Equivariant Chern characters

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Outline

- Dolds rational computation of a generalized homology theory in terms of singular homology.
- Equivariant homology theories and Chern characters
- Applications to the Farrell-Jones and the Baum-Connes Conjecture
- Rational computation of the topological K-theory of BG for a group G.

Theorem (Dold)

Let $\mathcal{H}_*$ be a generalized homology theory with values in Λ-modules for $\mathbb{Q} \subseteq \Lambda$.
Then there exists for every $n \in \mathbb{Z}$ and every CW-complex X a natural isomorphism

$$\bigoplus_{p+q=n} H_p(X;\Lambda) \otimes_\Lambda \mathcal{H}_q(pt) \xrightarrow{\cong} \mathcal{H}_n(X).$$

- This means that the Atiyah-Hirzebruch spectral sequence collapses in the strongest sense.
- The assumption $\mathbb{Q} \subseteq \Lambda$ is necessary.

Dolds' Chern character for a CW-complex X is given by the following composite

$$\mathrm{ch}_n \colon \bigoplus_{p+q=n} H_p(X;\mathcal{H}_q(*)) \xleftarrow{\alpha} \bigoplus_{p+q=n} H_p(X;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathcal{H}_q(*)$$

$$\xleftarrow[\cong]{\bigoplus_{p+q=n} \mathrm{hur} \otimes \mathrm{id}} \bigoplus_{p+q=n} \pi_p^s(X_+,*) \otimes_{\mathbb{Z}} \mathcal{H}_q(*)$$

$$\xrightarrow{\bigoplus_{p+q=n} D_{p,q}} \mathcal{H}_n(X).$$

Definition (*G*-homology theory)

A *G-homology theory* $\mathcal{H}_*$ is a covariant functor from the category of G-CW-pairs to the category of $\mathbb{Z}$-graded Λ-modules together with natural transformations

$$\partial_n(X,A)\colon \mathcal{H}_n(X,A) \to \mathcal{H}_{n-1}(A)$$

for $n \in \mathbb{Z}$ satisfying the following axioms:

- G-homotopy invariance;
- Long exact sequence of a pair;
- Excision;
- Disjoint union axiom.

Definition (Equivariant homology theory)

An *equivariant homology theory* $\mathcal{H}_*^?$ assigns to every group G a G-homology theory $\mathcal{H}_*^G$. These are linked together with the following so called *induction structure*: given a group homomorphism $\alpha\colon H \to G$ and a H-CW-pair (X, A) there are for all $n \in \mathbb{Z}$ natural homomorphisms

$$\mathrm{ind}_\alpha\colon \mathcal{H}_n^H(X, A) \quad \to \quad \mathcal{H}_n^G(\mathrm{ind}_\alpha(X, A))$$

satisfying

- Bijectivity
  If $\ker(\alpha)$ acts freely on X, then ind_α is a bijection;
- Compatibility with the boundary homomorphisms
- Functoriality in α
- Compatibility with conjugation

Examples for equivariant homology theories are

- Given a $\mathcal{K}_*$ non-equivariant homology theory, put

$$\begin{aligned} \mathcal{H}_*^G(X) &:= \mathcal{K}_*(X/G); \\ \mathcal{H}_*^G(X) &:= \mathcal{K}_*(EG \times_G X) \quad \text{Borel homology}. \end{aligned}$$

- Equivariant bordism $\Omega_*^?(X)$;
- Equivariant topological K-theory $K_*^?(X)$;
- Given a functor $\mathbf{E}\colon \mathcal{G}roupoids \to \mathcal{S}pectra$ sending equivalences to weak equivalences, there exists an equivariant homology theory $\mathcal{H}_*^?(-;\mathbf{E})$ satisfying

$$\mathcal{H}_n^H(\mathrm{pt}) \cong \mathcal{H}_n^G(G/H) \cong \pi_n(\mathbf{E}(H)).$$

Theorem (L.)

Let $\mathcal{H}_*^?$ be a proper equivariant homology theory with values in Λ-modules for $\mathbb{Q} \subseteq \Lambda$. Suppose that $\mathcal{H}_*^?$ has a Mackey extension. Let I be the set of conjugacy classes (H) of finite subgroups H of G.
Then there is for every group G, every proper G-CW-complex X and every $n \in \mathbb{Z}$ a natural isomorphism called equivariant Chern character

$$\mathrm{ch}_n^G\colon \bigoplus_{p+q=n} \bigoplus_{(H)\in I} H_p(C_GH\backslash X^H;\Lambda) \otimes_{\Lambda[W_GH]} S_H\left(\mathcal{H}_q^H(*)\right) \xrightarrow{\cong} \mathcal{H}_n^G(X)$$

- C_GH is the centralizer and N_GH the normalizer of $H \subseteq G$;
- $W_GH := N_GH/H \cdot C_GH$ (This is always a finite group);
- $S_H\left(\mathcal{H}_q^H(*)\right) := \mathrm{cok}\left(\bigoplus_{\substack{K\subset H\\ K\neq H}} \mathrm{ind}_K^H : \bigoplus_{\substack{K\subset H\\ K\neq H}} \mathcal{H}_q^K(*) \to \mathcal{H}_q^H(*)\right)$.
- $\mathrm{ch}_*^?$ is an equivalence of equivariant homology theories.

Theorem (Artin's Theorem)

Let G be finite. Then the map

$$\bigoplus_{C \subset G} \mathrm{ind}_C^G \colon \bigoplus_{C \subset G} \mathrm{Rep}_{\mathbb{C}}(C) \to \mathrm{Rep}_{\mathbb{C}}(G)$$

is surjective after inverting $|G|$, where $C \subset G$ runs through the cyclic subgroups of G.

Let C be a finite cyclic group. The Artin defect is the cokernel of the map

$$\bigoplus_{D \subset C, D \neq C} \mathrm{ind}_D^C \colon \bigoplus_{D \subset C, D \neq C} \mathrm{Rep}_{\mathbb{C}}(D) \to \mathrm{Rep}_{\mathbb{C}}(C).$$

For an appropriate idempotent $\theta_C \in \mathrm{Rep}_{\mathbb{Q}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right]$ the Artin defect is after inverting the order of $|C|$ canonically isomorphic to

$$\theta_C \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|C|}\right].$$

Let K_*^G be equivariant topological K-theory. We get for a finite subgroup $H \subseteq G$

$$K_n^G(G/H) = K_n^H(\mathrm{pt}) = \begin{cases} \mathrm{Rep}_{\mathbb{C}}(H) & \text{if } n \text{ is even;} \\ \{0\} & \text{if } n \text{ is odd.} \end{cases}$$

Example

Let G be finite, $X = \{*\}$ and $\mathcal{H}_*^? = K_*^?$. Then we get an improvement of Artin's theorem, namely, the equivariant Chern character induces to an isomorphism

$$\mathrm{ch}_0^G(pt)\colon \bigoplus_{(C)} \mathbb{Z} \otimes_{\mathbb{Z}[W_G C]} \theta_C \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|G|}\right] \xrightarrow{\cong} \mathrm{Rep}_{\mathbb{C}}(G) \otimes_{\mathbb{Z}} \mathbb{Z}\left[\frac{1}{|G|}\right]$$

Theorem (Davis-L)

Let R be a ring (with involution). There exist covariant functors

$$\begin{aligned} \mathbf{K}_R\colon \mathrm{Groupoids} &\to \mathrm{Spectra}; \\ \mathbf{L}_R^{\langle j \rangle}\colon \mathrm{Groupoids} &\to \mathrm{Spectra}; \\ \mathbf{K}^{\mathrm{top}}\colon \mathrm{Groupoids}^{\mathrm{inj}} &\to \mathrm{Spectra} \end{aligned}$$

with the following properties:

- *They send equivalences of groupoids to weak equivalences of spectra;*
- *For every group G and all $n \in \mathbb{Z}$ we have*

$$\begin{aligned} \pi_n(\mathbf{K}_R(G)) &\cong K_n(RG); \\ \pi_n(\mathbf{L}_R^{\langle j \rangle}(G)) &\cong L_n^{\langle j \rangle}(RG); \\ \pi_n(\mathbf{K}^{\mathrm{top}}(G)) &\cong K_n(C_r^*(G)). \end{aligned}$$

Definition (Family of subgroups)

A *family $\mathcal{F}$ of subgroups* of the group G is a set of subgroups of G which is closed under conjugation and taking subgroups.

Examples for families are

$\{1\}$	trivial subgroup
$\mathcal{FIN}$	finite subgroups
$\mathcal{VCYC}$	virtually cyclic subgroups
$\mathcal{ALL}$	all subgroups

Definition (Classifying space of a family)

Let $\mathcal{F}$ be a family of subgroups of G. A model for the *classifying space of the family $\mathcal{F}$* is a G-CW-complex $E_{\mathcal{F}}(G)$ such that $E_{\mathcal{F}}(G)^H$ is contractible if $H \in \mathcal{F}$ and is empty if $H \notin \mathcal{F}$.
Sometimes $\underline{E}G := E_{\mathcal{FIN}}(G)$ is called the *classifying space for proper G-actions*.

Theorem (tom Dieck)

The G-CW-complex $E_{\mathcal{F}}(G)$ is characterized uniquely up to G-homotopy by the property that for every G-CW-complex X whose isotropy groups belong to $\mathcal{F}$ there is up to G-homotopy precisely one G-map $X \to E_{\mathcal{F}}(G)$.

Obviously $E_{\{1\}}(G) = EG$ and $E_{\mathcal{ALL}}(G) = G/G$.

The spaces $\underline{E}G$ are interesting in their own right and have often very nice geometric models which are rather small. For instance

- Rips complex for word hyperbolic groups;
- Teichmüller space for mapping class groups;
- Outer space for the group of outer automorphisms of free groups;
- L/K for a connected Lie group L, a maximal compact subgroup $K \subseteq L$ and $G \subseteq L$ a discrete subgroup;
- CAT(0)-spaces with proper isometric G-actions, e.g., Riemannian manifolds with non-positive sectional curvature or trees.

Conjecture (Farrell-Jones)

The Farrell-Jones Conjecture for algebraic K-theory with coefficients in R for the group G predicts that the assembly map

$$H_n^G(E_{\mathcal{VCYC}}(G), \mathbf{K}_R) \to H_n^G(pt, \mathbf{K}_R) = K_n(RG)$$

is bijective for all $n \in \mathbb{Z}$.

The Farrell-Jones Conjecture gives a way to compute $K_n(RGH)$ in terms of $K_m(RV)$ for all virtually cyclic subgroups $V \subseteq G$ and all $m \leq n$.

Theorem (Bartels-L.-Reich)

The (Fibered) Farrell-Jones Conjecture for algebraic K-theory with (G-twisted) coefficients in any ring R is true for word-hyperbolic groups G.

It is analogous to the Baum-Connes Conjecture which is the version for topological K-theory of (reduced) group C^*-algebras.

Conjecture (Baum-Connes)

The Baum-Connes Conjecture predicts that the assembly map

$$K_n^G(\underline{E}G) = H_n^G(E_{\mathcal{F}in}(G), \mathbf{K}^{\mathrm{top}}) \to H_n^G(pt, \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*(G))$$

is bijective for all $n \in \mathbb{Z}$.

Theorem (L.)

Let G be a group. Let T be the set of conjugacy classes (g) of elements g ∈ G of finite order. There is a commutative diagram

$$\begin{array}{ccc} \bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q(\mathbb{C}) & \longrightarrow & K_n(\mathbb{C}G) \otimes_{\mathbb{Z}} \mathbb{C} \\ \downarrow & & \downarrow \\ \bigoplus_{p+q=n} \bigoplus_{(g)\in T} H_p(BC_G\langle g\rangle;\mathbb{C}) \otimes_{\mathbb{Z}} K_q^{\text{top}}(\mathbb{C}) & \longrightarrow & K_n^{\text{top}}(C_r^*(G)) \otimes_{\mathbb{Z}} \mathbb{C} \end{array}$$

- The vertical arrows come from the obvious change of rings and of K-theory maps.
- The horizontal arrows can be identified with the assembly maps occurring in the Farrell-Jones Conjecture and the Baum-Connes Conjecture by the equivariant Chern character.
- Splitting principle.

- One can spell out the Farrell-Jones Conjecture also for other theories like topological Hochschild homology and topological cyclic homology and compute the source of assembly map rationally using equivariant Chern characters.
- Injectivity and Bijectivity results have been obtained for such theories by L.-Rognes-Reich-Varisco.
- In particular L.-Rognes-Reich-Varisco extend the result of Bökstedt-Hsiang-Madsen for $\{1\}$ to $\mathcal{FIN}$ thus proving rational injectivity of the K-theoretic Farrell-Jones assembly map for coefficients in $\mathbb{Z}$ under mild homological assumptions.

Theorem (Atiyah-Segal)

Let G be a finite group.
Then there are isomorphisms of abelian groups

$$\begin{aligned} K^0(BG) &\cong \operatorname{Rep}_{\mathbb{C}}(G)\widehat{_{\mathbb{I}_G}} \\ &\cong \mathbb{Z} \times \prod_{p \text{ prime}} \mathbb{I}_p(G) \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}_p} \cong \mathbb{Z} \times \prod_{p \text{ prime}} (\widehat{\mathbb{Z}_p})^{r(p)}; \\ K^1(BG) &\cong 0. \end{aligned}$$

- For a prime p denote by $r(p) = |\operatorname{con}_p(G)|$ the number of conjugacy classes (g) of elements $g \neq 1$ in G of p-power order.
- $\mathbb{I}_G$ is the augmentation ideal of $\operatorname{Rep}_{\mathbb{C}}(G)$.
- Let $\mathbb{I}_p(G)$ be the image of the restriction homomorphism $\mathbb{I}(G) \to \mathbb{I}(G_p)$.

Theorem (L.)

Let G be a discrete group. Denote by $K^*(BG)$ the topological (complex) K-theory of its classifying space BG. Suppose that there is a cocompact G-CW-model for the classifying space $\underline{E}G$ for proper G-actions.

Then there is a $\mathbb{Q}$-isomorphism

$$\overline{\mathrm{ch}}^n_G \colon K^n(BG) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\cong} \left(\prod_{i \in \mathbb{Z}} H^{2i+n}(BG;\mathbb{Q})\right) \times \prod_{p \text{ prime}} \prod_{(g) \in \mathrm{con}_p(G)} \left(\prod_{i \in \mathbb{Z}} H^{2i+n}(BC_G\langle g\rangle;\mathbb{Q}\widehat{_p})\right),$$

- The multiplicative structure can also be determined.

Theorem (L.)

Let X be a proper G-CW-complex. Let $\mathbb{Z} \subseteq \Lambda^G \subset \mathbb{Q}$ be the subring of $\mathbb{Q}$ obtained by inverting the orders of all the finite subgroups of G. Then there is a natural isomorphism

$$\mathrm{ch}^G \colon \bigoplus_{(C)} K_n(C_G C \backslash X^C) \otimes_{\mathbb{Z}[W_G C]} \theta_C \cdot \mathrm{Rep}_{\mathbb{C}}(C) \otimes_{\mathbb{Z}} \Lambda^G \xrightarrow{\cong} K_n^G(X) \otimes_{\mathbb{Z}} \Lambda^G,$$

where (C) runs through the conjugacy classes of finite cyclic subgroups.

Here is a problem concerning the theorem above.

- Take $X = \underline{E}G$. Elements in $K_0(\underline{E}G)$ are given by elliptic G-operators P over cocompact proper G-manifolds with Riemannian metrics.
- What is the concrete preimage of its class under ch_0^G?
- One term could be the index of P^C on M^C giving an element in $K_0(C_GC\backslash\underline{E}^C)$ which is $K_0(BC_GC)$ after tensoring with Λ^G.
- Another term could come from the normal data of M^C in M which yields an element in $\theta_C \cdot \mathrm{Rep}_{\mathbb{C}}(C)$.
- Strategy: Use the pairing

$$K_0^G(X) \otimes K_G^0(X) \to \mathbb{Z}$$

given by twisting a G-operator with a G-vector bundle and then taking its index and the cohomological Chern character for K_G^0 which has K_G^0 as source and which is compatible with the obvious pairing on the "easy´´ sides of the two Chern characters.