note that, when $f = i : M \hookrightarrow N$ is an inclusion of a submanifold M of N, then i^*E is also denoted $E|_N$ and is called the restriction of E to M.

4.13. Differential forms with coefficients in vector bundles: Vector bundles also allows us to talk about more general differential forms: with coefficients. The standard differential forms are those with coefficeints in the trivial vector bundle of rank 1. Recall here that the space of (standard) differential forms of degree p on a manifold M, $\Omega^p(M)$, is defined as the space of sections of the bundle $\Lambda^p T^*M$. Equivalently, a p-form on M is the same thing as a $C^\infty(M)$-multilinear, antisymmetric map

$$\omega : \underbrace{\mathcal{X}(M) \times \ldots \times \mathcal{X}(M)}_{p \text{ times}} \to C^\infty(M), \tag{11}$$

where $\mathcal{X}(M)$ is the space of vector fields on M. Such p-forms can be written locally, over the domain U of a coordinate chart $(U, \chi_1, \ldots, \chi_n)$ as:

$$\omega = \sum_{i_1,\ldots,i_p} f^{i_1,\ldots,i_p} d\chi_{i_1} \ldots d\chi_{i_p}, \tag{12}$$

with $f^{i_1,\ldots,i_p}$-smooth functions on U.

Assume now that E is a vector bundle over M. We define the space of E-valued p-differential forms on M

$$\Omega^p(M; E) = \Gamma(\Lambda^p T^*M \otimes E).$$

As before, an element $\omega \in \Omega^p(M; E)$ can be thought of as a $C^\infty(M)$-multilinear antisymmetric map

$$\omega : \underbrace{\mathcal{X}(M) \times \ldots \times \mathcal{X}(M)}_{p \text{ times}} \to \Gamma(E). \tag{13}$$

Also, locally, with respect to a coordinate chart $(U, \chi_1, \ldots, \chi_n)$, one can write

$$\omega = \sum_{i_1,\ldots,i_p} d\chi_{i_1} \ldots d\chi_{i_p} \otimes e^{i_1,\ldots,i_p}. \tag{14}$$

with $e^{i_1,\ldots,i_p}$ local sections of E (defined on U). Using also a local frame $e = \{e_1, \ldots, e_r\}$ for E, we obtain expressions of type

$$\sum_{i_1,\ldots,i_p,i} f_i^{i_1,\ldots,i_p} dx_{i_1} \ldots dx_{i_p} \otimes e_i.$$

Recall also that

$$\Omega(M) = \bigoplus_p \Omega^p(M)$$

is an algebra with respect to the wedge product: given $\omega \in \Omega^p(M)$, $\eta \in \Omega^q(M)$, their wedge product $\omega \wedge \eta \in \Omega^{p+q}(M)$, also denoted $\omega\eta$, is given by

$$(\omega \wedge \eta)(X_1, \ldots, X_{p+q}) = \sum_\sigma sign(\sigma) \omega(X_{\sigma(1)}, \ldots, X_{\sigma(p)}) \cdot \eta(X_{\sigma(p+1)}, \ldots, X_{\sigma(p+q)}), \tag{15}$$

where the sum is over all (p, q)-shuffles σ, i.e. all permutations σ with $\sigma(1) < \ldots < \sigma(p)$ and $\sigma(p+1) < \ldots < \sigma(p+q)$. Although this formula no longer makes sense when ω and η are both E-valued differential forms, it does make sense when one of them is E-valued and the other one is a usual form. The resulting operation makes

$$\Omega(M, E) = \bigoplus_p \Omega^p(M, E)$$

into a (left and right) module over $\Omega(M)$. Keeping in mind the fact that the spaces Ω are graded (i.e are direct sums indexed by integers) and the fact that the wedge products involved are compatible with the grading (i.e. $\Omega^p \wedge \Omega^q \subset \Omega^{p+q}$), we say that $\Omega(M)$ is a graded algebra and $\Omega(M, E)$ is a graded bimodule over $\Omega(M)$. As for the usual wedge product of forms, the left and right actions are related by[1]

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega \quad \forall\, \omega \in \Omega^p(M), \eta \in \Omega^q(M, E).$$

4.2 Principal bundles

And here are some basics on principal bundles.

Definition 4.14. *Let M be a manifold and G a Lie group. A principal G-bundle over M consists of*

- *a manifold P together with a right action of G on P*

$$P \times G \to P, \ (p, g) \mapsto pg$$

- *a surjective map $\pi_P : P \to M$ which is G-invariant (i.e. $\pi_P(pg) = \pi_P(p)$ for all p and g).*

satisfying the following local triviality condition: for each $x_0 \in M$, there exists an open neighborhood U of x_0 and a diffeomorphism

$$\Psi : \pi_P^{-1}(U) \to U \times G$$

which maps each fiber $\pi_P^{-1}(x)$ to the fiber $\{x\} \times G$ and which is G-equivariant.

The G-equivariance means that

$$\Psi(pg) = \Psi(p)g$$

for all $p \in P$, $g \in G$, where the right action of G on $U \times G$ is

$$(x, a)g = (x, ag). \tag{16}$$

As for vector bundles, there are several concepts/constructions to be discussed.

4.15. Fibers: Given a principal G-bundle $\pi : P \to M$ one can talk about the fiber $P_x := \pi_P^{-1}(x)$ above any point $x \in M$. Since π_P is G-invariant, each P_x is stable under the action of G, hence each fibers is a (right) G-space. The axioms imply that any such fiber is, as a G-space, isomorphic to G itself (endowed with

[1] Important: this is the first manifestation of what is known as the "graded sign rule": in an formula that involves graded elements, if two elements a and b of degrees p and q are interchanged, then the sign $(-1)^{pq}$ is introduced

the right action coming from the multiplication of G); more precisely, choosing $p \in P_x$, one has an induced diffeomorphism

$$\phi_p : G \to P_x, \quad \phi_p(g) = pg$$

which is compatible with the right action of G:

$$\phi_p(gh) = \phi_p(g)h$$

for all $g, h \in G$. Keep in mind however that the isomorphism between P_x and G is not canonical: it depends on the choice of an element $p \in P_x$. Actually, P may fail to be diffeomorphic to $M \times G$.

4.16. Basic example: the frame bundle of a vector bundle: Start with a vector bundle E over M, of rank r. One forms the frame bundle associated to E

$$\mathrm{Fr}(E) = \{(x, u) : x \in M, u - \text{ a frame of } E_x\}.$$

Proceeding exactly as in the case of $\mathrm{Fr}(M)$ (which is obtained in teh particular case when $E = TM$), we find that $\mathrm{Fr}(E)$ is a principal GL_r-bundle over M, whose fiber over $x \in M$ is the space $\mathrm{Fr}(E_x)$ of frames on the vector space E_x.

4.17. Isomorphisms: Isomorphisms $F : P \to Q$ between two principal G-bundles over M are diffeomorphisms which map each fiber P_x into the fiber Q_x (or, in equation form: $\pi_Q(F(p)) = \pi_P(p)$) and which are G-equivariant, i.e. $F(pg) = F(p)g$ for all $p \in P$, $g \in G$. Of course, one could also try to define the notion of "morphisms between principal G-bundles" by requiring F to be smooth instead of being a diffeomorphism. However, it is not difficult to see that any such F must be a diffeomorphism.

4.18. The trivial principal G-bundle: For any Lie group G, $M \times G$ is itself a principal G-bundle over M, with the action given by (16) and with π_P being the projection on the first factor. This is called the trivial principal G-bundle.

We see that the local triviality axiom says that, around each point of M, P is isomorphic to the trivial principal G-bundle.

4.19. Sections: Sections of a principal G-bundle P are smooth maps $\sigma : M \to P$ which send each $x \in M$ into an element $\sigma(x)$ in the fiber P_x (equation-wise: $\pi_P \circ \sigma = \mathrm{Id}_M$). Similalry one can talk about local sections.

Note that the choice of a section σ of P is equivalent to a trivialization of P, i.e. the choice of an isomorphisms between P and $M \times G$. Indeed, because of the G-invariance of isomorphisms $F : M \to G \to P$ we see that any such F is determined by what it does on elements of type $(x, 1)$, i.e. by the values

$$\sigma(x) := F(x, 1) \in P_x$$

(then $F(x, g) = \sigma(x)g$ for all x and g).

In particular, the local triviality axiom says that, around each point of M, π admits local sections. Actually, we see that the local triviality condition is equivalent to the condition that π_P is a submersion! Indeed, using the local form of a submersion, we see that any submersion admits such local sections.

4.20. More general morphisms: To have a more sensible notion of morphisms one allows the change of the base manifold and of the group. More precisely, let $\rho : G_1 \to G_2$ be a morphism of Lie groups and let $f : M_1 \to M_2$ be a smooth map. Given principal G_i-bundles $\pi_i : P_i \to M_i$, $i \in \{1, 2\}$, one can talk about ρ-morphisms that cover f- which are smooth maps

$$F : P_1 \to P_2$$

with the property that

$$\pi_2(F(p)) = f(\pi_1(p))$$

for all $p \in P_1$ (i.e. F sends each fiber $(P_1)_x$ into $(P_2)_{f(x)}$), and which is ρ-equivariant in the sense that

$$F(pg) = F(p)\rho(g)$$

for all $p \in P_1$, $g \in G_1$.

4.21. Remark on the definition: So, what can we say for a general principal G-bundle $\pi : P \to M$? Here are some consequences of the axioms:

- π is a G-invariant submersion.
- the action of G on P is free and proper.
- the action of G on P restricts to an action on each fiber P_x; each two points p, q in the same fiber P_x determine an unique element $g \in G$ with the property that $p = qg$. We will denote this element by $[p : q]$ (read: p divided by q).
- Introducing the fibered product

  $$P \times_M P := \{(p, q) : p, q \in P, \pi(p) = \pi(q)\}$$

  (a submanifold of $P \times P$!), one obtains an operation

  $$P \times_M P \to G, \quad (p, q) \mapsto [p : q].$$

  This gives rise to a bijection (actually a diffeomorphism)

  $$P \times_M P \to P \times G, \quad (p, q) \to (p, [p : q]),$$

  whose inverse is simply

  $$P \times G \to P \times_M P, \quad (p, g) \mapsto (p, pg). \tag{17}$$

One can slightly change the point of view and take some of these consequences as axioms- giving rise to slightly different (but equivalent) definition of the notion of principal G-bundle. For instance, we have already remarked that the local triviality axiom is equivalent to the condition that π is a surjective submersion. Here is another possibility:

Exercise 4.22. Assume that G acts on the manifold P from the right and that $\pi : P \to M$ is a surjective submersion. Show that P is a principal G-bundle if and only if (17) is well-defined and is a diffeomorphism.

Yet another variation arises by realizing that the entire information is contained in P and the action of G. More precisely, one can show that: starting with any manifold P together with a right action of G on P which is free and proper, then the quotient P/G carries a natural smooth structure, uniquely determined by the condition that the quotient map $p : P \to P/G$ is a submersion, and P becomes a principal G-bundle over P/G. Hence a principal G-bundle over a manifold M can be thought of such a P (endowed if a free and proper action of G) together with a diffeomorphism between P/G and M.

4.23. Pull-backs: As for vector bundles, one can pull-back principal bundles along smooth maps: given $f : M \to M$ smooth and a principal G-bundle P over N, one forms

$$f^*(P) := M \times_N P := \{(x, p) \in M \times P : f(x) = \pi(p)$$

with projection the standard projection on the first coordinate (built so that its fiber at $x \in M$ is basically just $P_{f(x)}$), with the right action of G:

$$(x, p)g = (x, pg).$$

It is not difficult to see that f^*P becomes a principal G-bundle over P.

4.24. Push-forwards along group homomorphisms: The other operations on vector bundles such as taking duals, direct sums, etc, do not have an immediate analogue for principal bundles (think e.g. that the rank of the direct sum of two vector bundles is the sums $p + q$ of their ranks; hence, at the level of the associated frame bundles, one has to deal with three groups at the same time: GL_p, GL_q and GL_{p+q}). The main operation on principal bundles that is related to such operations on vector bundles is the push-forward along morphisms of Lie groups: a morphism $\rho : G \to \tilde{G}$ of Lie groups induces an operation

$$\rho_* : \mathrm{Bun}_G(M) \to \mathrm{Bun}_{\tilde{G}}(M)$$

which associates to a principal G-bundle $\pi : P \to M$ a principal $\tilde{G}$-bundle $\rho_*(P) \to M$. Explicitely,

$$\rho_*(P) := (P \times \tilde{G})/G$$

is the quotient of $P \times G$ modulo the action of g given by

$$(p, a)g = (pg, a\rho(g)).$$

With the same arguments as before, this is a smooth manifold. It comes with a projection

$$\tilde{\pi} : \rho_*(P) \to M, \; [p, a] \mapsto \pi(p)$$

and an action of $\tilde{G}$ given by

$$[p, a] \cdot b = [p, ab].$$

It is not difficult to check that $\rho_*(P)$ becomes a principal $\tilde{G}$-bundle over M.

Exercise 4.25. Consider the group homomorphism

$$\rho : GL_r \to GL_r, \quad \rho(A) = {}^t(A^{-1}).$$

Show that for any vector bundle E, the frames bundles associated to E and to itse dual are related by

$$\mathrm{Fr}(E^*) \cong \rho_*(\mathrm{Fr}(E)).$$

Exercise 4.26. Show that if $\pi_i : P_i \to M$ are principal G_i-bundles, for $i \in \{1, 2\}$, then

$$P_1 \times_M P_2$$

carries a natural structure of principal $G_1 \times G_2$-bundle.

Assume now that $P_i = \mathrm{Fr}(E_i)$ is the frame bundles associated to a vector bundles E_i of rank r_i (hence also $G_i = GL_{r_i}$), $i \in \{1, 2\}$. Consider $r = r_1 + r_2$ and the group homomorphism

$$\rho : GL_{r_1} \times GL_{r_2} \to GL_r, \quad (A, B) \mapsto A \oplus B := \mathrm{diag}(A, B).$$

Show that the frame bundle associated to $E_1 \oplus E_2$ is isomorphic to the push-forward of $P_1 \times_M P_2$ via ρ.

4.27. Reduction of the structure group: Of particular interest is the case when $\rho : G \to \tilde{G}$ is an inclusion of a subgropup G of $\tilde{G}$. Given a principal $\tilde{G}$ -bundle $\tilde{P} \to M$, the question of whether it comes from (i.e. is isomorphic to) the pushforward of a principal G-bundle $P \to M$ reflects the complexity of $\tilde{P}$. If the answer is positive, and G is rather small, it means that $\tilde{P}$ is, in its essence (i.e. as a principal bundle), simple. For instance, if the answer is positive for the trivial subgroup $G = \{e\}$, it simply means that $\tilde{P}$ is trivializable.

In general, given a principal $\tilde{G}$-bundle $\tilde{P}$ and a subgroup $G \subset \tilde{G}$, a reduction of $\tilde{P}$ to G is (by definition) choice of a principal G-bundle $P \to M$ and of an isomorphism between $i_*(P)$ and $\tilde{P}$, where $i : G \hookrightarrow \tilde{G}$ is the inclusion.

Lemma 4.28. *Up to isomorphisms, a reduction of a principal $\tilde{G}$-bundle $\tilde{\pi} : \tilde{P} \to M$ to a subgroup $G \subset \tilde{G}$ is the same thing as the choice of a subspace*

$$P \subset \tilde{P}$$

which is G-invariant and, together with the restriction $\pi := \tilde{\pi}|_P : P \to M$, is a principal G-bundle. In turn, the choice of such a P is equivalent to the choice of a (smooth) section σ of the resulting projection $\tilde{P}/G \to M$.

Proof. To come. □

Remark 4.29. Although it is rather simple, the last part of the lemma is quite important conceptually. One reason is that it allows us to interpret reductions as (smooth) sections. This is important since functions are easier to handle. Another use of the previous corollary comes from general properties a locally trivial fiber bundles. By a locally trivial fiber bundle with fiber N (N is a manifold) we mean a submersion $\pi : Q \to M$ with the property that any point $x \in M$ has a neighborhood U with the property that $\pi^{-1}(U)$ is diffeomorphic to $U \times N$, by a diffeomorphism that sends the fibers $\pi^{-1}(x)$ to the fibers $\{x\} \times N$. A basic but non-trivial property of locally trivial fiber bundles is that, if the fiber N is contractible, then it automatically admits global sections. Of course, this can be applied to the projection $\tilde{P}/G \to M$ to deduce:

Corollary 4.30. *If the quotient $\tilde{G}/G$ is contractible, then any principal $\tilde{G}$-bundle admits a reduction to G.*

4.3 Vector bundles versus principal bundles

We have seen that any vector bundle E of rank r over M gives rise to a principal GL_r-bundle over M- the frame bundle $\mathrm{Fr}(E)$ of E. We now describe a reverse construction.

What we need is a principal G-bundle $\pi : P \to M$ as well as a representation V of G, i.e. a vector space V together with a group homomorphism $\rho : G \to GL(V)$ from G to the group of all linear automorphisms of V. To such a data we will associate a vector bundle

$$E(P, V)$$

over M- called the vector bundle obtained by attaching to P the fiber V, via the representation ρ (a more faithful notation would be $E(P, G, V, \rho)$, but G and ρ are usually clear from the context).

First note that the representation encodes a linear action of G on V from the left:

$$G \times V \to V,\ (g, v) \mapsto g \cdot v := \rho(g)(v).$$

One then defines

$$E(P, V) := (P \times V)/G$$

the quotient of $P \times V$ modulo the action of G given by

$$(p, v)g\cdot := (pg, g^{-1}v).$$

Denotiong by $[p, v] \in E(P, V)$ the element induced by $(p, v) \in P$, we see that the identifications that are made in the quotient are

$$[pg, v] = [p, gv]$$

for all $p \in P$, $v \in V$, $g \in G$. We endow $E(P, V)$ with the projection

$$\tilde{\pi} : E(P, V) \to M, \quad \tilde{\pi}(p, v) = \pi(v).$$

Note that the fiber of $\tilde{\pi}$ above an arbitrary point $x \in M$,

$$E(P, V)_x = \{[p, v] : p \in P_x, v \in V\},$$

has a natural structure of vector space obtained by requiring

$$[p, v] + [p, w] = [p, v + w], \lambda[p, v] = [p, \lambda v].$$

Here it is important to note that, for arbitrary element $[p, v], [q, w]$ which sit in the same fiber, since p and q sit in the same fiber of π one can find $g \in G$ such that $p = qg$ and then $[q, w] = [p, gw]$- so that the sum between $[p, v]$ and $[q, w]$ can be defined as $[p, v + gw]$ (check that these are well-defined and do not depend on choices). Rephrasing a bit: given $x \in M$, the choice of an elemement $p \in P_x$ induces an bijection

$$\phi_p : V \to E(P, V)_x, \quad \phi_p(v) = [p, v];$$

this bijection is used to transfer the vector space structure from V to $E(P, V)_x$, and the result does not depend on the choice of $p \in P_x$. Of course, $E(P, V)_x$ is isomorphic to V as a vector space- but it is important to note that one needs to choose a point $p \in P_x$ in order to obtain such an isomorphism. Hence, when looking at $E(P, V)$ as a vector bundle over M, it is in general not the trivial vector bundle $M \times V$.

To describe the smooth structure on $E(P, V)$ one can proceed in several ways. One is to use again the fact that the quotient N/G of a manifold N modulo a free and proper action of a Lie group G on N carries itself a natural structure of smooth manifold (uniquely determined by the requirement that the quotient map $N \mapsto N/G$ is a submersion); one just needs the simple remark that, since the action of G on P is free and proper, so is the action on $P \times V$. Another way is to use local trivializations of P:

$$\Psi : \pi^{-1}(U) \to U \times G$$

and note that any such trivialization induces a trivialization of $E(P, V)$:

$$\tilde{\Psi} : \tilde{\pi}^{-1}(U) \to U \times V, \ p \mapsto (\pi(p), g_{\Psi(p)}v),$$

where $g_{\Psi(p)} \in G$ is the second component of $\Psi(p)$. Of course, similar to what we have already seen, the local trivializations $\tilde{\Psi}$ will serve as charts of a smooth atlas for $E(P, V)$.

4.31. The bijection between vector bundles and principal bundles: The previous construction is particularly interesting in the case when

$$G = GL_r, \ V = \mathbb{R}^r, \ \rho = \mathrm{Id},$$

when it associates to a principal GL_r bundle P the vector bundle $E(P, \mathbb{R}^r)$. Let us apply this to the frame bundle

$$P = \mathrm{Fr}(E)$$

of a vector bundle E of rank r over M. In this case, since an element $p \in P_x$ is the same thing as the choice of an isomorphism $i_p : \mathbb{R}^r \to E_x$, we see that that a pair $[p, v] \in E(P, \mathbb{R}^r)_x$ can be identified with $i_p(v) \in E_x$. In other words, one has an isomorphism of vector bundles over M:

$$E(P, \mathbb{R}^r) \xrightarrow{\sim} E, \quad [p, v] \mapsto i_p(v).$$

Putting everything together, we find:

Theorem 4.32. *The constructions which associate:*

- *to a vector bundle E its frame bundle $Fr(E)$*
- *to a principal GL_r-bundle P the vector bundle $E(P, \mathbb{R}^r)$*

define a 1-1 correspondence between (isomorphism classes of) vector bundles of rank r over M and principal GL_r-bundles over M.

4.33. Sections of the associated vector bundle: Let us now return to the general discussion and describe the space of sections of the vector bundle $E(P, V)$ associated to a principal G-bundle $\pi : P \to M$ and a representation

$\rho : G \to GL(V)$. We consider the (vector) space $C^\infty(P, V)$ of all smooth functions $f : P \to V$. We say that f is G-equivariant if

$$f(pg) = g^{-1} f(p)$$

for all $p \in P$, $g \in G$; we denote by $C^\infty(P, V)^G$ the resulting (vector) space of G-invariant functions. Equivalently, the actions of G on P and V we introduce the action of G on $C^\infty(P, V)$ given by

$$(g \cdot f)(p) := g f(pg)$$

and then

$$C^\infty(P, V)^G = \{f \in C^\infty(P, V) : g \cdot f = f \ \ \forall\, g \in G\}.$$

Note that any G-equivariant function f induces a section of $E(P, V)$: first of all it induces

$$(\mathrm{id}, f) : P \to P \times V, \ p \mapsto (p, f(p))$$

which is equivariant (where the right action of G on $P \times V$ is precisely the one used to define $E(P, V)$); hence, passing to the quotient modulo G, it induces a map

$$s_f : M \to E(P, V).$$

Lemma 4.34. *If $E = E(P, V)$ is the associated vector bundle, then one has a bijection*

$$C^\infty(P, V)^G \xrightarrow{\sim} \Gamma(E), \quad f \mapsto s_f.$$

Exercise 4.35. Now given a section s of the associated vector bundle $E(P, V)$, construct a G-invariant function f_s and check that the constructions are inverse to each other.

4.36. Differential forms with coefficients in the associated vector bundle: The previous discussion can be generalized further, obtaining a description directly in terms of P and V of the space of forms on M with coefficients in the vector bundle $E(P, V)$. We need some terminology:

- as above, there is an action of G on $\Omega^k(P, V)$ obtained by combininig the action of G on P and one V:

  $$(g \cdot \omega)(-) := \rho(g)(R_g^*(\omega)(-)),$$

  where $\rho : G \to GL(V)$ is the given representation and R_g^* is the pull-back of forms via the right multiplication by g, $R_g : P \to P$, $R_g(p) = pg$. Hence, on vectors $X_p^i \in T_pP$, $1 \leq i \leq k$:

  $$(g \cdot \omega)_p(X^1, \ldots, X^k) = \rho(g)(\omega_{pg}((dR_g)_p(X^1), \ldots, (dR_g)_p(X^k))).$$

  We say that $\omega \in \Omega^k(P, V)$ is G-invariant if $g \cdot \omega = \omega$ or, equivalently,

  $$R_g^*(\omega) = \rho(g^{-1})(\omega)$$

  for all $g \in G$. We denote by $\Omega^k(P, V)^G$ the space of invariant differential forms.

- we say that a tangent vector $X \in TP$ is vertical if it is tangent to the fiber of π or, equivalently, if $(d\pi)(X) = 0$. We denote by $T^{\vee}P$ the space of vertical vectors (a vector sub-bunlde of TP). Note that for each $p \in P_x$ ($x \in M$)

$$l_p : G \to P, \ l_p(g) = pg$$

is a diffeomorphism between G and P_x. It is natural to introduce the maps

$$a_p : \mathfrak{g} \to T_pP, \ a_p(v) = (dl_p)_I(v) = \left.\frac{d}{dt}\right|_{t=0} p\exp(tv)$$

which, when varrying p, give rise to a map

$$a : \mathfrak{g} \to \mathcal{X}(P)$$

which is called the infinitesimal action of $\mathfrak{g}$ on P. It is clear that each a_p is an isomorphism between $\mathfrak{g}$ and $T_p^{\vee}$.

Exercise 4.37. Check that a defines an isomorphism of vector bundles over P, from the trivial vector bundle with fiber $\mathfrak{g}$ to $T^{\vee}P$.

- we say that a differential form $\omega \in \Omega^k(P, V)$ is horizontal if $i_X(\omega) = 0$ for all vertical vectors X or, equivalently, if

$$i_{a(v)}(\omega) = 0 \quad \forall \ v \in \mathfrak{g}.$$

We denote by $\Omega^k(P, V)_{\mathrm{hor}}$ the space of horizontal differential forms.

- we say that a differential form $\omega \in \Omega^k(P, V)$ is basic if it is horizontal and G-invariant. Denote by $\Omega^k(P, V)_{\mathrm{bas}}$ the resulting space.

Proposition 4.38. *If $E = E(P, V)$ is the associated vector bundle, then one has a linear isomorphism*

$$\pi^{\bullet} : \Omega^k(M, E) \xrightarrow{\sim} \Omega^k(P, V)_{bas}.$$

Proof. We start with two remarks. The first one is that that the pull-back π^*E of the vector bundle E via the projection $\pi : P \to M$ is, as a vector bundle over P, isomorphic to the trivial vector bundle $P \times V$, by a canonical isomorphism

$$i : P \times V \to \pi^*(E).$$

At the fiber at $p \in P$, this is simply

$$i_p : V \to E_{\pi(p)}, \ v \mapsto [p, v].$$

The second remark is that the usual pull-back of forms along $\pi : P \to M$ makes sense for forms with coefficients, giving rise to

$$\pi^* : \Omega^k(M, E) \to \Omega^k(P, \pi^*E).$$

Explicitely, for $X^1, \ldots, X^k \in T_pP$,

$$\pi^*(\omega)(X^1, \ldots, X^k) = \omega((d\pi)_p(X^1), \ldots, (d\pi)_p(X^k)) \in E_{\pi(p)} = (\pi^*E)_p. \tag{18}$$

Combining the isomorphism from the previous remark, we consider

$$\pi^{\bullet} := i^{-1} \circ \pi^* : \Omega^k(M, E) \xrightarrow{\sim} \Omega^k(P, V).$$

The rest of the proof is left to the reader in the form of a (guided) exercise.

Exercise 4.39.

1. Check directly that $\pi^\bullet$ actually takes values in $\Omega^k(P,V)_{\text{bas}}$ (for instance, since $(d\pi)$ kills the vertical vector fields, all forms of type $\pi^*\omega$ are horizontal).

2. The map $\pi^\bullet$ is also easily seen to be injective (since the differential of π is surjective).

3. One still has to show that each $\eta \in \Omega^k(P,V)_{\text{bas}}$ can be written as $\pi^*\omega$ for some ω. Explicitely, one wants

$$\omega_{\pi(p)}((d\pi)_p(X^1),\ldots,(d\pi)_p(X^k)) = i_p(\eta_p(X^1,\ldots,X^k)).$$

   for all $p \in P$ $X^i \in T_pP$. But this forces the definition of ω: for $x \in M$ and $V^i \in T_xM$, choosing $p \in P$ with $\pi(p) = x$ and choosing $X^i \in T_pP$ with $(d\pi)_p(X^i) = V^i$, we must have

$$\omega_x(V^1,\ldots,V^k) = i_p(\eta_p(X^1,\ldots,X^k)).$$

4. What is left to check is that this definition does not depend on the choices we made. First of all, once $p \in \pi^{-1}(x)$ is chosen, the formula does not depend on the choice of the X^i's- this follows from the fact that η is horizontal. The independence of the choice of the point p in $\pi^{-1}(x)$ follows from the G-invariance of η and the fact that any other $q \in \pi^{-1}(x)$ can be written as $q = pg$ for some $g \in G$.

□

Finally, let us also discuss the compatibility of this construction with the push-forward operation along group homomorphisms.

Proposition 4.40. *Let $\tilde{P} = \rho_*(P)$- the pushforward of a principal G-bundle $P \to M$ along a morphism of Lie groups $\rho : G \to \tilde{G}$. Consider also a representation V of $\tilde{G}$; using ρ, we interpret V also as a representation of G. Then the resulting vector bundles $E(\tilde{P},V)$ and $E(P,V)$ are isomorphic.*

Proof. Direct checking (the identifications are tautological!). □

As a warm up for the next lecture, do the following exercise:

Exercise 4.41. Prove that:

1. every principal bundle over a contractible manifold is trivial;

2. two Riemannian metrics on R^n are isomorphic (as $O(n)$-structures) if and only if they are isometric, whereas the corresponding principal $O(n)$-bundles are always isomorphic.

4.4 Applications to G-structures

Of course, the main examples that are relevant in the discussion of G-structures on M are

- for vector bundles: TM.
- for principal bundles: $\mathrm{Fr}(M)$ (principal GL_n-bundle).

The two correspond to each other by the bijection of Theorem 4.32.

Furthermore, if we are given a G-structure, encoded in

$$\mathcal{S} \subset \mathrm{Fr}(M),$$

then $\mathcal{S}$ is itself a principal G-bundle. Hence one can apply the constructions/notions from principal bundles to the study of G-structures. Here are some illustrations of this principle.

4.42. A first characterization of principal bundles that come from G-structures: Of course, one interesting question is to understand which principal bundles come from G-structures. The first answer is related to another natural question: given a G-structure $\mathcal{S}$ on M (hence $G \subset GL_n$), seing $\mathbb{R}^n$ as a representation of G we can form the vector bundle $E(\mathcal{S}, \mathbb{R}^n)$; what is it?

Proposition 4.43. *Assume that $G \subset GL_n$ is a closed subgroup. Then a principal G-bundle P over M is isomorphic to a G-structure $\mathcal{S}$ if and only if the associated vector bundle $E(P, \mathbb{R}^n)$ is isomorphic to TM.*

4.44. Infinitesimal automorphisms of G-structures: Viewing a G-structure $\mathcal{S}$ on M as a principal G-bundle, we know that every representation V of G induces a vector bundle $E(\mathcal{S}, V)$ over M. Half of the previous proposition tells us that, for $V = \mathbb{R}^n$, we recover TM. Another interesting representation of G is its Lie algebra $\mathfrak{g}$, with the adjoint representation. What is the resulting vector bundle $E(\mathcal{S}, \mathfrak{g})$? We will show that it is related to the bundle of infinitesimal autommorphisms of the G-structure.

To explain this, note that any vector space V endowed with a linear G-structure $\mathcal{S}$ has an associated group of automorphisms

$$\mathrm{Aut}(\mathcal{S}) \subset GL(V)$$

and then also an associated Lie algebra (of infinitesimal automorphisms)

$$\mathrm{aut}(\mathcal{S}) \subset \mathrm{End}(V),$$

consisting of those linear maps $A : V \to V$ with the property that

$$\exp(A) := \mathrm{Id} + \frac{A}{1!} + \frac{A^2}{2!} + \ldots \in GL(V)$$

is an automorphism of $(V, \mathcal{S})$. Of course, for $\mathbb{R}^n$ with the standard G-structure one receovers the Lie group G and its Lie algebra $\mathfrak{g}$. Moreover, any $e \in \mathcal{S}$, interpreted as an isomorphism between $\mathbb{R}^n$ with standard G-structure and $(V, \mathcal{S})$, induces an isomorphism between the (infinitesimal) automorphisms of $(V, \mathcal{S})$ and G $(\mathfrak{g})$.

In the case of a G-structure $\mathcal{S}$ on a manifold M, one obtains the bundle of infinitesimal automorphisms of $\mathcal{S}$

$$\mathrm{aut}(\mathcal{S}) \subset \mathrm{End}(TM),$$

which is a vector bundle over M (actually a bundle of Lie algebras) whose fiber at $x \in M$ consists of infinitesimal automorphisms of $(T_xM, \mathcal{S}_x)$:

$$\mathrm{aut}(\mathcal{S})_x = \mathrm{aut}(T_xM, \mathcal{S}_x) \subset \mathrm{End}(T_xM).$$

Proposition 4.45. *For any G-structure $\mathcal{S}$ on M one has*

$$E(\mathcal{S}, \mathfrak{g}) \cong \mathit{aut}(\mathcal{S}).$$

Proof. □

4.46. G-structures as reductions: Lemma 4.28 tells us the G-structures are just reductions of the frame bundle $\mathrm{Fr}(M)$ to G. Hence the (rather trivial) last part of Lemma 4.28 gives us:

Corollary 4.47. *Given $G \subset GL_n$ and an n-dimensional manifold M, there is a 1-1 correspondence between G-structures $\mathcal{S}$ on M and (smooth) sections σ of the projection $Fr(M)/G \to M$.*

Remark 4.48. As we have already mentioned in the case of general principal bundles, the importance of this corollary comes from the is that it allows us to handle G-structures as (smooth) sections. This is particularly important if we want to linearize G-structures or study their deformations (just because, in principle, we know how to do that for functions).

Moving now to Corollary 4.30, one obtains a deeper consequence:

Corollary 4.49. *If the quotient GL_n/G is contractible, then any n-dimnesional manifold M admits a G-structure.*

The nicest illustration of this corollary is when $G = O(n)$; indeed, in this case one can show that the resulting quotient $GL_n/O(n)$ is contractible, hence we re-discover the fact that any manifold admits a Riemannian metric.

4.50. What is special about G-structures: the tautological form: Here is anotehr characterization of the principal bundles that come from G-structure, characterization that is extremely useful. And this brings us to one of the most important objects in the study of G-structures: the so called tautological form.

Definition 4.51. *Given a G-structure $\mathcal{S}$ on M, the tautological form (also called the Cartan form, or the soldering form) of $\mathcal{S}$ is the 1-form*

$$\theta_{\mathcal{S}} \in \Omega^1(\mathcal{S}, \mathbb{R}^n)$$

which associates to a tangent vector $X_u \in T_u\mathcal{S}$ ($u \in \mathcal{S}$) the vector

$$u^{-1}((d\pi)_u(X_u)) \in \mathbb{R}^n$$

(if u is a frame at x, then $(d\pi)_u(X_u) \in T_xM$, while u is interpreted as an isomorphism $u : \mathbb{R}^n \to T_xM$).

To describe the main (abstract) properties of θ_0 we will use the terminology from 4.36 on invariant, horizontal and basic forms. Furthermore, we will say that a form $\theta \in \Omega^1(P, V)$ (on a principal G-bundle P with coefficients in a representation V of G) is strictly horizontal if the kernel of each θ_p ($p \in P$) is precisely the space $T_p^{\vee}P$ of vertical vectors at p. It is straightforward to check that the tautological form is G-invariant and strictly horizontal. The main point is that this property, as well as the one from Lemma **??**, characterize the principal G-bundles that come from G-structures.

Proposition 4.52. *Let $G \subset GL_n$ be a closed subgroup and let $\pi : P \to M$ be a principal G-bundle. Then the following are equivalent:*

(i) P is (isomorphic to) the principal G-bundle of a G-structure on M.

(ii) there exists a G-invariant, strictly horizontal 1-form $\theta \in \Omega^1(P, \mathbb{R}^n)$.

Moreover, given θ as in (ii), the realization of P as a G-structure (i.e. the choice of the embedding $P \hookrightarrow Fr(M)$) can be done in such a way that θ becomes the tautological form $\theta_{\mathcal{S}}$.

The main message of the previous proposition is that, if we want to say something about G-structure that depends on more than just the principal bundle structure, then one has to use the tautological form $\theta_{\mathcal{S}}$ which, together with its properties, fully encodes the situation.

Proof. To come (or leave it as an exercise since it is easy). □

4.53. The relevance of the tautological forms to the equivalence problem: Here is another (but related) property that shows the importance of the tautological form. It shows that, when interested in the equivalence problem (deciding when two G-structures are isomorphic), one can forget about the base manifold and the G-structure and just concentrate on the manifold $\mathcal{S}$ and the tautological form on it.

Proposition 4.54. *Let $G \subset GL_n$ be a closed, connected, subgroup and assume that $\mathcal{S}_i$ is a G-structure on M_i, for $i \in \{1, 2\}$. Then the two G-structures are isomorphic if and only if there exists a diffeommorphism*

$$\Psi : \mathcal{S}_1 \to \mathcal{S}_2$$

with the property that

$$\Psi^*(\theta_{\mathcal{S}_2}) = \theta_{\mathcal{S}_1}.$$

Proof. To come (this is not so easy). □

5 Connections

5.1 Connections of vector bundles

Throughout this section E is a vector bundle over a manifold M. Unlike the case of smooth functions on manifolds (which are sections of the trivial line bundle!), there is no canonical way of taking derivatives of sections of (an arbitrary) E along vector fields. That is where connections come in.

Definition 5.1. *A connection on E is a bilinear map ∇*

$$\mathcal{X}(M) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X(s),$$

satisfying

$$\nabla_{fX}(s) = f\nabla_X(s), \ \nabla_X(fs) = f\nabla_X(s) + L_X(f)s,$$

for all $f \in C^\infty(M)$, $X \in \mathcal{X}(M)$, $s \in \Gamma(E)$.

Remark 5.2. In the case when E is trivial, with trivialization frame

$$e = \{e_1, \ldots, e_r\},$$

giving a connection on E is the same thing as giving an r by r matrix whose entries are 1-forms on M:

$$\omega := (\omega_i^j)_{i,j} \in M_r(\Omega^1(M)).$$

Given ∇, ω is defined by

$$\nabla_X(e_j) = \sum_{i=1}^{r} \omega_j^i(X) e_i.$$

Conversely, for any matrix ω, one has a unique connection ∇ on E for which the previous formula holds: this follows from the Leibniz identity (the last equation in the definition of connections).

Please be aware of our conventions: for the matrix $\omega = \{\omega_j^i\}_{i,j}$, the upper indices i count the rows, while the lower ones the columns:

$$\omega = \begin{pmatrix} \omega_1^1 & \ldots & \omega_r^1 \\ \ldots & \ldots & \ldots \\ \omega_1^r & \ldots & \omega_r^r \end{pmatrix}$$

The other convention (switching the rows and the columns) would correspond to considering the transpose matrix ${}^t(\omega)$; that convention is taken in some text-books and accounts for the sign changes between our formulas involving ω and the ones in those text-books.

5.3. Locality; connection matrices: Connections are local in the sense that, for a connection ∇ and $x \in M$,

$$\nabla_X(s)(x) = 0$$

for any $X \in \mathcal{X}(M)$, $s \in \Gamma(E)$ such that $X = 0$ or $s = 0$ in a neighborhood U of x. This can be checked directly, or can be derived from the remark that ∇ is a differential operator of order one in X and of order zero in f.

Locality implies that, for $U \subset M$ open, ∇ induces a connection ∇^U on the vector bundle $E|_U$ over U, uniquely determined by the condition

$$\nabla_X(s)|_U = \nabla^U_{X|_U}(s_U).$$

Choosing U the domain of a trivialization of E, with corresponding local frame $e = \{e_1, \ldots, e_r\}$, the previous remark shows that, over U, ∇ is uniquely determined by a matrix

$$\omega := (\omega_j^i)_{i,j} \in M_r(\Omega^1(U)).$$

This matrix is called the connection matrix of ∇ with respect to the local frame e and, when we want to emphasize this aspect, we write

$$\omega = \omega(\nabla, e) \in M_r(\Omega^1(U)) = \Omega^1(U, gl_r). \tag{19}$$

Proposition 5.4. *Any vector bundle E admits a connection.*

Proof. Start with a partition of unity η_i subordinated to an open cover $\{U_i\}$ such that $E|_{U_i}$ is trivializable. On each $E|_{U_i}$ we consider a connection ∇^i (e.g., in the previous remark consider the zero matrix). Define ∇ by

$$\nabla_X(s) := \sum_i (\nabla_{X|_{U_i}})(\eta_i s).$$

□

Exercise 5.5.

1. Let E^* be the dual bundle of the vector bundle E and suppose ∇ is a connection on E. For t a section of E^* and s a section of E define:

$$(\nabla^*_X t)(s) = X(t(s)) - t(\nabla_X s).$$

   Prove that ∇^* is a connection on E^*.

2. Let E_1 and E_2 be vector bundles over the same manifold M, with connections ∇^1 and ∇^2‘, respectively. Define a connection on the direct sum $E_1 \oplus E_2$. (Additivity of connections leaves no choice in the definition!)

Exercise 5.6. Prove that any convex linear combination of two connections is again a connection, i.e., given ∇^1 and ∇^2 connections on E and ρ_1, ρ_2 smooth functions on M (the base manifold) such that $\rho_1 + \rho_2 = 1$, then

$$\nabla = \rho_1 \nabla^1 + \rho_2 \nabla^2$$

is also a connection.

5.7. More than locality: derivatives of paths: We have seen that ∇ is local: if we want to know $\nabla_X(s)$ at the point $x \in X$, then it suffices to know X and s in a neighborhhod of x. However, much more is true.

Lemma 5.8. *Given a connection ∇ on a vector bundle E over M, $X \in \mathcal{X}(M)$, $s \in \Gamma(E)$ and $x_0 \in M$, then $\nabla_X(s)(x)$ vanishes in each of the cases:*

1. *s-arbitrary but $X(x) = 0$.*
2. *X-arbitrary but there exists a curve γ in M starting at x with speed X_x (i.e. $\gamma(0) = x$, $\dot{\gamma}(0) = X_x$) such that $s(\gamma(t)) = 0$ for all t near 0.*

Proof. We deal with a local problem and we can concentrate on an open U containing x on which we have a given local frame e of E; ∇ will then be specified by its connection matrix. An arbitrary section s can now be written on U as

$$s = \sum_{i=1}^{r} f^i e_i$$

with $f^i \in C^\infty(U)$; on such a section we find using the Leibniz identity and then, using the connection matrix:

$$\nabla_X(s)(x) = \sum_i (df^i)(X_x)e_i(x) + \sum_{i,j} f^j(x)\omega_j^i(X_x)e_i(x).$$

It is clear that this is zero when $X(x) = 0$. In the second case we find

$$\nabla_X(s)(x) = \sum_i \frac{df^i \circ \gamma}{dt}(0)e_i(x) + \sum_{i,j} f^j(\gamma(0))\omega_j^i(X_x)e_i(x) \tag{20}$$

which clearly vanishe under the condition that $f^i(\gamma(t)) = 0$ for t near 0. □

The first type of condition in the lemma tells us that, given $s \in \Gamma(E)$, it makes sense to talk about

$$\nabla_{X_x}(s) \in E_x$$

for all $X_x \in T_xM$. In other words, ∇ can be reinterpreted as an operator

$$d_\nabla : \Gamma(E) \to \Omega^1(M, E), \quad d_\nabla(s)(X) := \nabla_X(s).$$

The axioms on ∇ are equivalent to the fact that d_∇ is linear and

$$d_\nabla(fs) = fd_\nabla(s) + df \otimes s$$

for all $f \in C^\infty(M)$ and $s \in \Gamma(E)$. More on this a bit later.

Let us now turn to the secon type of condition in the lemma and give to it a more conceptual face. Given a path

$$\gamma : I \to M$$

(i.e. a smooth map, defined on some interval I, typically $[0,1]$ or of type $(-\epsilon, \epsilon)$), by a path in E above γ we mean any path $u : I \to E$ with the property that

$$u(t) \in E_{\gamma(t)} \quad \forall\, t \in I.$$

One way to produce such paths above γ is by using sections of E: any section $s \in \Gamma(E)$ induces the path

$$s \circ \gamma : I \to E$$

above γ. The previous lemma implies that the expression

$$\nabla_{\dot\gamma}(s)(\gamma(t))$$

makes sense, depends on the path $s \circ \gamma$ and defines is a path above γ. It is denoted

$$\frac{\nabla(s \circ \gamma)}{dt}.$$

Slightly more generally, for any path $u : I \to E$ above γ one can define the new path above γ

$$\frac{\nabla u}{dt} : I \to E.$$

Locally with respect to a frame e, writing $u(t) = \sum_j u^j(t)e_j(\gamma(t))$, the formula is just the obvious version of (20) and we find the components

$$\left(\frac{\nabla u}{dt}\right)^i = \frac{du^i}{dt}(t) + \sum_j u^j(t)\omega_j^i(\dot\gamma(t)). \tag{21}$$

Exercise 5.9. On the tangent bundle of $\mathbb{R}^n$ consider the connection

$$\nabla_X Y = \sum_i X(Y^i)\frac{\partial}{\partial x_i}.$$

Let γ be a curve in $\mathbb{R}^n$ and let $\frac{\nabla}{dt}$ be the derivative induced along γ by the connection. What is $\frac{\nabla \dot{\gamma}}{dt}$?

5.10. Connections and DeRham-like operators: Next, we point out a slightly different way of looking at connections, in terms of differential forms on M. Recall that the standard DeRham differential d acts on the space $\Omega(M)$ of differential forms on M, incresing the degree by one

$$d : \Omega^*(M) \to \Omega^{*+1}(M),$$

and satisfying the Leibniz identity:

$$d(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge d(\eta),$$

where $|\omega|$ is the degree of ω[2], and is a differential (i.e. $d \circ d = 0$). Locally, writing ω as in (12) in 4.13, we have

$$d\omega = \sum_i \sum_{i_1,\ldots,i_p} \frac{\partial f^{i_1,\ldots,i_p}}{\partial \chi_k} d\chi_i d\chi_{i_1} \ldots d\chi_{i_p}.$$

Globally, thinking of ω as a $C^\infty(M)$-linear map as in (11), one has

$$\begin{aligned} d(\omega)(X_1,\ldots,X_{p+1}) &= \sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_1,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots X_{p+1})) \\ &+ \sum_{i=1}^{p+1}(-1)^{i+1}L_{X_i}(\omega(X_1,\ldots,\hat{X}_i,\ldots,X_{p+1})). \end{aligned} \tag{22}$$

where L_X denotes the Lie derivative along the vector field X.

Let us now pass to differential forms with coefficients in a vector bundle E (see 4.13). The key remark here is that, while there is no canonical (i.e. free of choices) analogue of DeRham differential on $\Omega(M,E)$, connections are precisely the piece that is needed in order to define such operators. Indeed, assuming that ∇ is a connection on E, and thinking of forms $\omega \in \Omega^p(M,E)$ as $C^\infty(M)$-multilinear maps as in 13, we see that the previous formula for the DeRham differential does makes sense if we replace the Lie derivatives L_{X_i} by ∇_{X_i}. Hence one has an induced operator

$$d_\nabla : \Omega^\bullet(M,E) \to \Omega^{\bullet+1}(M,E).$$

As in the case of DeRham operator, d_∇ satisfies the Leibniz identity

$$d_\nabla(\omega \wedge \eta) = d(\omega) \wedge \eta + (-1)^{|\omega|}\omega \wedge d_\nabla(\eta)$$

for all $\omega \in \Omega(M)$, $\eta \in \Omega(M,E)$. the correspondence $\nabla \leftrightarrow d_\nabla$ is a bijection between connections on E and operators d_∇ as above (increasing the degree by one and satisfying the Leibniz identity).

[2]Note: the sign in the formula agrees with the graded sign rule: we interchange d which has degree 1 and ω

Note also that, as for the DeRham operator, d_∇ can be described in locally, using the connection matrices. First of all, if U is the domain of a local frame $e = \{e_1, \ldots, e_r\}$ with connection matrix ω, then one can write

$$d_\nabla(e_j) = \sum_{i=1}^{r} \omega_j^i e_j, \tag{23}$$

Assume now that U is also the domain of a coordinate chart $(U, \chi_1, \ldots, \chi_n)$. Representing $\omega \in \Omega^p(M, E)$ locally as in (14), the Leibniz identity gives the formula

$$d_\nabla(\omega) = \sum_{i_1, \ldots, i_p} (-1)^p dx_{i_1} \ldots dx_{i_p} \otimes d_\nabla(e^{i_1, \ldots, i_p}).$$

hence it suffices to describe d_∇ on sections of E. The same Leibniz formula implies that it suffices to describe d_∇ on the frame e- and that what (23) does.

5.11. Parallel transport: One of the main use of connections comes from the fact that a connection ∇ on E can be used to move from one fiber of E to another, along paths in the base. This is the so called parallel transport. To explain this, let us return to paths $u : I \to E$. We say that u is parallel (with respect to ∇) if

$$\frac{\nabla u}{dt} = 0 \quad \forall\, t \in I.$$

Lemma 5.12. *Let ∇ be a connection on the vector bundle E and $\gamma : I \to M$ a curve in M, $t_0 \in I$. Then for any $u_0 \in E_{\gamma(t_0)}$ there exists and is unque a parallel path above γ, $u : I \to E$, with $u(t_0) = u_0$.*

Proof. We can proceed locally (also because the uniqueness locally implies that the local pieces can be glued), on the domain of a local frame e. By formula (21), we have to find

$$u = (u^1, \ldots, u^r) : I \to \mathbb{R}^r$$

satisfying

$$\frac{du^i}{dt}(t) = -\sum_j u^j(t)\omega_j^i(\dot{\gamma}(t)),\ u(0) = u_0.$$

In a matricial form (with u viewed as a column matrix), writing $A(t)$ for the matrix $-\omega(\dot{\gamma}(t))$, we deal with the equation

$$\dot{u}(t) = A(t)u(t), \quad u(t_0) = u_0$$

and the existence and uniqueness is a standard result about first order linear ODE's. □

Definition 5.13. *Given a connection ∇ on E and a curve $\gamma : I \to M$, $t_0, t_1 \in I$, the parallel transport along γ (with respect to ∇) from time t_0 to time t_1 is the map*

$$T_\gamma^{t_0, t_1} : E_{\gamma(t_0)} \to E_{\gamma(t_1)}$$

which associates to $u_0 \in E_{\gamma(t_0)}$ the vector $u(t_1) \in E_{\gamma(t_1)}$, where u is the unique parallel curve above γ with $u(t_0) = u_0$.

Exercise 5.14. *Show that*

1. *each* $T_\gamma^{t_0,t_1}$ *is a linear isomorphism.*

2. $T_\gamma^{t_1,t_2} \circ T_\gamma^{t_0,t_1} = T_\gamma^{t_0,t_2}$ *for all* $t_0, t_1, t_2 \in I$.

Then try to guess how ∇ *can be recovered from all these parallel transports (there is a natural guess!). Then prove it.*

5.15. Curvature: Recall that, for the standard Lie derivatives of functions along vector fields,

$$L_{[X,Y]} = L_X L_Y(f) - L_Y L_X(f).$$

Of course, this can be seen just as the definition of the Lie bracket $[X, Y]$ of vector fields but, even so, it still says something: the right hand side is a derivation on f (i.e., indeed, it comes from a vector field). The similar formula for connections fails dramatically (i.e. there are few vector bundles which admit a connection for which the analogue of this formula holds). The failure is measured by the curvature of the connection.

Proposition 5.16. *For any connection* ∇*, the expression*

$$k_\nabla(X,Y)s = \nabla_X \nabla_Y(s) - \nabla_Y \nabla_X(s) - \nabla_{[X,Y]}(s), \tag{24}$$

is $C^\infty(M)$*-linear in the entries* $X, Y \in \mathcal{X}(M)$, $s \in \Gamma(E)$*. Hence it defines an element*

$$k_\nabla \in \Gamma(\Lambda^2 T^*M \otimes End(E)) = \Omega^2(M; End(E)),$$

called the curvature of ∇.

Proof. It follows from the properties of ∇. For instance, we have

$$\begin{aligned} \nabla_X \nabla_Y(fs) &= \nabla_X(f\nabla_Y(s) + L_Y(f)s) \\ &= f\nabla_X\nabla_Y(s) + L_X(f)\nabla_Y(s) + L_X(f)\nabla_Y(s) + L_X L_Y(f)s, \end{aligned}$$

and the similar formula for $\nabla_X \nabla_Y(fs)$, while

$$\nabla_{[X,Y]}(fs) = f\nabla_{[X,Y]}(s) + L_{[X,Y]}(f)s.$$

Hence, using $L_{[X,Y]} = L_X L_Y - L_Y L_X$, we deduce that

$$k_\nabla(X,Y)(fs) = fk_\nabla(X,Y)(s),$$

and similarly the others. □

Remark 5.17. One can express the curvature locally, with respect to a local frame $e = \{e_1, \ldots, e_r\}$ of E over an open U, as

$$k_\nabla(X,Y)e_j = \sum_{j=1}^{r} k_j^i(X,Y)e_i,$$

where $k_j^i(X,Y) \in C^\infty(U)$ are smooth functions on U depending on $X, Y \in \mathcal{X}(M)$. The previous proposition implies that each k_i^j is a differential form (of degree two). Hence k_∇ is locally determined by a matrix

$$k = k(\nabla, e) := (k_j^i)_{i,j} \in M_r(\Omega^2(U)),$$

called the curvature matrix of ∇ over U, with respect to the local frame e. A simple computation (exercise!) gives the explicit formula for k in terms of the connection matrix ω:

$$k = d\omega + \omega \wedge \omega,$$

where $\omega \wedge \omega$ is the matrix of 2-forms given by

$$(\omega \wedge \omega)^i_j = \sum_k \omega^i_k \wedge \omega^k_j.$$

5.2 Connections on principal bundles

Throughout this subsection G is a Lie group and $\pi : P \to M$ is a principal G-bundle. There are several different ways of looking at connections on P. We start here with the more intuitive one.

5.18. Connections as horizontal distributions: A horizontal subspace of P at $p \in P$ is a subspace

$$\mathcal{H}_p \subset T_pP$$

with the property that

$$(d\pi)_p|_{\mathcal{H}_p} : \mathcal{H}_p \to T_{\pi(p)}M \tag{25}$$

is an isomorphism. Note that in this case, for all $g \in G$,

$$R_g(\mathcal{H}_p) := \{(dR_g)_p(X_p) : X_p \in \mathcal{H}_p\} \subset T_{gp}P$$

is a horizontal subspace at gp.

By distribution on P we mean a vector sub-bundle

$$\mathcal{H} \subset TP.$$

Equivalently, it is a collection $\{\mathcal{H}_p\}_{p\in P}$ of vector sub-spaces $\mathcal{H}_p$ of T_pP, all of the same dimension, which fits together into a smooth submanifold $\mathcal{H}$ of TP.

Definition 5.19. *A connection on P is a distribution $\mathcal{H}$ on P, with the property that each $\mathcal{H}_p$ is a horizontal subspace of P and*

$$\mathcal{H}_{pg} = R_g(\mathcal{H}_p) \quad \forall\ p \in P, g \in G.$$

Note that here is a slightly different way of looking at horizontal subspaces $\mathcal{H}_p \subset T_pP$, by encoding them in the inverse of isomorphism (25), viewed as a linear map

$$h_p : T_{\pi(p)}M \to T_pP$$

with the property that it is a right inverse to $(d\pi)_p$:

$$(d\pi)_p \circ h_p = \mathrm{Id} : T_{\pi(p)}M \to T_{\pi(p)}M.$$

Maps with this property will be called horizontal lifting at p and h_p will be called the horizontal lifting associated to $\mathcal{H}_p$. Note that the subspace $\mathcal{H}_p$ can be recovered as the image of h_p. Of course, given a connection $\mathcal{H}$ one obtains an induced horizontal lifting at the level of vector fields

$$h : \mathcal{X} \to \mathcal{X}(P), \quad h(X)_p := h_p(X_{\pi(p)}).$$

Putting all the possible horizontal liftings together we obtain the so-called first order jet of P,

$$J^1(P) := \{(p, h_p) : p \in P, h_p \text{is a horizontal lifting of } P \text{ at } p\}.$$

This can be seen as a (discrete for now) bundle over P (using the projection on the first coordinate) and also as a bundle over M by further composing with the projection $\pi : P \to M$. The terminology comes from the fact that this set encodes all the possible first order data (jets) associated to (local) sections of P. More precisely, given $x \in M$, any local section σ of P defined around x, induces:

- the 0-th order data at x: just the value $\sigma(x) \in P$.
- the 1-st order data at x: the value $\sigma(x)$ together with the differential of σ at p (which encodes the first order derivatives of σ at x),

$$(d\sigma)_x : T_xM \to T_{\sigma(x)}P.$$

Of course, $(d\sigma)_x$ is a horizontal lifting at $\sigma(x)$ and it is not difficult to see that

$$J^1(P) = \{(\sigma(x), (d\sigma)_x) : x \in M, \sigma - \text{local section of } P \text{ defined around } p\}.$$

Note that for a section σ of P, the resulting map

$$j^1(\sigma) : M \to J^1(P), \quad x \mapsto (\sigma(x), (d\sigma)_x))$$

is a section of $J^1(P)$ viewed as a bundle over P. Similalrly for local sections. There is a natural smooth structure on $J^1(P)$ uniquely determined by the condition that all local sections of type $j^1(\sigma)$ are smooth.

5.20. Connection forms: Here is a slightly different point of view on connections, bot so natural but often easr to work with. Let $\mathfrak{g}$ be the Lie algebra of G. Recall that

$$a : \mathfrak{g} \to \mathcal{X}(P)$$

denotes the induced infinitesimal action of $\mathfrak{g}$ on P- given by

$$a(v)_p = \left.\frac{d}{dt}\right|_{t=0} p\exp(tv).$$

Also, we consider the adjoint representation of G on $\mathfrak{g}$,

$$\text{Ad} : G \to GL(\mathfrak{g}), \quad g \mapsto \text{Ad}_g$$

which associates to $g \in G$ the differential at the identity of the map $G \to G, a \mapsto aga^{-1}$. When $G \subset GL_n$ is a linear group (so that $\mathfrak{g} \subset gl_n$), Ad_g is simply the matrix conjugation by g:

$$\text{Ad}_g(v) = gvg^{-1}.$$

The general discussion from 4.36 gives us the notion of G-invariant forms on P with values in $\mathfrak{g}$- as those $\theta \in \Omega^k(P, \mathfrak{g})$ with the property that

$$R_g^*(\omega) = \text{Ad}_{g^{-1}}(\omega)$$

for all $g \in G$.

Definition 5.21. *A connection form on P is a 1-form*

$$\omega \in \Omega^1(P, \mathfrak{g})$$

with the property that it is G-invariant and satisfies

$$\omega(a(v)) = v \quad \forall\, v \in \mathfrak{g}.$$

Proposition 5.22. *There is a bijection between connection 1-forms $\omega \in \Omega^1(P, \mathfrak{g})$ and connections $\mathcal{H}$ on P. The bijection associates to ω its kernels:*

$$\mathcal{H}_p := \{X_p \in T_pP : \omega_p(X_p) = 0\}.$$

Proof. For each $p \in P$ we consider the short exact sequence of vector spaces

$$\mathfrak{g} \stackrel{a_p}{\to} T_pP \stackrel{(d\pi)_p}{\to} T_{\pi(p)}M.$$

A right splitting of this sequence is a right inverse of $(d\pi)_p$; as we have noticed, such right splittings h_p correspond to horizontal subspaces $\mathcal{H}_p$. On the other hand one can also talk about left splitting of the sequence, by which we mean a left inverse of a_p or, equivalently, linear maps $\omega_p : T_pP \to \mathfrak{g}$ satisfying $\omega_p(a_p(v)) = v$ for all $v \in \mathfrak{g}$. What we need here is a simple linear algebra fact which says that, for a short exact sequence of vector space, there is a 1-1 correspondence between the choice of right splittings h_p and the choice of left splittings ω_p. Explicitely, this correspondence is described by the equation

$$a_p \circ \omega_p + h_p \circ (d\pi)_p = \mathrm{Id}.$$

After applying this at all $p \in P$, one obtains the 1-1 correspondence from the statement. Of course, one still has to check that the smoothness of ω is equivalent to the one of h and similarly for the G-invariance, but we leave these as an exercise. □

5.23. Parallel transport As in the case of vector bundles, connections on principal bundles have associated parallel transports. Assume we fix a connection $\mathcal{H}$ on the principal G-bundle $\pi : P \to M$, with associated connection form ω. Then we say that a curve $u : I \to P$ is horizontal if

$$\dot{u}(t) \in \mathcal{H}_{\gamma(t)} \quad \forall\, t \in I,$$

where $\gamma(t) = \pi(u(t))$. Then, completely similar to Lemma 5.12, we have:

Lemma 5.24. *Let $\gamma : I \to M$ be a curve in M, $t_0 \in I$. Then for any $u_0 \in P_{\gamma(t_0)}$ there exists and is unque a horizontal path above γ, $u : I \to P$, with $u(t_0) = u_0$.*

Proof. Again, we may work locally, i.e. we may just assume that $P = M \times G$ is the trivial principal bundle. Let us now see what a connection form on this bundle looks like. First of all, the invariance condition means that, for all $(x, a) \in M \times G$, $(X_x, V_a) \in T_{x,a}(M \times G) = T_xM \times T_aG$, and for all $g \in G$,

$$\omega_{x,ag}(X_x, R_g(V_a)) = \mathrm{Ad}_{g^{-1}}(\omega_{x,a}(X_x, V_a)),$$

where we denote by $R_g : G \to G$ the right multiplication by g and also its differential $(dR_g)_a : T_aG \to T_{ag}G$. Taking $a = 1$, we see that ω is determined by all the expressions of type $\omega_{x,1}(X_x, v)$ with $1 \in G$ the unit, $v \in T_1G = \mathfrak{g}$. In turn, the condition that $\omega(a(v)) = v$ for all $v \in \mathfrak{g}$ implies that ω is uniquely determined by all expressions of type

$$\omega_{x,1}(X_x, 0) \in \mathfrak{g}$$

i.e. precisely by the restriction $\eta \in \Omega^1(M, \mathfrak{g})$ of ω to M via the inclusion $M \hookrightarrow M \times G$, $x \mapsto (x, 1)$. Writing down the explicit formulas we find that the formula for ω in terms of η:

$$\omega_{x,g}(X_x, V_g) = \mathrm{Ad}_{g^{-1}}(\eta_x(X_x)) + L_{g^{-1}}(V_g),$$

where, as before, $L_{g^{-1}} : G \to G$ stands for the left multiplication by g^{-1} and also for its differential $(dL_{g^{-1}})_g : T_gG \to T_1G = \mathfrak{g}$.

A curve in $P = M \times G$ above γ can be written as

$$u(t) = (\gamma(t), g(t)).$$

The fact that it is horizontal means that $\dot{u}(t)$ takes values in the horizontal space or, equivalently, that it is killed by ω:

$$\omega_{\gamma(t),g(t)}(\dot{\gamma}(t), \dot{g}(t)) = 0 \quad \forall\, t \in I.$$

Using the previous formula for ω we see that this equation becomes

$$\dot{g}(t) = -R_{g(t)}(\eta_{\gamma(t)}(\dot{\gamma}(t)).$$

For fixed γ we need existence and uniqueness of g satisfying this equation, with an initial contion $g(t_0) = g_0$ given. We denote $v(t) = -\eta_{\gamma(t)}(\dot{\gamma}(t)$ (curve in $\mathfrak{g}$, so that the equation reads $\dot{g}(t) = R_{g(t)}(v(t))$. At least for linear groups $G \subset GL_r$ we deal again (as in the case of vector bundles) with an ordinary ODE. For a general Lie group one can just interpret this equation as the flow equation of the time-dependent vector field $X(t, g)$ on G given by

$$X(t, g) = R_g(v(t))$$

and use the standard results about flows of time-dependent vector fields. However, one still has to make sure that the equation with the given initial condition $g(t_0) = g_0$ exists on the entire interval I on which v is defined; for that one uses the remark that, if $g(t)$ is an integral curve then so is $h(t) = g(t)a$ for all $a \in G$:

$$\dot{h}(t) = R_a(\dot{g}(t)) = R_a(R_{g(t)}(v(t)) = R_{h(t)}(v(t)).$$

□

Finally, as in the case of vector bundles, for curves $\gamma : I \to M$, $t_0, t_1 \in M$, one obtains the parallel transport (with respect to the connection) along γ:

$$T_\gamma^{t_0,t_1} : P_{\gamma(t_0)} \to P_{\gamma(t_1)}$$

which associates to $u_0 \in P_{\gamma(t_0)}$ the element $u(t_1) \in\in P_{\gamma(t_1)}$, where u is the horizontal path above γ with $u(t_0) = u_0$.

Exercise 5.25. *Show that $T_\gamma^{t_0,t_1}$ are diffeomorphisms and are G-equivariant.*

5.26. Curvature Let $\mathcal{H}$ be a connection on P, with associated horizontal lifting h and connection 1-form ω. The curvature of $\mathcal{H}$ measure the failure of $\mathcal{H}$ to be involutive or, equivalently, the failure of the horizontal lifting h to preserve the Lie bracket of vector fields. However, let us introduce it in the form that is easier to work with. For that we note that the Lie bracket $[\cdot,\cdot]$ of the Lie algebra $\mathfrak{g}$ extends to a bracket

$$[\cdot,\cdot] : \Omega^1(P,\mathfrak{g}) \times \Omega^1(P,\mathfrak{g}) \to \Omega^2(P,\mathfrak{g}),$$

$$[\alpha,\beta](X,Y) := [\alpha(X),\beta(Y)] - [\alpha(Y),\beta(X)].$$

Note that when $\beta = \alpha \in \Omega^1(P,\mathfrak{g})$ one obtains

$$[\alpha,\alpha](X,Y) = 2[\alpha(X),\alpha(Y)].$$

Definition 5.27. *The curvature of the connection ω is defined as*

$$K := d\omega + \frac{1}{2}[\omega,\omega] \in \Omega^2(P,\mathfrak{g}).$$

The main property of the curvature is that (see 4.36 for the notion of basic forms):

Lemma 5.28. *The curvature is a basic form:*

$$K \in \Omega^2(P,\mathfrak{g})_{bas}.$$

Proof. The G-invariance follows from the G-invariance of ω and the fact that all the operations involved in construction K (d and the bracket) are natural (functorial), hence are respected by the action of G (give the details!). The fact that K is horizontal □

Note that the fact that Ω is horizontal implies that, in order to know Ω, it suffices to know all the expressions of type

$$K(h(X),h(Y)) \in C^\infty(P,\mathfrak{g})$$

with $X,Y \in \mathcal{X}(M)$, where $h(X) \in \mathcal{X}(M)$ is the horizontal lift of X with respect to the connection. For the following, recall that $a : \mathfrak{g} \to \mathcal{X}(P)$ denotes the induced infinitesimal action (see above).

Proposition 5.29. *The curvature can also be characterized as:*

1. *the horizontal part of $d\omega$, i.e. the horizontal form on P (with values in $\mathfrak{g}$) which coincides with $d\omega$ on horizontal vectors.*

2. *the unique horizontal form Ω with the property that, after applying the infinitesimal action $a : \mathfrak{g} \to \mathcal{X}(P)$,*

$$a(K(h(X),h(Y))) = h([X,Y]) - [h(X),h(Y)]$$

for all $X,Y \in \mathcal{X}(M)$.

Proof. We have already shown that K is horizontal and then the first part is clear from the very definition of K. For the second part, by the comments before the proposition and the injectivity of a, the uniqueness follows. Hence we just have to show that the actual curvature does have this property. For that one just compute $K(h(X), h(Y))$ using the definition of K and the fact that $\omega \circ h = 0$ and we find:

$$K(h(X), h(Y)) = -\omega([h(X), h(Y)]) = \omega(h([X, Y]) - [h(X), h(Y)]).$$

On the other hand, since $h(X)$ is π-projectable to X, we have that $[h(X), h(Y)]$ is π-projectable to $[X, Y]$ and then $h([X, Y]) - [h(X), h(Y)]$ is π-projectable to $[X, Y] - [X, Y] = 0$, i.e. it is vertical. Hence, at each point p, it is of type $a(v_p)$ with some $v_p \in \mathfrak{g}$. Plugging in the previous equation, we find that

$$K(h(X), h(Y))(p) = \omega(a(v_p)) = v_p,$$

and then the desired equation follows is just the defining property of v_p. $\square$

5.3 Vector bundle connections versus principal bundle connections

We have seen the 1-1 correspondence between vector bundles of rank r over M and principal GL_r-bundles over M. We show that this bijection extends at the level of connections.

5.30. From vector bundle connections to principal bundle ones: Start with a vector bundle E and a connection ∇ on E. We consider the associated frame bundle $\mathrm{Fr}(E)$, a principal GL_r-bundle, and we want to associated to ∇ a connection on $\mathrm{Fr}(E)$. We will do that by constructing a connection 1-form

$$\omega_\nabla \in \Omega^1(\mathrm{Fr}(E), gl_r).$$

We have to say what it does on an arbitrary vector

$$\frac{d}{dt}\Big|_{t=0} e(t) \in T_{e(0)}\mathrm{Fr}(E),$$

where

$$\mathbb{R} \ni t \mapsto e(t) = (e_1(t), \ldots, e_r(t)) \in \mathrm{Fr}(E)$$

is a curve in $\mathrm{Fr}(E)$. Let $\gamma : \mathbb{R} \to M$ be the base curve of e. Consider the derivatives of the paths e_i with respect to ∇ (see 5.7) and decompose them with respect to the frame $e(t)$:

$$\frac{\nabla e_i}{dt}(t) = \sum_j a_i^j(t) e_j(t)$$

giving rise to the matrix

$$e(t)^{-1} \cdot \frac{\nabla e}{dt}(t) := (a_i^j(t))_{i,j} \in gl_r$$

(think that, while $e(t) : \mathbb{R}^r \xrightarrow{\sim} E_{\gamma(t)}$, $\frac{\nabla e}{dt}(t)$ is a linear map from $\mathbb{R}^r$ to $E_{\gamma(t)}$). Set now

$$\omega_\nabla(\frac{d}{dt}\Big|_{t=0} e(t)) := e(0)^{-1} \cdot \frac{\nabla e}{dt}(0) \in gl_r.$$

Theorem 5.31. *For any connection ∇ on E, ω_∇ is a connection 1-form on $Fr(E)$, and this defines a bijection between connections on E and connections on the principal GL_r-bundle $Fr(E)$.*

Proof. We first show that, for any ∇, ω_∇ is a connection 1-form. Let us first prove that $\omega(a_p(A)) = A$ for all $e \in \mathrm{Fr}(E)$, $A \in gl_r$. Recall that

$$a_p : gl_r \to T_e\mathrm{Fr}(M),\ a_e(A) = \frac{d}{dt}\Big|_{t=0} e \cdot \exp(tA).$$

In the definition of ω above, we deal with the path $t \mapsto e\exp(tA)$ (which at $t = 0$ is e), hence

$$\omega(a_p(A)) = e^{-1}\frac{\nabla e \cdot exp(tA)}{dt}(0).$$

Using the definitions (of $\frac{\nabla}{dt}$, as well as the fact that this operation on frames is defined componentwise), the standard derivation rules apply and, since e is constant on t, we find

$$\omega(a_p(A)) = e^{-1}e \cdot \frac{dexp(tA)}{dt}(0) = A.$$

Next, we prove the invariance of ω. We have to show that for any path $e(t)$ of frames,

$$\omega((dR_g)(\frac{de}{dt}(0)) = Ad_{g^{-1}}\omega(\frac{de}{dt}(0)).$$

We start from the left hand side. We see that we deal with ω evaluated on the curve of frames $e(t)g$ hence, by the definition of ω, the expression is

$$(e(0)g)^{-1}\frac{\nabla eg}{dt}(0) = (g^{-1}e(0)^{-1})\frac{\nabla e}{dt}(0)g$$

i.e. precisely

$$g^{-1}\omega(\frac{de}{dt}(0))g = Ad_{g^{-1}}(\omega(\frac{de}{dt}(0)).$$

We still have to show that $\nabla \mapsto \omega_\nabla$ is a bijection. This can be shown in several ways. One way is to basically read the previous argument backwards. Below we present a different argument, in a slightly more general context. □

5.32. From principal bundle connections to vector bundle ones: Let us now start with a Lie group G, a representation $\rho : G \to GL(V)$ and a principal G-bundle $\pi : P \to M$, so that we can form the vector bundle $E = E(P, V)$. We will associate to a connection ω on P, a connection ∇ on the vector bundle E. It suffices to construct

$$d_\nabla : \Gamma(E) \to \Omega^1(M, E)$$

or, equivalently (cf. Proposition 4.38),

$$D_\nabla := h^\bullet d_\nabla (h^\bullet)^{-1} : C^\infty(P, V)^G \to \Omega^1(P, V)_{\mathrm{bas}}.$$

Proposition 5.33. *For any connection ω on P,*

$$D_\nabla : C^\infty(P, V)^G \to \Omega^1(P, V)_{bas},\ D(f) = df + \rho(\omega(-))f$$

(where $\rho(\omega(-))f$ is the V-valued 1-form on P given by $X \mapsto \rho(\omega(X))(f)$) induces a connection ∇ on the vector bundle E.

Proof. Direct check. □

5.4 Application: connections compatible with G-structures

Throughout this section we fix a G-structure $\mathcal{S}$ on the manifold M.

Definition 5.34. *A connection*

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

on M is said to be compatible with the G-structure $\mathcal{S}$ if the equivalent properties from the next proposition are satisfied.

Proposition 5.35. *For a connection ∇ the following are equivalent:*

(i) For any local frame e over U, that belongs to $\mathcal{S}$ the resulting connection matrix (19) takes values in $\mathfrak{g}$:

$$\omega(\nabla, e) \in \Omega^1(U, \mathfrak{g}).$$

(ii) For any curve $\gamma : [0,1] \to M$, the parallel transport along γ gives isomorphisms

$$T_\gamma^{0,1} : (E_{\gamma(0)}, \mathcal{S}_{\gamma(0)}) \to (E_{\gamma(1)}, \mathcal{S}_{\gamma(1)}).$$

(iii) The induced principal bundle connection on $Fr(M)$, represented by the connection 1-form

$$\omega \in \Omega^1(Fr(M), gl_r)$$

has the property that

$$\omega|_{\mathcal{S}} \in \Omega^1(\mathcal{S}, \mathfrak{g}).$$

Moreover, the correspondences $\nabla \mapsto \omega \mapsto \omega|_{\mathcal{S}}$ define bijections between:

- *connections ∇ satisfying (i) (or (ii)).*
- *connections on $Fr(M)$ satisfying (iii).*
- *connections on the principal G-bundle $\mathcal{S}$.*

Proof. To come. □

Let us draw some conlcusions coming from the general theory of connections. From now on we fix

$$G \subset GL_n, \ V := \mathbb{R}^n,$$

and we also consider the Lie algebra $\mathfrak{g}$ of G. Denote by N the dimension of G. When we want to write expressions in their coordinates, we will use

$$\{e_1, \ldots, e_n\} - \text{ the standard basis of } V = \mathbb{R}^n$$

and

$$\{A(1), \ldots, A(N)\} - \text{ a fixed basis of } \mathfrak{g},$$

writing,

$$A(\alpha) = (A^i_j(\alpha))_{1 \leq i,j \leq n} \in \mathfrak{g} \subset gl_n.$$

5.36. The curvature (of connections on G-structures): Given a connection ∇ on M compatible with a G-structure $\mathcal{S}$ and the associated connection 1-form ω on $\mathcal{S}$, we can form:

1. The curvature of ∇:

$$K_\nabla \in \Omega^2(M, \mathrm{End}(TM)).$$

2. The curvature of the principal bundle connection ω:

$$K_\omega \in \Omega^2(P, \mathfrak{g})_{\mathrm{bas}}.$$

From the general theory (see also 4.44 for the bundle of infinitesimal autonorphisms) we deduce:

Corollary 5.37. *For any connection ∇ compatible with the G-structure $\mathcal{S}$ on M,*

$$K_\nabla \in \Omega^2(M, aut(\mathcal{S}))$$

and, modulo the isomorphism

$$\pi^\bullet : \Omega(M, aut(\mathcal{S})) \xrightarrow{\sim} \Omega(\mathcal{S}, \mathfrak{g})_{bas}$$

(see Proposition 4.38 and Proposition 4.45), K_∇ is identified with

$$K_\omega \in \Omega^2(\mathcal{S}, \mathfrak{g})_{bas}.$$

Exercise 5.38. Let $\mathcal{S}$ be a G-structure on M. Prove that $\mathrm{aut}(\mathcal{S})$ and $E[\mathcal{S}, \mathfrak{g}]$ are isomorphic as vector bundles on M. **Hint**: recall that the representation of G in $GL(\mathfrak{g})$ is the adjoint one. One can identify sections of $E[\mathcal{S}, \mathfrak{g}]$ with functions $f : P \to \mathfrak{g}$ and thus, via $a : \mathfrak{g} \to \mathcal{X}(P)$, with G-invariant vector fields on P.

In the case of connections compatible with G-structures their curvatures can be expressed in more "down to earth" ways. For instance, for any $e \in \mathcal{S}_x$, due to its horizontality, $K_\omega(e)$ can be seen as a map from $\Lambda^2 T_x^*M$ to $\mathfrak{g}$; moreover, since e defines an identification of T_xM with V, we find that K can be seen as a smooth map

$$K \in C^\infty(\mathcal{S}, \mathrm{Hom}(\Lambda^2 V, \mathfrak{g})).$$

Using the bases fixed above, K can be seen as a collection of maps

$$K = \{K^\alpha_{i,j} \in C^\infty(\mathcal{S}) : 1 \le i, j \le n, 1 \le \alpha \le N\}$$

(globally) so that

$$K(e_i, e_j) = \sum_\alpha K^\alpha_{i,j} A(\alpha).$$

5.39. The torsion (of connections on G-structures): Let us now take further advantage of the fact that we deal with TM. In this case, next to the curvature, the other important tensor that is associated to a connection is its torsion. As before, one can work either at the level of TM or that of $\mathcal{S}$:

- one has the torsion of ∇,

  $$T_\nabla : \mathcal{X}(M) \times \mathcal{X}(M),\ T_\nabla(X, Y) = \nabla_X(Y) - \nabla_Y(X) - [X, Y],$$

  viewed as a tensor

  $$T_\nabla \in \Omega^2(M, TM).$$

- one can also define the torsion of the principal connection ω directly. Since torsion is specific to TM and frame bundles, it is not surprising that it is related to the tautological form $\theta \in \Omega^1(\mathcal{S}, V)$ (see Proposition 4.52 and the comment thereafter). More precisely, while the curvature was the horizontal part of $d\omega$ (see Proposition 5.29), the torsion is the horizontal part of $d\theta$. Hence the torsion of ω,

$$T_\omega \in \Omega^2(\mathcal{S}, V)_{\text{bas}},$$

is the horizontal form which, on horizontal vectors, coincides with $d\theta$:

$$T_\omega(h(X), h(Y)) = (d\theta)(h(X), h(Y)).$$

Again, it is not difficult to see (exercise!) that the two torsions are basically the same thing, modulo the isomorphism

$$\pi^\bullet : \Omega(M, TM) \xrightarrow{\sim} \Omega(\mathcal{S}, V)_{\text{bas}}$$

coming from Proposition 4.38. And, as for the curvature, one can take further advantage of the fact that we deal with frames of TM and reinterpret T as a smooth map

$$T \in C^\infty(\mathcal{S}, \text{Hom}(\Lambda^2 V, V)),$$

or as a collection

$$T = \{T^k_{i,j} \in C^\infty(\mathcal{S}) : 1 \leq i, j, k \leq n\}$$

so that

$$T(e_i, e_j) = \sum_k T^k_{i,j} e_k.$$

5.40. The point of view of coframes: The following is now immediate, but it is important conceptually since it allows us to pass from G-structures and connections to coframes.

Proposition 5.41. *The choice of a connection compatible with the G-structure induces a coframe on $\mathcal{S}$, namely*

$$\omega \in \Omega^1(\mathcal{S}, \mathfrak{g}), \quad \theta \in \Omega^1(\mathcal{S}, V).$$

More precisely, using the basis $\{A_\alpha\}$ of $\mathfrak{g}$ and the standard basis of V, then the coframe is the one given by the components of ω and θ:

$$\omega^1, \ldots, \omega^N, \theta^1, \ldots, \theta^n.$$

Of course, the interesting question is now: which are the structure functions of the coframe? It is nice recognize all of them:

1. the Lie algebra structure of $\mathfrak{g}$, encoded in the structure constants $c^\alpha_{\beta,\gamma}$ defined by

$$[A(\beta), A(\gamma)] = \sum^{\alpha}_{\beta,\gamma} c^\alpha_{\beta,\gamma} A(\alpha).$$

2. the representation of $\mathfrak{g}$ on $\mathbb{R}^n$ (or the way that $\mathfrak{g}$ sits inside gl_n), encoded in the matricial representration of the basis $\{A(\alpha)\}$

$$A(\alpha) = (A^i_j(\alpha))_{1\leq i,j\leq n}.$$

3. the torsion and the curvature of the connection.

I.e., two types of coefficents depend only on $\mathfrak{g} \subset gl_n$, while the other give rise precisely to the torsion and the curvature:

Proposition 5.42. *One has*

$$d\theta = T(\theta \wedge \theta) + \omega \wedge \theta \quad (\textit{in } \Omega^2(\mathcal{S}, V),$$

$$d\omega = K(\theta \wedge \theta) - \frac{1}{2}[\omega, \omega] \quad (\textit{in } \Omega^2(\mathcal{S}, \mathfrak{g}).$$

In terms of their components:

$$d\theta^i = \sum_{j,k} T^i_{j,k}\theta^j \wedge \theta^k + \sum A^i_j(\alpha)\omega^\alpha \wedge \theta^j,$$

$$d\omega^\alpha = \sum_{j,k} K^\alpha_{j,k}\theta^j \wedge \theta^k - \sum_{\beta,\gamma} c^\alpha_{\beta,\gamma}\omega^\beta \wedge \omega^\gamma.$$

Maybe one word explaining the notations in the first type of formulas is appropriate. Whenever one has three vector spaces (or even vector bundles) W_1, W_2 and W and a bilinear map

$$f : W_1 \times W_2 \to W$$

one has an wedge-product operations

$$\Omega^p(\mathcal{S}, W_1) \otimes \Omega^q(\mathcal{S}, W_2) \to \Omega^{p+q}(\mathcal{S}, W),\ (\omega, \eta) \mapsto \omega \wedge_f \eta$$

given by the standard formula (15), but using f to pair the values of ω with those of η. Most of the times f is clear from the context and one just ommits it from the notation. This applies in particular to

- the evaluation operation

  $$\mathfrak{g} \times V \to V,\ (A, v) \mapsto A(v)$$

  which gives rise to

  $$\Omega^1(\mathcal{S}, \mathfrak{g}) \times \Omega^1(\mathcal{S}, V) \to \Omega^2(\mathcal{S}, V)$$

  and which explains the meaning of $\omega \wedge \theta$ (sometimes also denoted $\omega \circ \theta$).

- the Lie algebra operation

  $$[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$$

  which gives rise to

  $$\Omega^1(\mathcal{S}, \mathfrak{g}) \times \Omega^1(\mathcal{S}, \mathfrak{g}) \to \Omega^2(\mathcal{S}, \mathfrak{g}),$$

  which is usually denoted still by $[\cdot, \cdot]$ (and we defined already) and which explains the term $[\omega, \omega]$.

- the triple operation

$$\mathrm{Hom}(\Lambda^2 V, V) \times V \times V \to V,\ (\xi, v_1, v_2) \mapsto \xi(v_1, v_2)$$

which gives rise to

$$C^\infty(\mathcal{S}, \Lambda^2 V, V) \times \Omega^1(\mathcal{S}, V) \times \Omega^1(\mathcal{S}, V) \to \Omega^2(\mathcal{S}, V).$$

explaining the meaning of $T(\theta \wedge \theta)$. Note that a more conceptual notation would be

$$T \wedge \theta \wedge \theta$$

since it indicates (and allows us to use) the fact that this operation is a combination of two operations, in two different ways:

$$T \wedge \theta \wedge \theta = T \wedge (\theta \wedge \theta) = (T \wedge \theta) \wedge \theta.$$

- similarly for

$$C^\infty(\mathcal{S}, \Lambda^2 V, \mathfrak{g}) \times \Omega^1(\mathcal{S}, V) \times \Omega^1(\mathcal{S}, V) \to \Omega^2(\mathcal{S}, \mathfrak{g}),$$

explaining the notation $K(\theta \wedge \theta)$.

5.43. The example of metric connections in Riemannian geometry: In the literature one comes across the definition of connections compatible with a given Riemannian, (almost) symplectic or (almost) complex structure. It should come as no surprise that all these definitions ultimately coincide with the compatibility from the point of view of G-structures. For instance, below is the definition of a metric connection in Riemannian geometry.

Definition 5.44. *Given a Riemannian manifold (M, g), a connection ∇ on TM is called metric (or g-compatible) if the following holds for all vector fields X, Y, Z on M:*

$$X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X, Z).$$

In view of the following lemma and the equivalent characterizations of a G-compatible connection given in Proposition 5.35, a metric connection is the same as an $O(n)$-compatible connection.

Lemma 5.45. *If a connection is metric, then the associated parallel transport is an isometry*

Proof. Let $\gamma : I \to M$ be a smooth curve, $v, w \in T_{\gamma(t_0)}M$ for some $t_0 \in I$. Suppose X and Y are parallel vector fields along γ (i.e., sections of $\gamma^* TM$), with $X(t_0) = v$ and $Y(t_0) = w$. Then we need to prove that $t \mapsto g_{\gamma(t)}(X(t), Y(t))$ is constant. By locality we may assume X and Y are in fact local vector fields on M and so we compute

$$\frac{d}{dt} g(X, Y)(\gamma(t)) = \dot{\gamma}(g(X, Y)) = g(\nabla_{\dot{\gamma}} X, Y) + g(X, \nabla_{\dot{\gamma}} Y) = 0.$$

□

Exercise 5.46. Recover the covariant derivative from parallel transport and prove the converse of the above lemma.

Exercise 5.47. In a completely analogous fashion, one defines an almost symplectic connection on (M, ω) as a smooth linear connection on M such that the almost symplectic form $\omega \in \Omega^2(M)$ is parallel, i.e.,

$$X(\omega(Y, Z)) = \omega(\nabla_X Y, Z) + \omega(Y, \nabla_X, Z)$$

(notice that the connection is called symplectic if its torsion also vanishes). Prove that in this case the parallel transport associated to the connection preserves ω.

5.5 The intrinsic torsion

Let's now discuss the intrinisc torsion of a G-structure $\mathcal{S}$; this will be extracted from the torsion of connections compatible with $\mathcal{S}$ as the part that actually does not depend on the connection. We first need to introduce some spaces that only depend on

$$V := \mathbb{R}^n, \mathfrak{g} \subset gl_n = gl(V)$$

(we prefere to work coordinate free in order to have simpler formulas).

Definition 5.48. *The first prolongation of* $\mathfrak{g}$ *is defined as*

$$\mathfrak{g}^{(1)} := \{\phi : V \to \mathfrak{g} : \phi(u)(v) = \phi(v)(u) \ \ \forall\ u, v \in \mathfrak{g}\}.$$

It is useful to think of the first prolongation as the kernel of the linear map

$$\partial : \mathrm{Hom}(V, \mathfrak{g}) \to \mathrm{Hom}(\Lambda^2 V, V),\ \partial(\phi)(u, v) = \phi)u)(v) - \phi(v)(u).$$

Also the cokernel of this map will be important:

Definition 5.49. *The torsion space of* $\}$ *is defined as*

$$\mathcal{T}(\mathfrak{g}) := \frac{Hom(\Lambda^2 V, V)}{\partial(Hom(V, \mathfrak{g}))}.$$

It may help to think of these spaces as part of an exact sequence:

$$0 \to \mathfrak{g}^{(1)} \hookrightarrow \mathrm{Hom}(V, \mathfrak{g}) \stackrel{\partial}{\to} \mathrm{Hom}(\Lambda^2 V, V) \to \mathcal{T}(\mathfrak{g}) \to 0.$$

Proposition 5.50. *For any connection* ∇ *compatible with the* G*-structure* $\mathcal{S}$*, denoting by*

$$T_\nabla : \mathcal{S} \to Hom(\Lambda^2 V, V)$$

its torsion, composing with the projection onto $\mathcal{T}(\mathfrak{g})$*, the resulting*

$$T : \mathcal{S} \to \mathcal{T}(\mathfrak{g})$$

does not depend on the choice of the connection.

Definition 5.51. T *from the previous proposition is called the intrinsic torsion of the* G*-structure* $\mathcal{S}$*.*

Theorem 5.52. *If the* G*-structure* $\mathcal{S}$ *is integrable, then its intrinsic torsion must vanish.*

In some interesting examples, but not allways, also the converse of this corollary holds.

Corollary 5.53. *For a G-structure $\mathcal{S}$, the follwong are equivalent:*

1. *its intrinsic torsion vanishes.*
2. *M admits a torsion free connection compatible with $\mathcal{S}$.*
3. *locally, M admits torsion free connections compatible with $\mathcal{S}$.*

Corollary 5.54. *Given G:*

1. *If $\mathcal{T}(\mathfrak{g}) = 0$ then for any G-structure $\mathcal{S}$ on M, M admits a torsion-free connection compatible with $\mathcal{S}$.*
2. *If $\mathfrak{g}^{(1)} = 0$, then for any G-structure $\mathcal{S}$ on M, M admits a at most one torsion-free connection compatible with $\mathcal{S}$.*

5.55. Example: metrics For Riemannian metrics, the relevant Lie algebra is $o(n)$.

Lemma 5.56. $o(n)^{(1)} = 0$ *and* $\mathcal{T}(o(n)) = 0$.

This lemma implies that, for Riemannian metrics, the intrinsic torsion vanishes. Actually, as shown by one of the previous corollaries, this is also the reason that any Riemannian structure admits a unique torsion free compatible connection.

5.57. Example: complex structures For complex structures the relevant Lie algebra is $gl_k(\mathbb{C}) \subset gl_n$ $(n = 2k)$ with the inclusion that we dscussed before. Intrinsically, it is

$$\{A \in gl(V) : AJ = JA\}$$

where (V, J) is $\mathbb{R}^{2k}$ with the standard (linear) complex structure. We consider the space

$$\mathrm{Hom}_J := \{\phi : \Lambda^2 V \to V : \phi(Ju, v) = \phi(u, Jv) = -J\phi(u, v)\}.$$

Lemma 5.58. *One has*

$$\mathcal{T} gl_k(\mathbb{C}) \cong Hom_J$$

by the isomorphism which associates to the class of $\phi \in Hom(\Lambda^2 V, V)$ modulo the image of ∂ the bilinear map $t_J(\phi)$ given by

$$t_J(\phi)(u, v) = \phi(u, v) + J(\phi(Ju, v) + \phi(u, Jv)) - \phi(Ju, Jv).$$

Proof. We claim there is an exact sequence

$$\mathfrak{g}^{(1)} \hookrightarrow \mathrm{Hom}(V, \mathfrak{g}) \stackrel{\partial}{\to} \mathrm{Hom}(\Lambda^2 V, V) \stackrel{t_J}{\to} \mathrm{Hom}(\Lambda^2 V, V) \stackrel{\partial}{\partial_J} \mathrm{Hom}(\Lambda^2 V, V)$$

where ∂_J is given by

$$\partial_J(\phi)(u, v) = \phi(Ju, v) + \phi(u, Jv) + 2J\phi(u, v)$$

and has as kernel precisely the space Hom_J from the statement. □

Note that this lemma implies that, for almost complex structures, the intrinsic torsion is an element

$$T \in \Omega^2_J(M, TM) = \{\omega \in \Omega^2(M, TM) : \omega(JX, Y) = \omega(X, JY) = -J\omega(X, Y)\}$$

and the explicit formula for t_J identifies this with the Nijenhuis torsion of J. In other words, the intrinisic torsion of an almost complex structure J is precisely its Nijenhuis torsion. Hence, for almost complex structures,the converse of Theorem 5.52 does hold (and is the Newlander-Nirenberg theorem).

5.59. Example: symplectic structures: Let's now move to the symplectic algebra $sp(V, \omega)$ associated to a symplectic vector space (V, ω).

Lemma 5.60. *For the symplectic Lie algebra $sp(V, \omega)$, $sp(V, \omega)^{(1)}$ is isomorphic to S^3V* and one has an isomorphism*

$$\mathcal{T}(sp(V, \omega)) \cong \Lambda^3 V^*$$

by the isomorphism which associates to the class of $\phi \in Hom(\Lambda^2 V, V)$ modulo the image of ∂ the bilinear map $\partial_\omega(\phi)$ given by

$$\partial_\omega(\phi)(u, v, w) = \omega(\phi(u, v), w) + \omega(\phi(v, w), u) + \omega(\phi(w, u), v).$$

Proof. We have an exact sequence

$$S^3V^* \xrightarrow{i} \mathrm{Hom}(V, sp(V, \omega)) \xrightarrow{\partial} \mathrm{Hom}(\Lambda^2 V, V) \xrightarrow{\partial_\omega} \Lambda^3 V^*.$$

□

As before, this implies that the intrinsic torsion of an almost symplectic structure ω is precisely its differential $d\omega$. Hence, for almost symplectic structures,the converse of Theorem 5.52 does hold (and is the Darboux theorem).

5.61. Example: foliations: Similarly (to be written down), also with the conclusion that, for distributions, the converse of Theorem 5.52 does hold (and is the Frobenius theorem).

5.62. An integrability result: Although the converse of Theorem 5.52 does not hold in general, let us at least prove a weaker version of it.

Theorem 5.63. *A G-structure $\mathcal{S}$ is integrable if and only if, locally, M admits flat torsion free connections compatible with $\mathcal{S}$.*

Proof. We choose a connection as in the statement, say defined over the entire M. Note that the structure equations simplify to

$$d\theta = \omega \wedge \theta \quad (\text{in } \Omega^2(\mathcal{S}, V),$$

$$d\omega = \frac{1}{2}[\omega, \omega] \quad (\text{in } \Omega^2(\mathcal{S}, \mathfrak{g}).$$

In components, we deal with structure equations with constant coefficients. These coefficients can be recognized as the structure constants of a larger Lie algebra; but let us proceed intrinisically. We consider the semi-direct product Lie algebra

$$\tilde{\mathfrak{g}} = \mathfrak{g} \oplus V$$

and the forms θ, ω are put together into a 1-form

$$\Omega = (\theta, \omega) \in \Omega^1(\mathcal{S}, \tilde{\mathfrak{g}}).$$

Then the structure equations can be re-written as

$$d\Omega + \frac{1}{2}[\Omega, \Omega] = 0$$

hence, by a general result (see below[3]), one finds a local difeomorphism

$$f : \mathcal{S} \to \tilde{G} = G \times V$$

so that Ω is the pull-back of the Maurer-Cartan form of $\tilde{G}$. Clearly, the map will be G-equivariant, hence we obtain a local diffeomorphism

$$f : \tilde{M} \to V$$

compatible with teh G-structures. Locally, one can restrict to diffeomorphisms, proving integrability. □

5.64. Flat *G*-structures: The last discussion brings us to some rather special G-structure: flat ones (which is more than integrable!). By a flat G-structure on M we mean a G-structure and a torsion-free connection that is compatible with $\mathcal{S}$ (so that the previous theorem implies integrability).

The previous proof shows that, by passing to the universal cover (where the G-structure clearly lifts), one has an associated "classifying map", called the developping map of the G-structure,

$$f : \tilde{M} \to \mathbb{R}^n$$

which is a local diffeomorphism with the property that $\mathcal{S}$ is the pull-back by f of the standard G-structure on $\mathbb{R}^n$. An interesting question is whether this map is a diffeomorphism when M is compact. The answer is know to be true for Riemannian structures, but it is still open for other structures (e.g. for integral affine structures it is know as the Markus conjecture).

6 Part 8: Prolongations; *G*-structures of finite type

Discuss the first and then higher order prolongations of $\mathfrak{g} \subset gl_n$.

Go on an define the (first) prolongation of a G-structure and indicate the higher order structure functions.

Give the definition of finite order structures and note that, after prolongations, finite order structures give rise to $\{e\}$-structures. As solution to the

[3]to be added. It is the one that says that, for a manifold M, a Lie algebra $\mathfrak{g}$, forms $\omega \in \Omega^1(M, \mathfrak{g})$ satisfying $d\omega + \frac{1}{2}[\omega, \omega] = 0$ are allways pull-backs, via a smooth map $f : M \to G$, of the Maurer-Cartan form.

integrability problem in this case, deduce Theorem 33 from page 339. Also deduce that the automorphism group of a finite type G-structure is a Lie group Corollary 4.2 page 348 in Sternberg). Don't forget to give the Riemannian example. Maybe say that things are more complicated in general (say something about the symplectic case?).

7 Part 9: Elliptic G-structures

Here I would do, in detail, Theorem 4.1 from Kobayashi's book. We need some analytic preparation here. It would be nice to explain better here the reason of the term "elliptic" in this case. For this, the paper by Singer-Sternberg (or, better: the few pages where they discuss G-structures and then the other few pages in which they discuss prolongations) may be helpful.

8 What else?

What I described so far would keep us busy for a while. It is not clear to me how much time we will have left after all these (if any). There some interesting/worth learning things one could still do here:

- More on the equivalence problem, the definition of exterior differential systems + explaining the Cartan-Kahler theorem and how to use it for (analytic) G-structures. (Part of) the paper by Singer-Sternberg may be, again, one of the best references for this (because is short and gets to the point); much more is done in the book by Bryant et comp on exterior differential systems. This would take 2-3 lectures.

- Something that would take one lecture would be the discussion of pseudogroup structures- from section 8 (starting on page 33) of Kobayashi's book. **This may be actually the best/nicest** since it would allow us to treat more geometric structures- such as affine structures, contact structures etc (going more into details, also with contact structures, would require one extra-lecture).