NOTES ON THE J-HOMOMORPHISM

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1. Introduction

The *J-homomorphism* is a morphism

$$\pi_i(O(n)) \to \pi_{n+i}(S^n).$$

It may be defined as follows. Let $H(n)$ be the group of homotopy self-equivalences of S^n preserving the point at ∞. There is a natural map $O(n) \to H(n)$, since an orthogonal transformation $\mathbb{R}^n \to \mathbb{R}^n$ extends to a homeomorphism of S^n onto itself. If we give $O(n)$ the basepoint which is the identity and similarly for $H(n)$, we have a map of pointed spaces.

We can identify $H(n)$ with the union of two components of $\Omega^n S^n$ (which has a $\mathbb{Z}$ worth of connected components). As a result, there is a natural map

$$O(n) \to H(n) \to \Omega^n S^n.$$

Consequently, we get a natural map

$$\pi_i(O(n)) \to \pi_i(\Omega^n S^n) = \pi_{n+i}(S^n).$$

Let us observe that these maps are compatible in the following sense. There is an inclusion $O(n) \to O(n+1)$, and there is a suspension morphism $\pi_{n+i}(S^n) \to \pi_{n+1+i}(S^{n+1})$. These two are compatible in there is a commutative diagram

$$\begin{array}{ccc} \pi_i(O(n)) & \longrightarrow & \pi_{n+i}(S^n) \\ \downarrow & & \downarrow \\ \pi_i(O(n+1)) & \longrightarrow & \pi_{n+1+i}(S^{n+1}) \end{array}.$$

In fact, we need only show that there is a commutative diagram

$$\begin{array}{ccc} O(n) & \longrightarrow & H(n) \\ \downarrow & & \downarrow \\ O(n+1) & \longrightarrow & H(n+1) \end{array}$$

where the right vertical map is suspension. But this is easy to see.

As a result, we can take direct limits to get maps $\pi_i(O) \to \pi_i^s$ where the latter denotes the stable homotopy groups of spheres; O is the infinite orthogonal group.

Definition 1. The map $\pi_i(O(n)) \to \pi_{n+i}(S^n)$ is called the **J-homomorphism.** We will mostly be interested in the stable version $J : \pi_i(O) \to \pi_i^s$.

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The homotopy groups of O are known by Bott periodicity. We can state them:

i	0	1	2	3	4	5	6	7
$\pi_i(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

In particular, the homotopy groups of the form $\pi_{4n-1}(O)$ are infinite cyclic. The image in the stable homotopy groups is necessarily finite, since the stable homotopy groups are finite.

Here is the main result describing the image of J:

Theorem 1 (Adams, Quillen)**.**
(1) For $r > 0$ divisible by eight, J's image in π_r^s is finite cyclic, and J is a monomorphism.
(2) If $r \equiv 1 \mod 8$ and $r > 1$, then J is a monomorphism, and there is another summand of $\mathbb{Z}/2$ in π_r^s.
(3) If $r \equiv 2 \mod 8$, then π_r^s contains a summand $\mathbb{Z}/2$.
(4) If $r = 4s - 1$, then the image of J is a cyclic group of order $m(2s)$ and is a direct summand of π_r^s.

The part of the theorem which is not yet elucidated concerns the function $m(2s)$.

Definition 2. $m(2s)$ is the denominator of $B_s/4s$ (for B_s the Bernoulli numbers.

In fact, in Adams's second paper, the actual values are computed explicitly. Adams shows that $m(2s)$ is the positive integer whose 2-adic valuation is $3+v_2(s)$ and whose p-adic valuation for p odd is 0 if $2s \not\equiv 0 \mod (p-1)$ and $1 + v_p(t)$ if $t \equiv 0 \mod (p-1)$.

2. The groups $J(X)$

Adams's strategy is to bound from below and above the image of the J-homomorphism. Before mentioning this, we need an alternative description of it, which actually makes sense in a more general context.

Let X be a finite CW complex. Recall that $KO(X)$ is the K-group of real vector bundles on X, and $\widetilde{KO}(X)$ is the reduced K-group. Classes in $\widetilde{KO}(X)$ can be represented by stable equivalence classes of vector bundles $E \to X$: we say that two vector bundles E, E' are *stably equivalent* if there exist integers n, m such that

$$E \oplus \mathbb{R}^n \simeq E' \oplus \mathbb{R}^m.$$

We can define a weaker notion of *stable fiber homotopy equivalence.*

Definition 3. We say that two vector bundles E, E' over X are **fiber homotopy equivalence** if there are continuous maps over X

$$f : S(E) \to S(E'), \quad g : S(E') \to S(E)$$

for $S(E), S(E')$ the sphere bundles, such that the composites fg, gf are homotopic to the respective identities over X.

It is a theorem of Dold-Lashof that we can detect fiber homotopy equivalences via the following criterion. If there is a fiberwise map $f : S(E) \to S(E')$ which induces a homotopy equivalence on each fiber, then $S(E), S(E')$ are fiberwise homotopy equivalent (in fact, f has a fiberwise homotopy inverse). Incidentally, there are analogs in HTT for simplicial sets in the context of the co(ntra)variant model structures.

Definition 4. The group $J(X)$ is defined to be the collection of classes of vector bundles E modulo the relation of stable fiber homotopy equivalence: that is, the classes of E, E' are identified if $E \oplus \mathbb{R}^n, E' \oplus \mathbb{R}^n$ have fiberwise homotopy equivalent sphere bundles.

One has to check that this is in fact an abelian group, i.e. that addition of vector bundles preserves fiber homotopy equivalence; however, addition of vector bundles is basically fiberwise join. So it's ok. There is a natural homomorphism

$$j : \widetilde{KO}(X) \to J(X)$$

given by quotienting.

Let us try to connect this with the old definition in case X is a sphere S^r. In this case, $\widetilde{KO}(S^r)$ can be identified with $\pi_{r-1}(O)$ by the "clutching" construction. The claim is that $J(X)$ can be identified with the image of the previously defined J-homomorphism

$$\pi_{r-1}(O) \to \pi_{r-1}^s.$$

In fact, let's work out exactly when two elements f, g in $\pi_{r-1}(O)$ are identified in $J(S^r)$: they are if, for some $N \gg 0$ and for some reduction to $f, g : S^{r-1} \to O(N)$, there is a *fiber homotopy equivalence*, between the sphere bundles defined by f, g. This corresponds to saying that f, g are homotopic in the space of homotopy equivalences $S^N \to S^N$, and this is precisely the condition that f and g are identified in π_{r-1}^s under the usual J-homomorphism.

As a result, one can study these groups $J(X)$ instead of simply studying the J-homomorphism. It is known, and proved in Atiyah's paper "Thom complexes," that they are always finite. In fact, it is known that $J(X)$ is contained in $[X, BH]$ for H the "stable" homotopy equivalences of the sphere. The homotopy groups of this are finite (they are the stable homotopy groups of spheres).

3. The groups $J''(X)$

The language of K-theory is convenient, though, because it gives us various other tools. For instance, we have the Adams operations ψ^k; these are given by raising to the kth power on a line bundle and are additive (even ring) operations $KO(X) \to KO(X)$ for any X. Assuming the following, Adams was able to bound the image of the J-homomorphism:

> **Adams conjecture**. If $k \in \mathbb{N}$, then for any $x \in \widetilde{KO}(X)$, we have $k^n(\psi^k(x) - x) = 0$ in $J(X)$ for some $n \gg 0$.

So the Adams conjecture is saying that when one localizes at k and quotients by the kernel of j, the operation ψ^k doesn't do anything.

The Adams conjecture was proved by Quillen. If we believe it, we can work out an upper bound for the J-homomorphism in the $4n-1$ case. That is, we can see:

Proposition 1. The image of $J : \pi_{4n-1}(O) \to \pi_{4n-1}^s$ has order dividing $m(2n)$.

In fact, Adams defines for any finite complex X, a group $J'(X)$: this is defined by taking $KO(X)$, and forming the subgroup H defined as follows. Consider any function $f : \mathbb{N} \to \mathbb{N}$, and the subgroup H_f generated by elements of the form $k^{f(k)}(\psi^k x - x)$. We let $H = \bigcap_f H_f$.

Definition 5. $J'(X) := \widetilde{KO}(X)/H$ where H is as above.

According to the Adams conjecture (and the finite generation of $KO(X)$), we find that there is a surjection

$$J'(X) \to J(X).$$

Proof of the proposition. We can calculate $J'(S^{4n})$ and thus find an "upper bound" for $J(S^{4n})$. By Bott periodicity, we know that $\widetilde{KO}(S^{4n}) = \mathbb{Z}$, and we know that the complexification homomorphism

$$\widetilde{KO}(S^{4n}) \to \widetilde{K}(S^{4n}) = \mathbb{Z}$$

is nonzero (in fact, its image is at least $2\mathbb{Z}$), and consequently $\widetilde{KO}(S^{4n})$ is generated by a class x whose complexification is nonzero. As a result, we can easily work out what the Adams operations on x are, in view of the fact that we know them for $\widetilde{K}(S^{2n})$ by ordinary Bott periodicity.

In fact, we use the fact from ordinary complex K-theory that $\psi^k(y) = k^{2n}y$ for $y \in \widetilde{K}(S^{4n})$. Consequently, the same holds for any element of $\widetilde{KO}(S^{4n})$.

In fact, we need only prove this in complex K-theory, but the above observations; but we can check this for S^2 using the generator $H-1$ of $\widetilde{K}(S^2)$. Then, everything else follows by induction and taking powers.

As a result, we can determine the group $J'(S^{4n})$. Let $x \in \widetilde{KO}(S^{4n})$ be a generator. For all functions $f : \mathbb{N} \to \mathbb{N}$, we need to consider the subgroup generated by

$$k^{f(k)}(k^{2n}-1)x, \quad k \in \mathbb{N}$$

and take the intersection over all f.

Here we need a little number theory. In Adams's second paper, it is shown that as f varies, the greatest common divisor of the set $\left\{k^{f(k)}(k^{2n}-1), k \in \mathbb{N}\right\}$ always divides the denominator $m(2n)$ of $B_{2n}/4n$, and choosing f large we can get precisely this. Consequently, it follows that $J'(S^{4n})$ is precisely $\mathbb{Z}/m(2n)\mathbb{Z}$. □

4. The cannibalistic classes ρ^k

To bound below the image of J (which had already been done in some cases by Milnor-Kervaire), Adams used a characterization of when something is zero in $J(X)$ in terms of characteristic classes: that is, he constructed a quotient $J''(X)$ of $J(X)$ and computed that.

We will need a general formalism of characteristic classes. Let F, T be cohomology theories. Suppose that they have a *natural theory of Thom classes* with respect to a certain class of vector bundles (e.g. complex vector bundles). That is, given such a vector bundle $E \to X$, we should have a Thom class $u_E \in \widetilde{F}(X^E)$ which is natural in E, and similarly $t_E \in \widetilde{T}(X^E)$. Suppose moreover that we have a natural transformation of cohomology theories $f : F \to T$. Then, we can construct characteristic classes in T of any vector bundle $E \to X$.

Construction: Let $E \to X$ be a vector bundle. Consider the Thom class $u_E \in \widetilde{F}(X^E)$, and its image $f(u_E) \in \widetilde{T}(X^E)$. Inverse Thom it back to $T(X)$ to get a characteristic class of E

$$f(u_E)/t_E \in T(X).$$

This is clearly natural in E, and gives a characteristic class.

Example. Let F, T be $\mathbb{Z}/2$-cohomology, and f be the Steenrod square Sq^i. Then the characteristic class thus obtained is the Stiefel-Whitney class w_i.

There is a natural choice of Thom classes in K-theory for complex vector bundles: that is, complex K-theory is *complex oriented.* The construction is convenient: it has the multiplicative property. That is, if $E \to X, E' \to Y$ are vector bundles, then we have

$$X^E \wedge Y^{E'} = (X \times Y)^{E \oplus E'},$$

and the product of the Thom classes of E, E' is the Thom class for $E \oplus E' \to X \times Y$.

An explicit construction of the complex orientation can be given as follows. If $\pi : E \to X$ is a vector bundle, we take the Koszul complex

$$0 \to \pi^* E \to \bigwedge^2 \pi^* E \to \dots$$

on E; the boundary map on (v, x) is given by wedging with $v \in E_x$. This is a complex of vector bundles on E, and outside of the zero section it is exact; by the "difference bundle" construction, it defines an element of $\widetilde{K}(X^E)$, which is the Thom class. We denote the Thom class by u_E.

Atiyah, Bott, and Shapiro have constructed (using Clifford theory) natural Thom classes in KO-theory for Spin-bundles, and natural Thom classes in K-theory for Spin^c-bundles (the latter is a weaker condition than having a complex structure). This is used in the construction of the ρ^k for real vector bundles, which we won't try to deal with here.

Example. Let F be K-theory, and let T be ordinary cohomology with $\mathbb{Q}$-coefficients. Let f be the Chern character. Then the associated characteristic class of complex vector bundles is the **Borel-Hirzebruch class.** If the q_i are the Chern roots of a vector bundle E, then we have

$$\mathrm{Bh}(E) = \prod_i \frac{e^{q_i} - 1}{q_i}.$$

(This is not quite the usual definition.) Let's prove this.

By multiplicativity of the Thom isomorphisms and the splitting principle, we can restrict to the case of a line bundle. Then we can even reduce to the "universal case" of the "universal" line bundle over $\mathbb{CP}^\infty$. In fact, we know that $K(\mathbb{CP}^\infty) = \mathbb{Z}[[x]]$ for x the Euler class (in K-theory) of the canonical line bundle: in other words, x corresponds to $L-1$ for L the class of the canonical line bundle. The Thom space $MU(1) = \mathbb{CP}^\infty$ is the same thing and the Thom class is $L-1$. The Thom isomorphism

$$K(\mathbb{CP}^\infty) \simeq \widetilde{K}(MU(1)) = \widetilde{K}(\mathbb{CP}^\infty)$$

is just multiplication by $H-1$.

The Thom class is $x = H - 1$, as before. The completed (rational) cohomology ring is $\mathbb{Q}[[y]]$ where y has degree two and $c_1(H) = y$ (so y is the Thom class in rational cohomology). Applying the Chern character to the Thom class gives $e^y - 1$, and then we have to divide by y for the inverse Thom isomorphism.

We have the Adams operations $\psi^k : K(X) \to K(X)$ for any space X. In view of these, and the complex orientation of K, we have:

Definition 6. The **cannibalistic classes** $\rho^k(E) \in K(X)$ of a complex vector bundle $E \to X$ are defined as $\psi^k(u_E)/u_E \in K(X)$, for u_E the Thom class.

So, in other words, we start with $1 \in K(X)$, apply the Thom isomorphism, apply ψ^k, and apply the inverse to the Thom isomorphism. In view of the multiplicativity properties, we have

$$\rho^k(E \oplus E') = \rho^k(E)\rho^k(E').$$

We can describe the cannibalistic classes explicitly using the following formalism.

Example. $\rho^k(L) = 1 + L + L^2 + \cdots + L^{k-1}$ when L is a line bundle.

To prove this, we may as well work with the universal line bundle H on $\mathbb{CP}^\infty$. The Thom space is, as before, $MU(1) = \mathbb{CP}^\infty$ and the Thom class is $H - 1 \in \widetilde{K}(\mathbb{CP}^\infty)$.

Now to compute $\rho^k(H)$, we need to divide $\psi^k(H-1)$ by $H-1$; this gives

$$\frac{H^k - 1}{H - 1} = 1 + H + \dots H^{k-1}.$$

As an example, we can figure out the cannibalistic classes for the vector bundles on S^{2n}.

This example completely determines the characteristic class ρ^k, in view of the splitting principle. Using it, and using the immediate consequence that $\rho^k(n) = k^n$, we can define the ρ^k as operations from K-theory to K-theory localized at k. In Adams's blue book, it is shown that the ρ^2 of a stably almost complex manifold can be used to compute the signature by evaluating on the fundamental class (this is the K-theoretic analog of the Hirzebruch signature formula).

We have now theoretically figured out all we need to know about ρ^k, but to help things out, we can obtain the Chern character of ρ^k. Let us make the observation that the Chern character induces an isomorphism

$$\mathrm{Ch} : K(X) \otimes \mathbb{Q} \to H^{even}(X; \mathbb{Q})$$

for X a finite complex. The corresponding operation to ψ^k is just multiplication by k^i on H^{2i}.

Definition 7. We write ψ^k_H for the operation in ordinary (even) cohomology which is just multiplication by k^i on H^{2i}. As a result, we have a commutative diagram

$$\begin{array}{ccc} K(X) & \xrightarrow{\psi^k} & K(X) \\ \downarrow{\scriptstyle \mathrm{Ch}} & & \downarrow \\ H^{even}(X;\mathbb{Q}) & \xrightarrow{\psi^k_H} & H^{even}(X;\mathbb{Q}) \end{array} .$$

The vertical maps become isomorphisms after tensoring with $\mathbb{Q}$.

The next result will be our basic computational tool.

Proposition 2. For a vector bundle E of dimension n, we have

$$\mathrm{Ch}\rho^k(E) = k^n \psi^k_H(\mathrm{Bh}(E))/\mathrm{Bh}(E) \tag{1}$$

for $\mathrm{Bh}(E)$ the Borel-Hirzebruch class of E.

Recall that the *Borel-Hirzebruch class* is the multiplicative characteristic class associated to $\frac{e^x - 1}{x}$. This is the inverse of the usual terminology.

Proof. We need only verify the result for a line bundle; everything here sends direct sums in E to products. Let x be the class in K-theory of a line bundle; then we have that $\psi^k(x) = x^k$. Let $\overline{x} = c_1(x)$. We have

$$\mathrm{Ch}\rho^k(E) = \mathrm{Ch}(1 + x + \cdots + x^{k-1}) = 1 + e^{\overline{x}} + e^{2\overline{x}} + \cdots + e^{(k-1)\overline{x}}.$$

We have, on the other hand,

$$\mathrm{Bh}(E) = \frac{e^{\overline{x}} - 1}{\overline{x}},$$

and since $\overline{x}$ has degree two,

$$\psi^k_H(\mathrm{Bh}(E)) = \frac{e^{k\overline{x}} - 1}{k\overline{x}}.$$

If we combine these, we have

$$k\psi^k(\mathrm{Bh}(E))/\mathrm{Bh}(E) = \frac{e^{k\overline{x}} - 1}{e^{\overline{x}} - 1} = 1 + e^{\overline{x}} + \cdots + \overline{x}^{k-1}.$$

These are the same, now. $\square$

5. Bounding below the image of J

For simplicity, we will do what Adams does for complex K-theory rather than real K-theory. In other words, we will consider the composite

$$\widetilde{K}(X) \to \widetilde{KO}(X) \to J(X)$$

and try to bound below the image by bounding above the kernel.

The key strategy is the observation that if a vector bundle is fiber-homotopically trivial, then its cannibalistic classes are strongly restricted (just by looking at the definitions). Applying the previous computation, we'll translate that into a statement on the Chern character. This will give a strong (though off by a factor of two) bound on the image.

Proposition 3. If a vector bundle E is fiber-homotopically trivial, then

$$\rho^k(E) = k^{\dim E} \frac{\psi^k(1+y)}{1+y} \tag{2}$$

for some $y \in \widetilde{K}(X)$.

This result is more complicated for real vector bundles because then one has to specify exactly when the ρ^k can even be defined.

Proof. In fact, we have a homotopy equivalence of the Thom spaces

$$X^E \to X^{E'}$$

where E' is a trivial bundle. We will show that more generally, if E, E' are any vector bundles, and we have a fiberwise homotopy equivalence

$$\phi : X^E \to X^{E'},$$

then $\rho^k(E) = \rho^k(E')\frac{\psi^k(1+y)}{1+y}$ for some $y \in \widetilde{K}(X)$ (note that $1+y$ is a unit). In fact, let's compare $\phi^* u_{E'}$ with u_E; clearly there is an element, which we see is of degree one by restricting to fibers, $1 + y \in K(X)$, such that

$$\phi^* u_{E'} = u_E(1+y).$$

Now we recall that

$$\rho^k(E) = \psi^k(u_E)/u_E,$$

and consequently

$$\rho^k(E') = \psi^k(u_{E'})/u_{E'} = \psi^k(\phi^* u_{E'})/\phi^* u_{E'} = (\psi^k(u_E)/u_E)\psi^k(1+y)/(1+y).$$

∎

Combining this result with the previous one, we can get a criterion for when a bundle is fiber homotopically trivial. As we have seen, the necessary condition is that $\rho^k(E) = k^{\dim E}\psi^k(1+y)/1+y$, for each k.

Henceforth, let us assume that we are working in a finite complex X such that $K(X)$ is *torsion-free* (this happens, e.g., if the cohomology of X is torsion-free). Let E be a complex vector bundle which is fiber homotopically trivial. Let $n = \dim E$. Thus, an equivalent restatement is that $E - n \in \widetilde{K}(X)$ maps to zero in $J(X)$. It follows that (if we cancel copies of k^n)

$$\psi^k_H(\mathrm{Bh}(E))/\mathrm{Bh}(E) = \psi^k_H(\mathrm{Ch}(1+y))/\mathrm{Ch}(1+y).$$

Then we have:

$$\psi^k_H(\mathrm{Bh}(E))/\mathrm{Bh}(E) = \psi^k_H(\mathrm{Ch}(1+y))/\mathrm{Ch}(1+y),$$

and this gives

$$\psi_H^k(\mathrm{Bh}(E)/\mathrm{Ch}(1+y)) = \mathrm{Bh}(E)/\mathrm{Ch}(1+y).$$

We are working in *rational* cohomology here, and the only fixed points of ψ_H^k are the multiples of 1. In fact, it follows that

$$\mathrm{Bh}(E) = \mathrm{Ch}(u_E).$$

Corollary 1. If E is a stably-fiber-homotopically vector bundle, then

$$\mathrm{Bh}(E) = \mathrm{Ch}(1+y) \tag{3}$$

for some $y \in \widetilde{K}(X)$.

If we take *logarithms* (formally), we can write this in a form that makes more transparent the relation with the Bernoulli numbers; it will imply integrality relations on the Borel-Hirzebruch classes. The *Bernoulli numbers* were defined via

$$\frac{x}{1-e^x} = \sum \frac{\beta_s x^s}{s!}, \quad B_s := \beta_{2s}.$$

We will just use the β_s, though. We can see that

$$\log\left(\frac{e^x-1}{x}\right) = -\sum \alpha_t \frac{x^t}{t!}$$

where $\alpha_t = \beta_t/t, t > 1$, by differentiation. It follows that if $x \in K(X)$, then we have

$$\log(\mathrm{Bh}(x)) = -\sum_{t=1}^{\infty} \alpha_t \mathrm{Ch}_t(x) \tag{4}$$

if $\mathrm{Ch}_t(x)$ is the component of $\mathrm{Ch}(x)$ of degree $2t$. Here the α_t as previously.

Proof. Both are additive in x, so we reduce to the case of a line bundle. Then it is clear, because $\mathrm{Ch}_t(x) = \frac{c_1(x)^t}{t!}$ and $\log(\mathrm{Bh}(x)) = \log \frac{e^{c_1(x)}-1}{c_1(x)}$. □

Corollary 2. Let X be a space where all cup products are zero. Let E be a vector bundle over X which is stably fiber-homotopically trivial; let $x = E - \dim E$ in $\widetilde{K}(X)$. Then there is $y \in \widetilde{K}(X)$ such that

$$(-1)^t \alpha_t \mathrm{Ch}_t(x) = \mathrm{Ch}_t(1+y), \forall t.$$

Proof. In fact, we see this from (3) (the subtraction of a trivial bundle does nothing) by taking logarithms; since all cup products are zero, we have $\log(1 + \mathrm{Ch}(y)) = \mathrm{Ch}(y)$. □

This puts fairly strong restrictions on what $\mathrm{Ch}_t(x)$ can be.

Let's now say that t is an integer which is divisible by four. Consider the J-homomorphism

$$\widetilde{K}(X) \to J(X);$$

we have seen that if x is in the kernel, then

$$\alpha_{t/2}\mathrm{Ch}_{t/2}(x)$$

is in the image of $\mathrm{Ch}_{t/2}$ (which is a monomorphism on $\widetilde{K}(S^t) = \mathbb{Z}$). In particular, x must be divisible by the denominator of $\alpha_{t/2}$.

We have proved:

Corollary 3. If $4 \mid t$, then $J(S^{4t})$, or equivalently the image of $\pi_{t-1}(O) \to \pi_{t-1}^s$, has image a cyclic group of order divisible by the denominator of $\alpha_{t/2}$.

Since $\alpha_u = 0$ if u is odd, this is not very interesting for things not divisible by four.

6. Bibliography

(1) Karoubi – K theory
(2) Adams – On the groups J(X)
(3) Bott – Lectures on K(X) (a different viewpoint).