Towards a Solution of Large N Double-Scaled SYK

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The Sachdev-Ye-Kitaev (SYK) model

SYK is a quantum mechanical model (0 + 1 dimensions) involving $N$ Majorana fermion $\psi_i$, $i = 1, \cdots, N$, with random all-to-all interactions

$$H = \frac{i^p}{2} \sum_{1 \leq i_1 < \cdots < i_p \leq N} J_{i_1 \cdots i_p} \psi_{i_1} \cdots \psi_{i_p}$$

The fermions satisfy the algebra

$$\{\psi_i, \psi_j\} = 2\delta_{ij}$$

Interesting because can do calculations at large $N$ and strong coupling, and find that it is maximally chaotic (chaos bounded [Maldacena, Shenker, and Stanford, 2016]). Features of holographic theories with semi-classical duals, so can study holography.
The SYK model

Usually studied using summation of Feynman diagrams, leading to Schwinger-Dyson equations. In the IR, a conformal ansatz is plugged in and solved (Polchinski and Rosenhaus, 2016, Maldacena and Stanford, 2016).

\[
\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega), \quad \Sigma(\tau) = J^2 G(\tau)^{p-1}
\]

**Goal**: We will take a combinatorial approach, allowing to do exact computations (at all energy scales).
Double-Scaled SYK

Usually $p$ is held fixed (independent of $N$) and $N \to \infty$.

Double-scaled SYK = we take $p$ (even) to scale as $\sqrt{N}$:

$$N \to \infty, \quad \lambda = \frac{2p^2}{N} = \text{fixed}$$  

[Erдёs and Schrёder, 2014], [Cotler et al., 2017], [Berkooz, Narayan, and Simon, 2018]

Will be natural to denote

$$q \equiv e^{-\lambda}$$

The more standard SYK $\leftrightarrow q \to 1$

The $J$’s are independent and Gaussian (actually enough to assume they are independent, have zero mean, uniformly bounded moments) with

$$\langle J_{i_1 \ldots i_p}^2 \rangle J = \left( \frac{N}{p} \right)^{-1}$$

In the double-scaling (1), this differs by a factor of $\lambda$ from usual convention [Maldacena and Stanford, 2016].
Chord diagrams
Chord diagrams for the partition function

Consider moments from which can get immediately the (averaged) partition function

$$\langle \text{tr} \ H^k \rangle_J$$

Denote \( \{i_1, \cdots, i_p\} \leftrightarrow I \), so

$$H = \frac{i^p}{2} \sum_I J_I \cdot \psi_I$$

where \( \psi_I = \psi_{i_1} \cdots \psi_{i_p} \).

$$\langle \text{tr} \ H^k \rangle_J = \frac{i^{kp}}{2} \sum_{I_1, \cdots, I_k} \langle J_{I_1} \cdots J_{I_k} \rangle_J \text{ tr } \psi_{I_1} \cdots \psi_{I_k}.$$  

By Wick’s theorem, the \( I_j \) come in pairs.
Chord diagrams for the partition function

Wick’s theorem $\rightarrow$ sum over pairings $\Leftrightarrow$ sum over chord diagrams (circular since trace).
Each node $\leftrightarrow H$ insertion.

For each chord diagram left with

\[
\left( \frac{N}{p} \right)^{-k/2} i^{kp/2} \sum_{I_1, \cdots, I_{k/2}} \text{tr} \, \psi_{I_1} \cdots \psi_{I_1} \cdots
\]

Now commute nodes to bring all pairs to be neighboring:

\[
\psi_{I_j} \quad \psi_{I_{j'}} = \psi_{I_{j'}} \quad \psi_{I_j} \times (-1)^{|I_j \cap I_{j'}|}
\]
Chord diagrams for the partition function

The $\binom{N}{p}^{-k/2}$ factor (number of terms in the sum) turns counting to probabilities.

For $p \ll N$ can do this by choosing independently the $p$ points of $I_j$ $(p$ trials, in intersection with probability $p/N$).

$$|I_j \cap I_{j'}| \sim \text{Pois} \left( \frac{p^2}{N} \right)$$

Since $p^2/N \sim O(1)$, the different intersections are independent.
Chord diagrams for the partition function

Each intersection then gives \( (n = |I_j \cap I_{j'}|) \)

\[
\sum_{n=0}^{\infty} \left( \frac{(p^2/N)^n}{n!} e^{-p^2/N} \right) (-1)^n = e^{-\lambda} = q
\]

Using \( i^{kp/2} \text{tr} \psi_{I_1} \psi_{I_1} \psi_{I_2} \psi_{I_2} \cdots = 1 \) we get

\[
\langle \text{tr} H^k \rangle_J = \sum_{\text{Chord diagrams}} q^{\# \text{ intersections}}
\]

For example

\[
\langle \text{tr} H^4 \rangle_J = 2 + q
\]
Operators

Similarly to the Hamiltonian, consider random operators with different $p_A \sim \sqrt{N}$

$$M_A = i^p_A/2 \sum_{1 \leq i_1 < \cdots < i_{p_A} \leq N} J_{i_1 \cdots i_{p_A}}^{(A)} \psi_{i_1} \cdots \psi_{i_{p_A}}$$

$A$ - flavor. The $J$’s are again random, independent, with zero mean and

$$\langle J_{i_1 \cdots i_{p_A}}^{(A)} J_{j_1 \cdots j_{p_B}}^{(B)} \rangle_J = \left( \frac{N}{p_A} \right)^{-1} \delta^{AB} \delta_{i_1,j_1} \delta_{i_2,j_2} \cdots$$

(and independent of the Hamiltonian couplings).
Correlation function moments

$$\langle \text{tr } H^{k_1} M_1 H^{k_2} M_1 \cdots \rangle_J$$

From the averaging, the Hamiltonian insertions are paired, the $M_1$ insertions are paired.
The only difference is the probability distribution of the number of sites in the intersection. For sets of size $p, p_A$ the intersection is distributed $\text{Pois} \left( \frac{pp_A}{N} \right)$. So

\[
\begin{align*}
\begin{array}{c}
\times \\
q = e^{-2p^2/N}
\end{array}
\quad = \begin{array}{c}
\begin{array}{c}
\times \\
\tilde{q}_A = e^{-2pp_A/N}
\end{array}
\end{array}
\end{align*}
\]

\[
\langle \text{tr} \ H^{k_1} M_1 H^{k_2} M_1 \cdots \rangle_J = \\
= \sum_{\text{Chord diagrams}} q^\# \ H-H \text{ intersections} \prod_A \tilde{q}_A^\# \ H-M_A \text{ intersections}
\]
Effective Hilbert space and analytic evaluation
Want to evaluate the sum over chord diagrams [Berkooz, Narayan, and Simon, 2018]. Cut open the chord diagrams at an arbitrary point.

Recall each node is a Hamiltonian insertion, and between each two insertions there is a propagating state $\cdots H H \cdots = \cdots H|l⟩⟨l|H \cdots$. 
Effective Hilbert space and analytic evaluation

**Partition function**

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Recall each node is a Hamiltonian insertion, and between each two insertions there is a propagating state \( \cdots HH \cdots = \cdots H|l\rangle\langle l|H \cdots \). Effective Hilbert space \( \mathcal{H} \) with basis \( |l\rangle \), number of chords \( l = 0, 1, 2, \cdots \).
Effective Hilbert space and analytic evaluation

Partition function

Node ↔ Hamiltonian insertion. 2 transitions only:

|l⟩ → |l + 1⟩

|l⟩ → |l − 1⟩
In the latter case can close either of the \(l\) chords. Crossings \(\rightarrow 1 + q + q^2 + \cdots q^{l-1} = \frac{1-q^l}{1-q}\). Effective Hamiltonian

\[
T = \begin{pmatrix}
0 & \frac{1-q}{1-q} & 0 & 0 & \cdots \\
1 & 0 & \frac{1-q^2}{1-q} & 0 & \cdots \\
0 & 1 & 0 & \frac{1-q^3}{1-q} & \cdots \\
0 & 0 & 1 & 0 & \cdots \\
\vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

Can simply diagonalize and get the energies

\[
E(\theta) = \frac{2\cos(\theta)}{\sqrt{1-q}}, \quad \theta \in [0, \pi).
\]

Partition function is just

\[
\langle \text{tr} e^{-\beta H} \rangle_J = \int_0^\pi d\mu(\theta) \ e^{-\beta E(\theta)}
\]

The measure is

\[
d\mu(\theta) \equiv \frac{d\theta}{2\pi} (q; q)_\infty (e^{2i\theta}; q)_\infty (e^{-2i\theta}; q)_\infty, \text{ where } (a; q)_n = \prod_{k=0}^{n-1} (1 - aq^k).
\]
Correlation functions
Consider a region enclosed by a contracted pair of $M$-nodes.

Time evolution over this region (before was $T^k$)? In the Hilbert space we keep only number of solid chords, can we do that? Yes!
Correlation functions

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Time evolution over this region (before was $T^k$)? In the Hilbert space we keep only number of solid chords, can we do that? Yes!
Similarly to [Mertens, Turiaci, and Verlinde, 2017], it is convenient to organize the results for correlation functions using non-perturbative diagrammatic rules. The diagrams arise naturally here; these are just chord diagrams.

- **Propagator**

\[
\theta
\]

\[
\tau_2 \quad \tau_1 = e^{-\Delta \tau \cdot E(\theta)}
\]

- **Sum over energy eigenstates that propagate, or equivalently over \( \theta \), with measure**

\[
d\mu(\theta) = \frac{d\theta}{2\pi} (q, e^{\pm 2i\theta}; q)_\infty.
\]

- **Vertex**

\[
\theta_1
\]

\[
\theta_2
\]

\[
\gamma_l(\theta_1, \theta_2) = \sqrt{\frac{(\tilde{q}_A^2; q)_\infty}{(\tilde{q}_A e^{i(\pm \theta_1 \pm \theta_2)}; q)_\infty}}, \quad \tilde{q}_A = q^{l_A}
\]
Correlation functions

Diagrammatic rules

These rules give precisely the 2-point function and the first 4-point function $\langle M_1 M_1 M_2 M_2 \rangle$.

$$
\langle M_1 M_2 M_1 M_2 \rangle = \int \prod_{j=1}^{4} d\mu(\theta_j) e^{-\sum_j \beta_j E(\theta_j)} \gamma_{l_1}(\theta_1, \theta_4) \gamma_{l_1}(\theta_2, \theta_3) \gamma_{l_2}(\theta_1, \theta_2) \gamma_{l_2}(\theta_3, \theta_4) \cdot R
$$

So $R$ is associated to the crossing of chords.
The R-matrix

The chord diagram is reminiscent of holography, representing the hyperbolic disc, boundary is exactly our QM system. The chords intersection is scattering in the bulk – the R-matrix.

For the Schwarzian, the R-matrix is the $6j$ symbol of $SU(1, 1)$.

In double-scaled SYK, the R-matrix is the $6j$ symbol of the quantum group $U_q(su(1, 1))$!

The spectrum also matches to this quantum group. Suggests that the theory can be **completely solved** by symmetry considerations.
More on the results

- In $q \to 1$ and low energies, these results reduce exactly to those of the Schwarzian. But the results above are at all energies and for any $q$. 

Using saddle point for the 4-point function [Lam et al., 2018], calculated the Lyapunov exponent for small $\lambda$ and low energies $T \ll \sqrt{\lambda}$

$$\lambda L = 2\pi T - \frac{4}{\pi \lambda} - \frac{1}{2} T^2 + \cdots$$

The analysis did not use the trace, so holds trivially also for pure states in agreement with [Kourkoulou and Maldacena, 2017].
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Summary and future directions

Calculated exact correlation functions, including the 4-point function, in large $N$ double-scaled SYK and saw an emerging quantum group.

- Solving the model by $U_q(su(1, 1))$ symmetry considerations.
- Computing chaos for large $\lambda$ and temperature.
- The leading order in $N$ is basically completely solved. Suggests we can go to subleading in $N$.
- Bulk dual.
- Non-thermal mixed states.
Thank you!