Locally compact groups

Paul Klee, Bunte Gruppe

Linus Kramer

December 13, 2017

Introduction

Contents

1 Topological Groups

This chapter contains basic results about the point-set topology of topological groups. Our convention that all topological spaces are assumed to be Hausdorff will not be in force for this chapter—whenever we use the Hausdorff condition, this will be mentioned explicitly.

Definition 1.1. A *topological group* $(G, \cdot, \mathcal{T})$ consists of a group $(G, \cdot)$ and a topology $\mathcal{T}$ on G (not necessarily Hausdorff) for which the map

$$q : G \times G \longrightarrow G, \quad (g, h) \longmapsto g^{-1}h$$

is continuous. Specializing this to $h = e$, we see that then the inversion map

$$i : g \longmapsto g^{-1}$$

is a homeomorphism. It follows that the multiplication map

$$m : (g, h) \longmapsto gh$$

is also continuous. For every $a \in G$, the right translation map

$$\rho_a(g) = ga^{-1},$$

the left translation map

$$\lambda_a(g) = ag,$$

and the conjugation map

$$\gamma_a(g) = aga^{-1}$$

are homeomorphisms of G onto itself, with inverses $\lambda_{a^{-1}}$, $\rho_{a^{-1}}$ and $\gamma_{a^{-1}}$, respectively. In particular, the homeomorphism group of G acts transitively on G. It follows that every neighborhood W of a group element $g \in G$ can be written as $W = gU = Vg$, where $U = \lambda_{g^{-1}}(W)$ and $V = \rho_g(W)$ are neighborhoods of the identity. In what follows, we will mostly write G for a topological group, without mentioning the topology $\mathcal{T}$ explicitly. A neighborhood of the identity element e will be called an *identity neighborhood* for short.

Definition 1.2. We define a *morphism* $f : G \longrightarrow K$ between topological groups G, K to be a continuous group homomorphism. Topological groups and their morphisms form a category which we denote by **TopGrp**.

Example 1.3. The following are simple examples of topological groups and morphisms.

(a) The additive and the multiplicative groups of the fields $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and the p-adic fields $\mathbb{Q}_p$, endowed with their usual field topologies, are examples of topological groups. The exponential maps $\exp : \mathbb{R} \longrightarrow \mathbb{R}^*$ and $\exp : \mathbb{C} \longrightarrow \mathbb{C}^*$ are morphisms.

(b) The *circle group* $\mathrm{U}(1) = \{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C}^*$ is another example of a topological group. The map $\mathbb{R} \longrightarrow \mathrm{U}(1)$ that maps t to $\exp(2\pi \boldsymbol{i} t) = \cos(2\pi t) + \boldsymbol{i} \sin(2\pi t)$ is a morphism.

(c) Every morphism $f : \mathbb{R} \longrightarrow \mathbb{R}$ is of the form $f(t) = rt$, for a unique real r. This follows from the fact that $\mathbb{Q}$ is dense in $\mathbb{R}$, and that an additive homomorphism $f : \mathbb{Q} \longrightarrow \mathbb{R}$ is determined uniquely determined by the element $r = f(1)$, since $\mathbb{Q}$ is uniquely divisible.

(d) Every morphism $f : \mathrm{U}(1) \longrightarrow \mathrm{U}(1)$ is of the form $f(z) = z^m$, for a unique integer $m \in \mathbb{Z}$.

(e) As a vectorspace over $\mathbb{Q}$, the group $(\mathbb{R}, +)$ has dimension $2^{\aleph_0}$. Hence the abelian group $\mathbb{R}$ has $2^{2^{\aleph_0}}$ additive endomorphisms, almost all of which are not continuous.

(f) Let H denote the additive group of the reals, endowed with the discrete topology. Then $\mathrm{id} : H \longrightarrow \mathbb{R}$ is a continuous bijective morphism, whose inverse is not continuous.

(g) Let F be a field and let $\mathrm{GL}_n F$ denote the group of invertible $n \times n$-matrices over F. For an $n \times n$-matrix g, let $g^\#$ denote the matrix with entries $(g^\#)_{i,j} = (-1)^{i+j} \det(g'(j,i))$, where $g'(j,i)$ is the $(n-1) \times (n-1)$-matrix obtained by removing column i and row j from the matrix g. Then $gg^\# = g^\# g = \det(g)\mathbb{1}$. Hence if F is a topological field, then $q(g,h) = g^{-1}h = \frac{1}{\det(g)} g^\# h$ depends continuously on g and h, and therefore $\mathrm{GL}_n F$ is a topological group. In particular, the matrix groups $\mathrm{GL}_n \mathbb{Q}$, $\mathrm{GL}_n \mathbb{R}$, $\mathrm{GL}_n \mathbb{C}$, and $\mathrm{GL}_n \mathbb{Q}_p$ are topological groups.

(h) Every group G, endowed either with the discrete or with the trivial nondiscrete topology, is a topological group.

Proposition 1.4. *Suppose that $(G_i)_{i \in I}$ is a family of topological groups. Then the product $K = \prod_{i \in I} G_i$, endowed with the product topology, is again a topological group. For each j,* *the projection map* $\mathrm{pr}_j : K \longrightarrow G_j$ *is an open morphism. If* H *is a topological group and if there are morphisms* $f_j : H \longrightarrow G_j$*, for every* $j \in J$*, then there is a unique morphism* $f : H \longrightarrow K$ *such that* $\mathrm{pr}_j \circ f = f_j$ *holds for all* $j \in J$*.*

Proof. We have to show that the map $q : K \times K \longrightarrow K$ that maps (g, h) to $g^{-1}h$ is continuous. Let $q_j : G_j \times G_j \longrightarrow G_j$ denote the corresponding maps, which are by assumption continuous. Then we have for each j a continuous map $\mathrm{pr}_j \circ q = q_j \circ (\mathrm{pr}_j \times \mathrm{pr}_j)$,

$$\begin{array}{ccc} K \times K & \xrightarrow{\quad q \quad} & K \\ {\scriptstyle \mathrm{pr}_j \times \mathrm{pr}_j}\Big\downarrow & & \Big\downarrow{\scriptstyle \mathrm{pr}_j} \\ G_j \times G_j & \xrightarrow{\quad q_j \quad} & G_j. \end{array}$$

By the universal property of the product topology, this implies that q is continuous.

The remaining claims follow, since these maps on the one hand are continuous (and open) as claimed by the properties of the product topology, and on the other hand are group homomorphisms. □

The following local criterion for morphisms is often useful.

Lemma 1.5. *Let* G, K *be topological groups and let* $f : G \longrightarrow K$ *be a (not necessarily continuous) group homomorphism. Then the following are equivalent.*

(i) The map f *is continuous and hence a morphism of topological groups.*

(ii) The map f *is continuous at one point* $a \in G$*, i.e. for every neighborhood* W *of* $f(a)$*, there exists a neighborhood* V *of* a *such that* $f(V) \subseteq W$*.*

Proof. It is clear that (i) implies (ii): a continuous map is continuous at every point. Suppose that (ii) holds and that $U \subseteq K$ is open. If $g \in f^{-1}(U)$, then $f(a) = f(ag^{-1}g) \in f(ag^{-1})U$. Hence there exists a neighborhood V of a with $f(V) \subseteq f(ag^{-1})U$. Then $ga^{-1}V$ is a neighborhood of g, with $f(ga^{-1}V) \subseteq U$. Hence $f^{-1}(U)$ is open. □

Below we will look into separation properties for topological groups more closely. At this point we just record the following.

Lemma 1.6. *A topological group* G *is Hausdorff if and only if some singleton* $\{a\} \subseteq G$ *is closed.*

Proof. Suppose that $\{a\} \subseteq G$ is closed. The preimage of $\{a\}$ under the continuous map $(g, h) \longmapsto g^{-1}ha$ is the diagonal $\{(g, g) \mid g \in G\} \subseteq G \times G$, which is therefore closed in $G \times G$. Thus G is Hausdorff. Conversely, every singleton in a Hausdorff space is closed. □

Subgroups

Now we study subgroups of topological groups.

Proposition 1.7. *Let H be a subgroup of a topological group G. Then H is a topological group with respect to the subspace topology. Moreover, the closure $\overline{H}$ is also a subgroup of G. If H is normal in G, then $\overline{H}$ is also normal.*

Proof. It is clear from the definition that a subgroup of a topological group is again a topological group. Let $H \subseteq G$ be a subgroup. The continuity of the map q from Definition 1.1 ensures that

$$q(\overline{H} \times \overline{H}) = q(\overline{H \times H}) \subseteq \overline{q(H \times H)} = \overline{H}.$$

Thus $\overline{H}$ is a subgroup. Suppose in addition that $H \trianglelefteq G$ is normal. For $a \in G$, put $\gamma_a(g) = aga^{-1}$. Since the conjugation map $\gamma_a : G \longrightarrow G$ is continuous, we have

$$\gamma_a(\overline{H}) \subseteq \overline{\gamma_a(H)} = \overline{H}.$$

This shows that $\overline{H}$ is normal in G. □

Lemma 1.8. *Let G be a topological group and let $A \subseteq G$ be a closed subset. Then the normalizer of A,*

$$\mathrm{Nor}_G(A) = \{g \in G \mid \gamma_g(A) = A\},$$

is a closed subgroup.

Proof. For $a \in A$ let $c_a(g) = gag^{-1}$. Then $c_a : G \longrightarrow G$ is continuous and hence $c_a^{-1}(A) = \{g \in G \mid gag^{-1} \in A\}$ is closed. Therefore

$$S = \bigcap \{c_a^{-1}(A) \mid a \in A\} = \{g \in G \mid \gamma_g(A) \subseteq A\}$$

is a closed semigroup in G, and $\mathrm{Nor}_G(A) = S \cap S^{-1}$ is closed as well. □

Lemma 1.9. *Let G be a Hausdorff topological group, and let $X \subseteq G$ be any subset. Then the centralizer*

$$\mathrm{Cen}_G(X) = \{g \in G \mid [g, X] = e\}$$

is closed. In particular, the center of G is closed.

Proof. Given $x \in X$, the map $g \longrightarrow [g, x] = gxg^{-1}x^{-1}$ is continuous. Therefore $\mathrm{Cen}_G(x) = \{g \in G \mid [g, x] = e\}$ is closed, provided that $\{e\} \subseteq G$ is closed. Then $\mathrm{Cen}_G(X) = \bigcap \{\mathrm{Cen}_G(x) \mid x \in X\}$ is closed as well. □

Lemma 1.10. *Let G be a Hausdorff topological group. If $A \subseteq G$ is an abelian subgroup, then $\overline{A}$ is an abelian subgroup.*

Proof. The commutator map $(g, h) \longrightarrow [g, h]$ is constant on $A \times A$ and hence also constant on the closure $\overline{A \times A} = \overline{A} \times \overline{A}$. □

Lemma 1.11. *Let G be a topological group and suppose that $U \subseteq G$ is an open subset. If $X \subseteq G$ is any subset, then UX and XU are open subsets. In particular, the multiplication map $m : G \times G \longrightarrow G$, $(g, h) \longmapsto gh$ and the map $q : (g, h) \longmapsto g^{-1}h$ are open.*

Proof. For each $x \in X$, the sets $Ux = \rho_{x^{-1}}(U)$ and $xU = \lambda_x(U)$ are open. Hence $UX = \bigcup\{Ux \mid x \in X\}$ and $XU = \bigcup\{xU \mid x \in X\}$ are open as well. □

Proposition 1.12. *Let G be a topological group and let $H \subseteq G$ be a subgroup.*

(i) *The subgroup H is open if and only if it contains a nonempty open set.*

(ii) *If H is open, then H is also closed.*

(iii) *The subgroup H is closed if and only if there exists an open set $U \subseteq G$ such that $U \cap H$ is nonempty and closed in U.*

Proof. For (i), suppose that H contains the nonempty open set U. Then $H = UH$ is open by Lemma 1.11. Conversely, if H is open then it contains the nonempty open set H. For (ii), suppose that $H \subseteq G$ is open. Then $G - H = \bigcup\{aH \mid a \in G - H\}$ is also open. For (iii), suppose that $U \cap H$ is nonempty and closed in the open set U. Then $U \cap H$ is also closed in the smaller set $U \cap \overline{H} \subseteq U$. Upon replacing G by $\overline{H}$, we may thus assume in addition that H is dense in the ambient group G, and we have to show that $H = G$. The set $U - H = U - (U \cap H)$ is open in U and hence open in G. On the other hand, H is dense in G. Therefore $U - H = \varnothing$ and thus $U \subseteq H$. By (i) and (ii), H is closed in G, whence $H = G$. Conversely, if H is closed, then H is closed in the open set G. □

Corollary 1.13. *Let G be a topological group and let $V \subseteq G$ be a neighborhood of some element $g \in G$. Then V generates an open subgroup of G.*

Corollary 1.14. *Suppose that G is a Hausdorff topological group and that $H \subseteq G$ is a subgroup. If H is locally compact in the subspace topology, then H is closed. In particular, every discrete subgroup of G is closed.*

Proof. Let $C \subseteq H$ be a compact set which is an identity neighborhood in the topological group H. Then there exists an open identity neighborhood U in G such that $U \cap H \subseteq C$. Since C is compact, C is closed in G and hence $U \cap H = U \cap C$ is closed in U and nonempty. Now we can apply Proposition 1.12(iii). □

A product of closed subsets in a topological group need not be closed. The standard example is the additive group of the reals $(\mathbb{R}, +)$, with the closed subgroups $A = \mathbb{Z}$ and $B = \sqrt{2}\mathbb{Z}$. Then $A+B$ is a countable dense subgroup of $\mathbb{R}$ which is not closed. However, we have the following.

Lemma 1.15. *Let G be a Hausdorff topological group, and let $A, B \subseteq G$ be closed subsets. If either A or B is compact, then $AB \subseteq G$ is closed.*

The proof uses a weak form of Wallace's Lemma, which we will use on several occasions.

Lemma 1.16 (Wallace). *Let $X_1, \ldots X_k$ be Hausdorff spaces containing compact sets $A_j \subseteq X_j$, for $j = 1, \ldots, k$. If $W \subseteq X_1 \times \cdots \times X_k$ is an open set containing $A_1 \times \cdots \times A_k$, then there exist open sets U_j with $A_j \subseteq U_j \subseteq X_j$, for $j = 1, \ldots, k$, such that*

$$A_1 \times \cdots \times A_k \subseteq U_1 \times \cdots \times U_k \subseteq W.$$

Proof. There is nothing to show for $k = 1$. Suppose that $k = 2$. We put $A = A_1$ and $B = A_2$ and we fix $a \in A$. For every point $b \in B$, we choose an open neighborhood $U_b \times V_b$ of (a, b) such that $U_b \times V_b \subseteq W$. Since $\{a\} \times B$ is compact, finitely many such neighborhoods $U_{b_1} \times V_{b_1}, \ldots, U_{b_m} \times V_{b_m}$ cover $\{a\} \times B$. We put $U_a = U_{b_1} \cap \cdots \cap U_{b_m}$ and $V_a = V_{b_1} \cup \cdots \cup V_{b_m}$. Then $\{a\} \times B \subseteq U_a \times V_a \subseteq W$. Now we let $a \in A$ vary. Since A is compact, finitely many such strips $U_{a_1} \times V_{a_1}, \ldots, U_{a_n} \times V_{a_n}$ cover $A \times B$. We put $U = U_{a_1} \cup \cdots \cup U_{a_n}$ and $V = V_{a_1} \cap \cdots \cap V_{a_n}$. Then $A \times B \subseteq U \times V \subseteq W$ and the claim is proved for $k = 2$.

For $k \geq 3$ we apply the previous argument to $A = A_1$ and $B = A_2 \times \cdots \times A_k$, and we obtain open sets $U \subseteq X_1$ and $V \subseteq X_2 \times \cdots \times X_k$ with $A \times B \subseteq U \times V \subseteq W$. By induction, we find now open sets $U_2, \ldots U_k$ such that $A_2 \times \cdots \times A_k \subseteq U_2 \times \cdots U_k \subseteq V$. Therefore $A_1 \times \cdots \times A_k \subseteq U \times U_2 \times \cdots U_k \subseteq W$. □

Proof of Lemma 1.15. Suppose that A is compact and B is closed, and that $g \in G - AB$. We have to show that $G - AB$ contains a neighborhood of g. By assumption, $A^{-1}g \cap B = \varnothing$. If we put $q(g, h) = g^{-1}h$, then $q(A \times \{g\}) \subseteq G - B$. By Wallace's Lemma 1.16 there exists an open neighborhood V of g such that $q(A \times V) \subseteq G - B$, i.e. $A^{-1}V \cap B = \varnothing$. Hence $V \cap AB = \varnothing$ and the claim follows. The case where B is compact and A is closed follows by taking inverses. □

Quotients

Suppose that H is a subgroup of a topological group G. We endow the set G/H of left cosets with the quotient topology with respect to the natural map

$$p : G \longrightarrow G/H, \qquad g \longmapsto gH.$$

Thus a subset of G/H is open if and only if its preimage is open. The next result is elementary, but important.

Proposition 1.17. *Let G be a topological group and let H be a subgroup. Then the quotient map*

$$p : G \longrightarrow G/H$$

is open. The quotient G/H is Hausdorff if and only if H is closed in G.

Proof. Suppose that $U \subseteq G$ is an open set. Then $p^{-1}(p(U)) = UH$ is open by Lemma 1.11, hence $p(U)$ is open by the definition of the quotient topology.

If G/H is Hausdorff, then $\{H\} \subseteq G/H$ is closed, hence $H = p^{-1}(\{H\}) \subseteq G$ is closed as well. Conversely, suppose that $H \subseteq G$ is closed. The map $p \times p : G \times G \longrightarrow G/H \times G/H$ is open, because p is open and because a cartesian product of two open maps is again open. The open set $W = \{(x, y) \in G \times G \mid x^{-1}y \in G - H\}$ maps under $p \times p$ onto the complement of the diagonal in $G/H \times G/H$. Hence the diagonal $\{(gH, gH) \mid g \in G\}$ is closed in $G/H \times G/H$, and therefore G/H is Hausdorff. □

Corresponding remarks apply to the set $H\backslash G$ of right cosets by taking inverses.

Proposition 1.18. *Let G be a topological group. If $N \trianglelefteq G$ is a normal subgroup, then the factor group G/N is a topological group with respect to the quotient topology on G/N. The quotient map $p : G \longrightarrow G/N$ is an open morphism. The factor group G/N is Hausdorff if and only if N is closed. In particular, $G/\overline{N}$ is a Hausdorff topological group.*

Proof. We put $\bar{q}(gN, hN) = g^{-1}hN$ and $p(g) = gN$. Then the diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\quad q \quad} & G \\ \downarrow{\scriptstyle p\times p} & & \downarrow{\scriptstyle p} \\ G/N \times G/N & \xrightarrow{\quad \bar{q} \quad} & G/N \end{array}$$

commutes, and $p \circ q$ is continuous. Since p is open, $p \times p$ is also open and hence a quotient map. It follows from the universal property of quotient maps that $\bar{q}$ is continuous, and therefore G/N is a topological group. The remaining claims follow from Proposition 1.17. □

The next result is the Homomorphism Theorem for topological groups.

Lemma 1.19. *Let $f : G \longrightarrow K$ be a morphism of topological groups, and put $N = \ker(f)$. Then f factors through the open morphism $p : G \longrightarrow G/N$ via a unique morphism $\bar{f}$,*

$$\begin{array}{ccccc} G & & \xrightarrow{\quad f \quad} & & K. \\ & {\scriptstyle p}\searrow & & \nearrow{\scriptstyle \bar{f}} & \\ & & G/N & & \end{array}$$

If f is open, then $\bar{f}$ is also open.

Proof. The group homomorphism $\bar{f}$ exists uniquely by the Homomorphism Theorem for groups. Since p is a quotient map, $\bar{f}$ is continuous and thus a morphism of topological groups. If f is open and if $W \subseteq G/N$ is an open set, then $f(p^{-1}(W)) = \bar{f}(W)$ is open as well. ☐

Corollary 1.20. *Suppose that G, K are topological groups and that K is Hausdorff. If $f : G \longrightarrow K$ is a morphism of topological groups, then f factors through the open morphism $p : G \longrightarrow G/\overline{\{e\}}$,*

$$\begin{array}{ccc} G & \xrightarrow{\qquad f \qquad} & K. \\ & {\scriptstyle p}\searrow \qquad \nearrow{\scriptstyle \bar{f}} & \\ & G/\overline{\{e\}} & \end{array}$$

Connected components

Definition 1.21. Let x be a point in a topological space X. The *connected component* of x is the union of all connected subsets of X containing x. This union is closed and connected. We call a topological space X *totally disconnected* if the only connected nonempty subsets of X are the singletons.

The connected component the identity element of a topological group G will be denoted by G°, and we call G° the *identity component* of G. Since the homeomorphism group of G acts transitively on G, the group G is totally disconnected if and only if $G^\circ = \{e\}$. We note that a totally disconnected group is automatically Hausdorff.

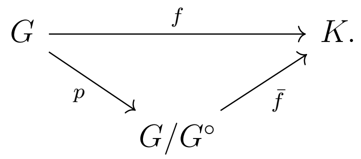
Proposition 1.22. *Let G be a topological group. Then the identity component G° is a closed normal subgroup, and G/G° is a totally disconnected Hausdorff topological group.*

Proof. We put $q(g, h) = g^{-1}h$ and we note that a continuous image of a connected set is connected. Since $G^\circ \times G^\circ$ is connected and contains the identity element, $q(G^\circ \times G^\circ) \subseteq G^\circ$. This shows that G° is a subgroup. By the remark above, G° is closed. For every $a \in G$, the set $\gamma_a(G^\circ) = aG^\circ a^{-1}$ is connected and contains the identity, whence $aG^\circ a^{-1} \subseteq G^\circ$. This shows that G° is a closed normal subgroup.

It remains to show that G/G° is totally disconnected. We put $H = (G/G^\circ)^\circ$ and $N = p^{-1}(H)$. Then N is a closed normal subgroup of G containing G°. We claim that $N = G^\circ$. If we have proved this claim, then $H = \{G^\circ\}$ and thus G/G° is totally disconnected. The restriction-corestriction map $p|_H^N : N \longrightarrow H$ is open, hence H carries the quotient topology with respect to $p|_H^N : N \longrightarrow H$. Suppose that $V \subseteq N$ is closed and open in N and contains the identity. Since G° is connected and contains e, we have

$vG^\circ \subseteq V$ for all $v \in V$. Hence $V = p^{-1}(p(V))$, and therefore $p(V)$ is closed and open in H. But H is connected, whence $H = p(V)$ and thus $V = N$. It follows that N is connected, whence $N = G^\circ$. □

Corollary 1.23. *Let $f : G \longrightarrow K$ be a morphism of topological groups. If K is totally disconnected, then f factors through the open morphism $p : G \longrightarrow G/G^\circ$,*



Proof. Since $f(G^\circ) \subseteq K$ is connected, G° is contained in the kernel of f. □

Metrizability of topological groups

At some point the metrizability of topological groups will become important.

Definition 1.24. A *pseudometric* d on a set X is a map $d : X \times X \longrightarrow \mathbb{R}_{\geq 0}$ satisfying

$$d(x,x) = 0, \quad 0 \leq d(x,y) = d(y,x), \quad d(x,z) \leq d(x,y) + d(y,z),$$

for all $x, y, z \in X$. If $d(x,y) = 0$ implies that $x = y$, then d is called a *metric*. A metric or pseudometric d on a group G is called *left invariant* if the left translation map $\lambda_a : G \longrightarrow G$ is an isometry of the metric or pseudometric space (G,d), for every $a \in G$. In other words, we require for a left invariant metric that

$$d(x,y) = d(ax,ay)$$

for all $a, x, y \in G$. A *length function* ℓ on a group is a map

$$\ell : G \longrightarrow \mathbb{R}_{\geq 0}$$

with the properties

$$\ell(e) = 0, \quad \ell(g) = \ell(g^{-1}), \quad \ell(gh) \leq \ell(g) + \ell(h),$$

for all $g, h \in G$. It follows that the set $\{g \in G \mid \ell(g) = 0\}$ is a subgroup of G. If ℓ is a length function, then

$$d(g,h) = \ell(g^{-1}h)$$

is a left invariant pseudometric. This pseudometric is a metric if and only if $\ell(g) = 0$ implies that $g = e$.

Theorem 1.25 (Birkhoff–Kakutani). *Let G be a Hausdorff topological group. The following are equivalent.*

(i) The topology on G is metrizable by a left invariant metric.

(ii) The topology on G is metrizable.

(iii) The identity element e has a countable neighborhood basis.

The proof relies on the following technical lemma. We call an identity neighborhood V symmetric if $V = V^{-1}$. If V is any identity neighborhood, then $V \cap V^{-1}$ is a symmetric identity neighborhood.

Lemma 1.26. *Let G be a topological group. Suppose that $(K_n)_{n\in\mathbb{Z}}$ is a family of symmetric identity neighborhoods with the property that $K_nK_nK_n \subseteq K_{n+1}$ holds for all $n \in \mathbb{Z}$, and with $\langle \bigcup_{n\in\mathbb{Z}} K_n\rangle = G$. For $g \in G$ we put*

$$\ell(g) = \inf\{t \geq 0 \mid \text{ there is some } k \geq 1 \text{ and } n_1, \ldots, n_k \in \mathbb{Z} \text{ with } t = 2^{n_1} + \cdots + 2^{n_k} \text{ and } g \in K_{n_1}K_{n_2}\cdots K_{n_k}\}.$$

Then ℓ is a continuous length function. Moreover, $\{g \in G \mid \ell(g) < 2^n\} \subseteq K_n$ and therefore $\bigcap_{n\in\mathbb{Z}} K_n = \{g \in G \mid \ell(g) = 0\}$.

Proof. First of all we note that for every $g \in G$ there exist numbers $n_1, \ldots, n_k$ with $g \in K_{n_1}\cdots K_{n_k}$, because the union of the K_n generates G and because every K_n is symmetric. Thus $\ell(g)$ is defined for all $g \in G$.

If $g \in K_{m_1}\cdots K_{m_r}$ and $h \in K_{n_1}\cdots K_{n_s}$, then $gh \in K_{m_1}\cdots K_{m_r}K_{n_1}\cdots K_{n_s}$. It follows that ℓ satisfies the triangle inequality. Since each K_n is symmetric, we have $\ell(g) = \ell(g^{-1})$ for all $g \in G$. Finally, $\ell(e) = 0$ since $e \in K_n$ holds for every $n \in \mathbb{Z}$. This shows that ℓ is a length function.

Next we show the continuity of ℓ. We note that $\ell(g) \leq 2^n$ holds whenever $g \in K_n$. Let $g \in G$ be any element, and let $\varepsilon > 0$. We choose $n \in \mathbb{Z}$ in such a way that $2^n \leq \varepsilon$ and we claim that $|\ell(g) - \ell(h)| \leq \varepsilon$ holds for all $h \in gK_n$. Since $g^{-1}h, h^{-1}g \in K_n$, we have $\ell(h) = \ell(gg^{-1}h) \leq \ell(g) + 2^n$ and $\ell(g) = \ell(hh^{-1}g) \leq \ell(h) + 2^n$. Therefore $|\ell(g) - \ell(h)| \leq 2^n \leq \varepsilon$ holds for all $h \in gK_n$, and hence ℓ is continuous.

For the last claim, we note first that $K_n \subseteq K_nK_nK_n \subseteq K_{n+1}$. Suppose now that $\ell(g) < 2^n$. We have to show that $g \in K_n$. There exists $k \geq 1$ and numbers $n_1, \ldots, n_k \in \mathbb{Z}$ with $g \in K_{n_1}\cdots K_{n_k}$ and with $2^{n_1} + \cdots + 2^{n_k} < 2^n$. It will suffice to prove the following claim.

Claim. Suppose that $2^{n_1} + \cdots + 2^{n_k} < 2^n$. Then $K_{n_1}\cdots K_{n_k} \subseteq K_n$.

Proof of the claim. We note that $n_j < n$ holds for $j = 1, \ldots, k$. Hence $K_{n_1} \cdots K_{n_k} \subseteq K_{n-1} \cdots K_{n-1}$. This proves the claim for $k = 1, 2, 3$. For $k \geq 4$ we proceed by induction on k. Suppose that $k \geq 4$. If $2^{n_1} + \cdots + 2^{n_k} < 2^{n-1}$, then $K_{n_1} \cdots K_{n_{k-1}} \subseteq K_{n-1}$ by the induction hypothesis, and $K_{n_k} \subseteq K_{n-1}$, whence $K_{n_1} \cdots K_{n_k} \subseteq K_{n-1}K_{n-1} \subseteq K_n$. There remains the case where $2^{n-1} \leq 2^{n_1} + \cdots + 2^{n_k} < 2^n$. We choose the smallest $r \in \{1, \ldots, k\}$ with $2^{n-1} \leq 2^{n_1} + \cdots + 2^{n_r}$. Then $2^{n_1} + \cdots + 2^{n_{r-1}} < 2^{n-1}$ and $2^{n_{r+1}} + \cdots + 2^{n_k} < 2^{n-1}$. By the induction hypotheses $K_{n_1} \cdots K_{n_{r-1}} \subseteq K_{n-1}$ and $K_{n_{r+1}} \cdots K_{n_k} \subseteq K_{n-1}$. Thus $K_{n_1} \cdots K_{n_k} \subseteq K_{n-1}K_{n_r}K_{n-1} \subseteq K_{n-1}K_{n-1}K_{n-1} \subseteq K_n$. Note that this is true also for the extremal cases $r = 1$ and $r = k$. $\square$

Proof of Theorem 1.25. It is clear that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and we have to show that (iii) implies (i). Let $(V_n)_{n \in \mathbb{N}}$ be neighborhood basis for the identity element. For $n \geq 1$ we put $K_n = G$. Next we choose, for every integer $n \leq 0$, inductively symmetric identity neighborhoods $K_n \subseteq G$ with $K_n \subseteq V_{-n}$, such that $K_{n-1}K_{n-1}K_{n-1} \subseteq K_n$. This is possible by the continuity of the 3-fold multiplication map $G \times G \times G \longmapsto G$. Let ℓ denote the continuous length function resulting from Lemma 1.26, applied to the family $(K_n)_{n \in \mathbb{Z}}$. Since $e \in \bigcap_{n \in \mathbb{Z}} K_n \subseteq \bigcap_{n \geq 0} V_n = \{e\}$, we have $\ell(g) = 0$ if and only if $g = e$. Thus $d(g, h) = \ell(g^{-1}h)$ is a left invariant continuous metric on G. Let $U \subseteq G$ be an open set and suppose that $g \in U$. Then there exists an integer $n \in \mathbb{N}$ such that $gV_n \subseteq U$. It follows that the metric ball $B_{2^{-n}}(g) = \{h \in G \mid d(g, h) < 2^{-n}\}$ is contained in $gK_{-n} \subseteq gV_n \subseteq U$. Therefore d metrizes the topology of G. $\square$

Another important consequence of Lemma 1.26 is the following. We recall that a Hausdorff space is called a *Tychonoff space*, or *completely regular*, or a $\mathrm{T}_{3\frac{1}{2}}$-space if for every closed set $A \subseteq X$ and every point $b \in X - A$ there is a continuous map

$$\varphi : X \longrightarrow [0, 1]$$

with $\varphi(b) = 0$ and $\varphi(A) \subseteq \{1\}$.

Theorem 1.27. *Every Hausdorff topological group is a Tychonoff space.*

Proof. Let $A \subseteq G$ be a closed set, and let $b \in G - A$. We have to show that there exists a continuous function $\varphi : G \longrightarrow [0, 1]$ with $\varphi(A) = \{1\}$ and $\varphi(b) = 0$. Since the homeomorphism λ_b of G maps the identity element e to b we may assume as well that $b = e$. We put $K_n = G$ for $n \geq 1$ and we choose a symmetric identity neighborhood $K_0 \subseteq G - A$. For $n < 0$ we choose inductively symmetric identity neighborhoods K_n in such a way that $K_{n-1}K_{n-1}K_{n-1} \subseteq K_n$. Let ℓ denote the continuous length function resulting from the family $(K_n)_{n \in \mathbb{Z}}$ by Lemma 1.26. Then $\ell(e) = 0$ and if $\ell(g) < 1$, then $g \in K_0 \subseteq G - A$. Thus $\ell(a) \geq 1$ for all $a \in A$. Hence $\varphi = \min\{\ell, 1\}$ is the desired continuous function. $\square$

The open mapping theorem and meager sets

We recall some notions related to the Baire Category Theorem. Let X be a topological space.

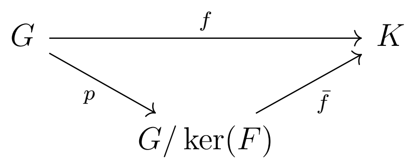
Definition 1.28. A subset $N \subseteq X$ is called *nowhere dense* if its closure $\overline{N}$ has empty interior. Equivalently, there exists a dense open set $U \subseteq X$ which is disjoint from N. A subset of a nowhere dense set is again nowhere dense.

A countable union of nowhere dense sets is called a *meager* set (a *set of first category* in the older literature). It follows that subsets of meager sets are meager, and that countable unions of meager sets are again meager. A subset $M \subseteq X$ is meager if and only if there exists a countable family of dense open sets $(U_n)_{n\geq 0}$ in X with $M \cap \bigcap_{n\geq 0} U_n = \varnothing$.

A topological space X is called a *Baire space* if for every countable family of dense open sets $(U_n)_{n\geq 0}$, the intersection $\bigcap_{n\geq 0} U_n$ is again dense. In particular, a Baire space is not meager in itself. Every completely metrizable space and every locally compact space is a Baire space by Baire's Category Theorem.

Proposition 1.29 (The Open Mapping Theorem). *Let $f : G \longrightarrow K$ be a surjective morphism of Hausdorff topological groups. If G is σ-compact and if K is not meager in itself, then f is open.*

Proof. From the diagram



and the fact that p is open and that $\bar{f}$ is a bijective map, we see that we may assume in addition that the morphism f itself is bijective. We have then to show that its inverse f^{-1} is continuous. We write $G = \bigcup_{n\in\mathbb{N}} A_n$, with A_n compact. For every $n \in \mathbb{N}$, the restriction-corestriction $f|_{f(A_n)}^{A_n} : A_n \longrightarrow f(A_n)$ is a continuous bijection and hence a homeomorphism. Moreover, each $f(A_n)$ is compact and therefore closed. Since $K = \bigcup_{n\in\mathbb{N}} f(A_n)$ is not meager in itself, there exists an index $m \in \mathbb{N}$ such that $f(A_m)$ contains a nonempty open set V. Let $U = f^{-1}(V)$. Then $U \subseteq G$ is open, and the restriction-corestriction $f|_V^U : U \longrightarrow V$ is a homeomorphism. It follows from Lemma 1.5 that f^{-1} is continuous. □

A Hausdorff topological group G is *compactly generated* if there exists a compact set $C \subseteq G$ which generates the group G. A compactly generated group is in particular σ-compact.

Corollary 1.30. *Suppose that $f : G \longrightarrow K$ is a surjective morphism of Hausdorff topological groups. If G is compactly generated and if K is a Baire space (eg. locally compact or completely metrizable), then f is open.*

We continue to study meager and nonmeager sets in topological groups.

Definition 1.31. Let X be a topological space. The complement of a meager set is called a *comeager* set. Note that this is not the same as a nonmeager set. Every subset of X containing a comeager set is again comeager, and countable intersections of comeager subsets are comeager.

Suppose that $V \subseteq X$ is open and that $A \subseteq X$ is any subset. We say that A is *comeager in V* if $A \cap V$ is comeager in the subspace V. Note that we do not require that $A \subseteq V$. We need the following localization result.

Lemma 1.32. *Suppose that V is an open set in the topological space X.*

(i) A subset $M \subseteq V$ is meager in the subspace V if and only if M is meager.

(ii) If A is a comeager subset of X, then A is comeager in V.

Proof. For the (i) it suffices to consider the case of nowhere dense sets. Suppose that $N \subseteq V$ is nowhere dense in the subspace V. Thus there exists an open dense subset $W \subseteq V$ which is disjoint from N. Since W is dense in V, we have $N \subseteq V \subseteq \overline{W}$. Then $W \cup (X - \overline{W})$ is dense in X and disjoint from N. Therefore N is nowhere dense in X. Conversely, if the closure of $N \subseteq V$ in V contains a nonempty open set $U \subseteq V$, then also $U \subseteq \overline{N}$. Claim (i) follows now by passing to countable unions. Claim (ii) is a special case of (i). If $M = X - A$ is meager, then $M \cap V = V - A$ is meager in the subspace V by (i), hence A is comeager in V. ☐

Lemma 1.33. *Suppose that $(X_i)_{i \in I}$ is a family of topological spaces. Let $X = \coprod_{i \in I} X_i$ denote their coproduct (disjoint union) in the category of topological spaces. Then a subset $M \subseteq X$ is meager if and only if $M \cap X_i$ is meager in X_i for every $i \in I$.*

Proof. If $M \subseteq X$ is meager, then $M \cap X_i$ is meager in X_i for every i by Lemma 1.32. Conversely, suppose that $N \subseteq X$ and that $N \cap X_i$ is nowhere dense in X_i for every $i \in I$. We claim that N is nowhere dense in X. Let $W_i \subseteq X_i$ be a dense open set which is disjoint from $N \cap X_i$. Then $W = \bigcup_{i \in I} W_i$ is open and disjoint from N and dense in X Therefore $N \subseteq X$ is nowhere dense. Hence if $M_i = M \cap X_i$ is meager for every $i \in I$, then we write $M_i = \bigcup_{n \geq 0} N_{n,i}$ as a countable union of nowhere dense sets in X_i. Then $N_n = \bigcup_{i \in I} N_{n,i}$ is nowhere dense and thus $M = \bigcup_{n \geq 0} N_n$ is meager. ☐

Definition 1.34. For a subset A of a topological space X we put

$$O(A) = \bigcup\{U \subseteq X \mid U \text{ is open and } A \text{ is comeager in } U\}.$$

Clearly, $O(A)$ is open and possibly empty. The next result is Banach's Category Theorem. It says that $O(A)$ is the unique maximal open set in which A is comeager.

Theorem 1.35. *If $O(A) \neq \varnothing$, then A is comeager in $O(A)$.*

Proof. We consider sets $\mathcal{C}$ of open subsets of X with the following property: the members of $\mathcal{C}$ are nonempty, pairwise disjoint, and A is comeager in every member of $\mathcal{C}$. If A is comeager in the open set U, then $\mathcal{C} = \{U\}$ is an example of such a set. The set P consisting of all such sets $\mathcal{C}$ is nonempty and partially ordered by inclusion. Every linearly ordered subset L of P has $\bigcup L \in P$ as an upper bound. Thus $(P, \subseteq)$ is inductive and has, by Zorn's Lemma, maximal elements. Let $\mathcal{C}$ be such a maximal element. We put $W = \bigcup \mathcal{C}$ and we note that $W \subseteq O(A)$.

Claim. The set $C = O(A) - W$ is nowhere dense.

We show that the closed set $B = \overline{O(A)} - W$ has empty interior. For otherwise there would exist a nonempty open set $V \subseteq B$. Then $V \cap W = \varnothing$, and there would exist an element $u \in O(A) \cap V$. Hence there would exist an open set U containing u in which A is comeager. But then A would be comeager in $V \cap U \subseteq N$ by Lemma 1.32(ii), and hence $\mathcal{C} \cup \{U \cap V\} \in P$ would be a bigger collection than $\mathcal{C}$, contradicting the maximality of $\mathcal{C}$. Hence $B = \overline{O(A)} - W$ has nonempty interior, and thus $C = O(A) - W$ is nowhere dense.

Claim. A is comeager in W.

The open subspace W is homeomorphic to the coproduct $\coprod\{U \mid U \in \mathcal{C}\}$ of the subspaces $U \in \mathcal{C}$. Let $M = W - A$. For every $U \in \mathcal{C}$, the set $U \cap M$ is meager in U by the definition of $\mathcal{C}$. Hence M is meager in W by Lemma 1.33.

Now we finish the proof. We have $O(A) = M \mathbin{\dot\cup} W$, for a set M which is meager in X and hence meager in $O(A)$ by Lemma 1.32(i). Now $O(A) - A = (M - A) \mathbin{\dot\cup} (W - A)$. Since $W - A$ is meager in W, this set is also meager in $O(A)$ by Lemma 1.32(ii). Thus $O(A) - A$ is a union of two sets which are meager in $O(A)$ and therefore A is comeager in $O(A)$. ☐

In the topological group $\mathbb{Q}$ every subset, including the empty set, is comeager. The following dichotomy will be used below.

Lemma 1.36. *Let G be a topological group. Then either $O(\varnothing) = G$ and G is meager or $O(\varnothing) = \varnothing$.*

Proof. The set $O(\varnothing)$ is invariant under all the left translation maps λ_a and hence either empty or all of G. If $G = O(\varnothing)$, then G is meager by Theorem 1.35. ☐

We recall that a subset A of a topological space is called *Baire measurable* or *almost open* if there exists an open set V such that the symmetric difference $M = (A \cup V) - (A \cap V)$ is meager. We will see in Proposition 1.40 below that these sets form a σ-algebra and that every Borel set is Baire measurable.

Lemma 1.37. *If $A \subseteq X$ is Baire measurable and not meager, then $O(A)$ is not meager and in particular nonempty.*

Proof. Let $V \subseteq X$ be open such that $M = (V \cup A) - (V \cap A)$ is meager. Since A itself is not meager and since $A \subseteq V \cup M$, the set V is not meager. □

The following result is known as Pettis' Lemma.

Theorem 1.38. *Let G be a topological group. Suppose that $A, B \subseteq G$ are nonmeager subsets. Then $O(A)O(B) \subseteq AB$. If A is Baire measurable and nonmeager, then $A^{-1}A$ is an identity neighborhood.*

Proof. Suppose that $g \in O(A)O(B)$. Then $O(A)$ and $O(B)$ are nonempty, and $O(A) \cap gO(B)^{-1} \neq \varnothing$. Now $gO(B)^{-1} = gO(B^{-1}) = O(gB^{-1})$, since λ_g and the inversion map are homeomorphisms. Hence $W = O(A) \cap O(gB^{-1}) \neq \varnothing$. By Lemma 1.32(i) and Theorem 1.35, both A and gB^{-1} are comeager in W. Hence $A \cap gB^{-1}$ is comeager in the nonempty open set W. By Lemma 1.36 the empty set is not comeager in W, because G, containing the nonmeager sets A, B, is not meager. Hence $A \cap gB^{-1} \neq \varnothing$ and therefore $g \in AB$. If A is Baire measurable and not meager, then $O(A) \neq \varnothing$ by Lemma 1.37. Thus $O(A)^{-1}O(A)$ is an open identity neighborhood. But $O(A)^{-1} = O(A^{-1})$, and $O(A^{-1})O(A) \subseteq A^{-1}A$. □

Corollary 1.39. *Suppose that G is a topological group and that $H \subseteq G$ is a subgroup. If H contains a Baire measurable set which is not meager, then H is open.*

The following is a useful property of Baire measurable sets.

Proposition 1.40. *The Baire measurable sets in a topological space X form a σ-algebra which contains all open sets. Hence every Borel set in X is Baire measurable.*

Proof. Every open set is Baire measurable. In particular, the empty set is among the Baire measurable sets. Suppose that $A \subseteq X$ is Baire measurable. We claim that the complement $B = X - A$ is also Baire measurable. Let $U \subseteq X$ be open such that $(U \cup A) - (U \cap A)$ is meager and put $V = X - \overline{U}$. We note that $M = \overline{U} - U$ is nowhere dense. The symmetric difference of two sets is not changed if we replace both sets by their complements. Hence

$$(V \cup B) - (V \cap B) = (\overline{U} \cup A) - (\overline{U} \cap A) \subseteq ((U \cup A) - (U \cap A)) \cup M$$

is meager, and therefore B is Baire measurable. Suppose that $(A_n)_{n\geq 0}$ is a family of Baire measurable sets. For every A_n there is an an open set U_n such that the symmetric difference $M_n = (A_n \cup U_n) - (A_n \cap U_n)$ is meager. We put $A = \bigcup_{n\geq 0} A_n$, $M = \bigcup_{n\geq 0} M_n$ and $U = \bigcup_{n\geq 0} U_n$. Then $A_n - U \subseteq A_n - U_n \subseteq M_n$, whence $A - U \subseteq M$. Likewise, $U - A \subseteq M$, and therefore

$$(A \cup U) - (A \cap U) = (A - U) \cup (U - A) \subseteq M$$

is meager. This shows that the Baire measurable sets form a σ-algebra. Since every open set is Baire measurable, every Borel set is Baire measurable. ☐

We recall that a topological space is called *Lindelöf* if every open covering has a countable subcovering. Examples of Lindelöf spaces are second countable spaces and σ-compact spaces.

Theorem 1.41. *Let G, K be topological groups and let $f : G \longrightarrow K$ be a group homomorphism. Assume also that K is Lindelöf and that G is not meager in itself. If for every open subset $U \subseteq K$ the preimage $f^{-1}(U)$ is Baire measurable, then f is continuous.*

Proof. A closed subset of a Lindelöf space is again Lindelöf. Replacing K by $\overline{f(G)}$, we may thus assume in addition that $f(G)$ is dense in K. Let $V \subseteq K$ be an identity neighborhood. We claim that $f^{-1}(V)$ contains an identity neighborhood. We choose an identity neighborhood $U \subseteq K$ such that $U^{-1}U \subseteq V$. By assumption, $E = f^{-1}(U)$ is Baire measurable. Since K is Lindelöf and $f(G)$ is dense, we find elements $g_n \in G$ such that $K = \bigcup_{n\geq 0} f(g_n)U$. Hence $G = \bigcup_{n\geq 0} g_n E$. Since G is not meager, E cannot be meager. Hence $E^{-1}E$ is an identity neighborhood by Theorem 1.38, and $f(E^{-1}E) \subseteq V$. It follows that f is continuous at the identity element of G. By Lemma 1.5, the map f is continuous and hence a morphism of topological groups. ☐

We recall that a map between topological spaces is called *Borel measurable* if the preimage of every open set is a Borel set.

Corollary 1.42. *Suppose that the topological group G is either locally compact or completely metrizable and that the topological group K is either σ-compact or second countable. Then a group homomorphism $f : G \longrightarrow K$ is continuous if and only if f is Borel measurable.*

Finite dimensional topological vector spaces over local fields

An *absolute value* on a field F is a nonzero map $|-| : F \longrightarrow \mathbb{R}_{\geq 0}$ with $|0| = 0$ which satisfies the triangle inequality and which is multiplicative,

$$|a + b| \leq |a| + |b| \quad \text{and} \quad |ab| = |a||b|,$$

for all $a, b \in F$. It follows that $|a| > 0$ for all $a \in F^\times$. Such an absolute value determines a metric and a topology on F. If $|a| = 1$ holds for all $a \in F^\times$, then the absolute value is called *trivial*. The field F is called a *local field* if this topology is locally compact and if $|-|$ is nontrivial. Examples of local fields are the real field $\mathbb{R}$ and the complex field $\mathbb{C}$ with their usual absolute values, but also the p-adic fields $\mathbb{Q}_p$ with $|a| = p^{-\nu_p(a)}$, where ν_p is the p-adic valuation.

Lemma 1.43. *Suppose that $\mathbb{K}$ is a local field. For every $r > 0$, the set*

$$B_r^{\mathbb{K}}(0) = \{a \in \mathbb{K} \mid |a| \le r\}$$

is compact. In particular, $\mathbb{K}$ is σ-compact.

Proof. Let $C \subseteq \mathbb{K}$ be a compact 0-neighborhood. There exists $\varepsilon > 0$ such that $B_\varepsilon^{\mathbb{K}}(0) \subseteq C$, and thus $B_\varepsilon^{\mathbb{K}}(0)$ is also compact. We choose $a \in B_\varepsilon^{\mathbb{K}}(0)$ in such a way that $|a| < 1/r$. This is possible because $|-|$ is nontrivial. Then $\frac{1}{a} B_\varepsilon^{\mathbb{K}}(0) = B_{\varepsilon/|a|}^{\mathbb{K}}(0)$ is compact, and hence $B_r^{\mathbb{K}}(0) \subseteq B_{\varepsilon/|a|}^{\mathbb{K}}(0)$ is also compact. □

Let $\mathbb{K}$ be a local field. A *topological vector space* over $\mathbb{K}$ is a vector space E over $\mathbb{K}$ which is a Hausdorff topological group, such that the scalar multiplication map

$$\mathbb{K} \times E \longrightarrow E, \qquad (t, u) \longmapsto tu$$

is continuous. A *morphism* of topological vector spaces over $\mathbb{K}$ is a continuous $\mathbb{K}$-linear map.

Theorem 1.44. *Let E be a topological vector space over a local field $\mathbb{K}$. Then the following are equivalent.*

(i) *E has finite dimension.*

(ii) *E is as a topological vector space isomorphic to $\mathbb{K}^m$, for some $m \ge 0$.*

(iii) *E is locally compact.*

Proof. Suppose that (i) holds and that $v_1, \ldots, v_m$ is a basis for E. The map $f : \mathbb{K}^m \longrightarrow E$ that maps $(z_1, \ldots, z_m)$ to $z_1 v_1 + \cdots + z_m v_m$ is a bijective morphism of topological vector spaces. We claim that the inverse of f is continuous at 0. We endow $\mathbb{K}^m$ with the box norm

$$\| (z_1, \ldots, z_m) \| = \max\{|z_1|, \ldots, |z_m|\}$$

and we put $V_\varepsilon = \{u \in \mathbb{K}^m \mid \|u\| < \varepsilon\}$, for $\varepsilon > 0$. For $\delta > 0$ we put $L_\delta = \{z \in \mathbb{K} \mid |z| < \delta\}$. We have to show that for every $\varepsilon > 0$, there is an open 0-neighborhood $W \subseteq E$ with

$W \subseteq f(V_\varepsilon)$. Passing to a smaller bound $\varepsilon > 0$ if necessary, we may assume that $\varepsilon < 1$ and that there exists $a \in \mathbb{K}$ with $|a| = \varepsilon$. Let $S_\varepsilon = \{u \in \mathbb{K}^m \mid \lVert u \rVert = \varepsilon\}$. Then S_ε is compact and therefore $f(S_\varepsilon)$ is closed in E. There exists $\delta > 0$ and an open 0-neighborhood $V \subseteq E$ such that the image W of $L_\delta \times V$ under the multiplication map $\mathbb{K} \times E \longrightarrow E$ is contained in $E - f(S_\varepsilon)$. Since

$$W = \bigcup \{zV \mid 0 < |z| < \delta\},$$

the set W is open. Moreover, we have $sw \in W$ for every $w \in W$ and every $s \in L_1$. Suppose that $u \in \mathbb{K}^m$ is a vector with $r = \lVert u \rVert > \varepsilon$. Then there exists $b \in \mathbb{K}$ with $|b| = \varepsilon r^{-1}$. Then $bu \in S_\varepsilon$ and $|b| < 1$. It follows that $f(u) \notin W$. This shows that $W \subseteq f(V_\varepsilon)$. By Lemma 1.5, the inverse of f is a morphism.

It is clear that (ii) implies (iii).

Now we show that (iii) implies (i). Suppose that E is locally compact. We showed already that (i) implies (ii), hence every finite dimensional subspace $F \subseteq E$ is locally compact. By Corollary 1.14, such a finite dimensional subspace is closed. Let $W \subseteq E$ be a compact 0-neighborhood, and let $a \in \mathbb{K}$ be an element with $0 < |a| < 1$. There is a finite set $A \subseteq E$ such that $W \subseteq A + aW$. Let F denote the linear span of A. We claim that $W \subseteq F$. We have $W \subseteq F + aW$. Iterating this inclusion, we see that $W \subseteq F + a^m W$ holds for all $m \geq 1$. Suppose that $w \in W - F$. Since F is closed, there exists a symmetric open 0-neighborhood $U \subseteq E$ such that $(w+U) \cap F = \varnothing$, whence $w \notin F+U$. On the other hand, Wallace's Lemma 1.16, applied to the compact set $\{0\} \times W \subseteq \mathbb{K} \times E$, shows that there exists $\delta > 0$ such that $zW \subseteq U$ holds for all $z \in L_\delta$. But for m sufficiently large, $|a^m| = |a|^m < \delta$. Hence $a^m W \subseteq U$ and thus $w \in F + a^m W \subseteq F + U$, a contradiction. Hence $W \subseteq F$. It remains to show that $E = F$. For every $u \in E$ there exists a $\delta > 0$ such that $zu \in W$ holds for all $z \in L_\delta$. Therefore W spans E, and hence $F = E$. □

2 | Locally compact groups and the Haar integral

We call a topological group G a *locally compact group* (a *compact group*) if the topology on G is Hausdorff and locally compact (compact).

General properties of locally compact groups

Proposition 2.1. *Let G be a locally compact group. Then a subgroup $H \subseteq G$ is closed if and only if it is locally compact. If $H \subseteq G$ is a closed subgroup, then G/H is locally compact.*

Proof. A closed subspace of a locally compact space is again locally compact. Conversely, a locally compact subgroup of a Hausdorff topological group is closed by Corollary 1.14. For the last claim, suppose that $H \subseteq G$ is a closed subgroup, and that $g \in G$ is any element. We have to show that gH has a compact neighborhood in G/H. Let $V \subseteq G$ be an open identity neighborhood with compact closure. Then $p(Vg)$ is an open neighborhood of g because $p : G \longrightarrow G/H$ is open, and thus $p(\overline{V}g)$ is a compact neighborhood of gH. □

It is straightforward to see that a closed subgroup of a compact group is open if and only if it has finite index. The following is somewhat more general

Lemma 2.2. *Let G be a locally compact group, and suppose that $H \subseteq G$ is a closed subgroup of countable index. Then H is open.*

Proof. Let $(g_n)_{n \geq 0}$ be a countable set of elements with $G = \bigcup_{n \geq 0} g_n H$. Since G is a Baire space and since the cosets $g_n H$ are closed, there is an index m such that $g_m H$ has nonempty interior. But then $H = \lambda_{g_m^{-1}} g_m H$ has nonempty interior and hence is open by Proposition 1.12. □

Lemma 2.3. *Let G be a locally compact group. Then G has a σ-compact open subgroup. In particular, every connected locally compact group is σ-compact.*

Proof. Let $C \subseteq G$ be a compact symmetric identity neighborhood. Then $H = \langle C \rangle = C \cup CC \cup CCC \cup \cdots$ is σ-compact. Since C contains a nonempty open set, H is open. □

Proposition 2.4. *Let G be a locally compact σ-compact group and let $(V_n)_{n\in\mathbb{N}}$ be a countable family of identity neighborhoods. Then there exists a compact normal subgroup $N \trianglelefteq G$ with $N \subseteq \bigcap_{n\in\mathbb{N}} V_n$ such that G/N is metrizable.*

Proof. The proof uses Lemma 1.26. Since G is σ-compact, there exists a family of compact sets $(A_n)_{n\in\mathbb{N}}$ of G with $G = \bigcup_{n\in\mathbb{N}} A_n$. We put $L_n = A_0 \cup \cdots \cup A_n$, for $n \geq 0$. We also put $K_n = G$ for $n \geq 1$. For the integers $n \leq 0$, we choose recursively symmetric identity neighborhoods K_n as follows. Given K_{n+1}, we choose an identity neighborhood W such that $\gamma_a(b) = aba^{-1} \in K_{n+1}$ holds for all $(a,b) \in L_{-n} \times W$. The existence of W is guaranteed by Wallace's Lemma 1.16, since L_{-n} is compact and $\gamma_a(e) = e$. We then choose a compact symmetric identity neighborhood $K_n \subseteq W \cap V_{-n}$ such that $K_nK_nK_n \subseteq K_{n+1}$. Let ℓ denote the corresponding continuous length function on G, as given by Lemma 1.26. Then $N = \bigcap_{n\in\mathbb{N}} K_n = \{g \in G \mid \ell(g) = 0\}$ is a compact subgroup of G. We claim that N is a normal subgroup. For $a \in L_m$ we have $a \in L_{m+s}$ for all $s \geq 0$. If $g \in N$, then likewise $g \in K_{-m-s}$ holds for all $s \geq 0$. Thus $\gamma_a(g) \in K_{-m-s+1}$ holds for all $s \geq 0$. Since $K_n \subseteq K_{n+1}$ holds for all $n \in \mathbb{Z}$, this implies that $\gamma_a(g) \in N$, and therefore N is normal in G.

Since $\ell(g) = \ell(h)$ holds whenever $g^{-1}h \in N$, we obtain a well-defined length function $\bar{\ell}$ on G/N, via $\bar{\ell}(gN) = \ell(g)$. Because $p : G \longrightarrow G/N$ is a quotient map, $\bar{\ell}$ is continuous. Also, $\bar{\ell}(gN) = 0$ holds if and only if $g \in N$. Therefore $\bar{d}(gN, hN) = \ell(g^{-1}h)$ is a continuous left invariant metric on G/N. It remains to show that $\bar{d}$ determines the topology on G/N. Suppose that $W \subseteq G/N$ is an open identity neighborhood. We claim that there exists an integer $m \leq 0$ with $p(K_m) \subseteq W$. Let $U = p^{-1}(W)$. Then U is an open identity neighborhood, and $N \subseteq U$. If there exists no $m \leq 0$ with $K_m \subseteq U$, then $(K_n - U)_{n\leq 0}$ is a nested family of nonempty compact sets. But then $\bigcap(K_n - U) = N - U$ is nonempty, a contradiction. Therefore there exists an integer m with $p(K_m) \subseteq W$. Hence if $g \in G$ is any element, then $\{hN \in G/N \mid \bar{d}(hN, gN) < 2^m\} \subseteq p(g)W$. This shows that the left invariant metric $\bar{d}$ determines the topology on G/N □

Definition 2.5. A Hausdorff topological group G has *no small subgroups* if there exists an identity neighborhood $U \subseteq G$ such that the only subgroup of G that is contained in U is the trivial group $\{e\}$. If G has not small subgroups, and if $K \longrightarrow G$ is an injective morphism of topological groups, then K also has no small subgroups.

Note that in the previous definition it does nor make a difference if we require only that some identity neighborhood does not contain a nontrivial closed subgroup, because G is regular.

Corollary 2.6. *Suppose that the locally compact group G has no small subgroups. Then G has an open metrizable subgroup.*

Proof. Let $U \subseteq G$ be an identity neighborhood which contains no nontrivial subgroup of G. By Lemma 2.3, the group G has an open σ-compact subgroup H. Now we apply Proposition 2.4 to H and the constant family $U_n = U \cap H$, for $n \in \mathbb{N}$. It follows that the compact normal subgroup $N \trianglelefteq H$ is trivial, and hence that H is metrizable. ☐

We now study the existence of small subgroups in the totally disconnected case.

Lemma 2.7. *Let G be a locally compact group und suppose that V is a compact open neighborhood of the identity. Then V contains an open subgroup $H \subseteq G$.*

Proof. By Wallace's Lemma 1.16, applied to the compact set $V \times \{e\} \subseteq V \times V$, there exists an open symmetric identity neighborhood $U \subseteq V$ such that $VU \subseteq V$. In particular, $UU \subseteq V$. By induction we conclude that for every $k \geq 1$ the k-fold product $U \cdots U$ is contained in V. Hence the open subgroup $H = U \cup UU \cup UUU \cup \cdots$ is contained in V. ☐

In order to put this result to work, we need two results about totally disconnected locally compact spaces.

Lemma 2.8. *Suppose that X is a compact space, and that $x \in X$. Then the set*

$$Q(x) = \bigcap \{D \subseteq X \mid D \text{ contains } x \text{ and } D \text{ is closed and open}\}$$

is connected.

In a general topological space, the set $Q(x)$ as defined above is called the *quasi-component* of x.

Proof. Clearly $Q(x)$ is closed and contains x. Suppose that $Q(x) = A \cup B$, with $x \in A$ and A, B closed and disjoint. We have to show that $B = \varnothing$. Since X, being compact, is normal, there exist disjoint open sets $U, V \subseteq X$ with $A \subseteq U$ and $B \subseteq V$. We put $C = X - (U \cup V)$ and we note that C and $Q(x)$ are disjoint. For every $c \in C$ we can therefore choose an an open and closed set W_c containing x, but not containing c. The open sets $X - W_c$ cover C. Since C is compact, there exists $c_1, \ldots, c_m \in C$ such that $C \subseteq (X - W_{c_1}) \cup \cdots \cup (X - W_{c_m})$. Hence C is disjoint from the set $W = W_{c_1} \cap \cdots \cap W_{c_m}$. Also, W is closed and open, and therefore $Q(x) \subseteq W$. Since W is disjoint from C, we have $W \subseteq U \cup V$. Now $Y = U \cup (X - W)$ is open and $Z = V \cap W$ is open and contains B. Since $X = Y \cup Z$ and $Y \cap Z = \varnothing$, the set Y is closed and open and thus $Q(x) \subseteq Y$. It follows that $B = \varnothing$. ☐

Lemma 2.9. *Suppose that X is a locally compact totally disconnected space, and that $x \in X$. Then x has arbitrarily small compact open neighborhoods.*

Proof. Let V be a neighborhood of x. We have to show that V contains a compact open neighborhood of x. Passing to a smaller neighborhood if necessary, we may assume in addition that V is open and that $\overline{V}$ is compact. We put $A = \overline{V} - V$ and we note that we are done if $A = \varnothing$, with $U = V$. If $A \neq \varnothing$, we make use of Lemma 2.8. Since X is totally disconnected, we have $Q(x) = \{x\}$. Hence for each $a \in A$, there exists a compact neighborhood $U_a \subseteq \overline{V}$ of x which does not contain a, and which is open in $\overline{V}$. Then $\bigcap\{U_a \mid a \in A\} \cap A = \varnothing$. Hence there exist finitely many points $a_1, \ldots, a_m$ such that $U = U_{a_1} \cap \cdots \cap U_{a_m}$ is disjoint from A. Then U is closed and open in $\overline{V}$, and $U \subseteq V$. But then U is also open in X. □

The next result is van Dantzig's Theorem.

Theorem 2.10. *Let G be a locally compact group. Then the following are equivalent.*

(i) *G is totally disconnected.*

(ii) *Every identity neighborhood in G contains an open subgroup.*

Proof. Suppose that (i) holds and that $U \subseteq G$ is an identity neighborhood. By Lemma 2.9, there exists a compact open identity neighborhood $V \subseteq U$ and by Lemma 2.7, there exists an open subgroup $H \subseteq V$. Hence (ii) follows. Suppose that (ii) holds and that $g \in G - \{e\}$. There exists an open subgroup $H \subseteq G - \{g\}$, and thus is no connected set $C \subseteq G$ containing e and g. Hence (i) follows. □

Corollary 2.11. *If a locally compact totally disconnected group G has no small subgroups, then G is discrete.*

For compact groups we obtain a stronger form of van Dantzig's Theorem.

Theorem 2.12. *Let G be a compact group. Then the following are equivalent.*

(i) *G is totally disconnected.*

(ii) *Every identity neighborhood contains a normal open subgroup.*

Proof. Suppose that (i) holds and that U is an identity neighborhood. By Theorem 2.10, there exist an open subgroup $H \subseteq G$ which is contained in U. Since $G = \bigcup G/H$ is compact, G/H is finite. Let N denote the kernel of the action of G on G/H. Then N has finite index in G, and $N = \bigcap\{\gamma_a(H) \mid a \in G\}$ is closed. Since G/N is finite, G/N is discrete and therefore N is open. Claim (ii) follows, since $N \subseteq H \subseteq U$. If (ii) holds, then G is totally disconnected by Theorem 2.10. □

Remarks on profinite groups

A compact group satisfying condition (ii) in Theorem 2.12 is commonly called a *profinite group.* Let $(F_i)_{i\in I}$ be a family of finite groups. If we endow each group F_i with the discrete topology, then the F_i are compact and the product

$$G = \prod_{i\in I} F_i$$

is a totally disconnected compact group. The fact that the topology is totally disconnected follows from the next lemma.

Lemma 2.13. *A product of totally disconnected spaces is again totally disconnected.*

Proof. Let $X = \prod_{i\in I} X_i$ be a product of totally disconnected spaces X_i. Suppose that $C \subseteq X$ is a nonempty connected set. Since the X_i are totally disconnected, $\mathrm{pr}_i(C) \subseteq X_i$ is for every $i \in I$ a singleton. But then C itself is a singleton. □

The next result is a very weak version of the Peter–Weyl Theorem.

Lemma 2.14. *Let G be a profinite group and suppose that $g \in G$ is not the identity element. Then there exist an integer $n \geq 1$ and a morphism of topological groups $\rho : G \longrightarrow \mathrm{GL}_n(\mathbb{C})$ with $\rho(g) \neq \mathbb{1}$.*

Proof. By Theorem 2.12 there exists a normal open subgroup $N \subseteq G - \{g\}$. Then $F = G/N$ is a finite group and $gN \neq N$. Let $f : F \longrightarrow \mathrm{GL}_n(\mathbb{C})$ be a faithful representation of F. Such a representation exists, for example via the embedding of F in the complex group ring $\mathbb{C}[F]$, see Definition 2.18 below. Then the composite

$$\rho = f \circ p : G \longrightarrow G/N \longrightarrow \mathrm{GL}_n(\mathbb{C})$$

is a morphism with $\rho(g) \neq \mathbb{1}$. □

Proposition 2.15. *Let G be a profinite group. Then there exists a family of finite groups $(F_i)_{i\in I}$ and a closed injective morphism $f : G \longrightarrow \prod_{i\in I} F_i$. Hence G is isomorphic as a topological group to a closed subgroup of a product of finite groups.*

Proof. Let I denote the set of all open normal subgroups of G. For every $N \in I$ we put $F_N = G/N$, and we put $f_N(g) = gN$. Then each F_N is a finite group and the f_N fit together to a morphism $f : G \longrightarrow \prod_{N\in I} F_N$. Since the open normal subgroups form a neighborhood base of the identity in G by Theorem 2.12, the morphism f is injective. Since G is compact, f is closed. □

A first major goal is to prove that a suitable version of Lemma 2.14 holds for all compact groups. This requires the Haar integral.

The Haar integral

Suppose that X is a Hausdorff space. The *support* of a continuous map $\varphi : X \longrightarrow \mathbb{R}$ is the closed set

$$\operatorname{supp}(\varphi) = \overline{\varphi^{-1}(\mathbb{R} - \{0\})}.$$

We say that φ has *compact support* if $\operatorname{supp}(\varphi)$ is compact. The real valued continuous functions with compact support form a real vector space which we denote by $C_c(X)$. For $\varphi, \psi \in C_c(X)$ we write $\varphi \leq \psi$ if $\varphi(x) \leq \psi(x)$ holds for all $x \in X$. The nonnegative functions in $C_c(X)$ form a positive cone which we denote by

$$C_c^+(X) = \{\varphi \in C_c(X) \mid 0 \leq \varphi\}.$$

We note $C_c(X)$ is a normed vector space with respect to the sup norm

$$\|\varphi\|_\infty = \sup\{|\varphi(x)| \mid x \in X\}.$$

Lemma 2.16. *Suppose that X is a locally compact space and that $C \subseteq X$ is compact. Then there exists a continuous map $\varphi : X \longrightarrow [0,1]$ with compact support, and with $\varphi(C) \subseteq \{1\}$.*

Proof. For every $c \in C$ we choose an open neighborhood U_c of c with compact closure. Since C is compact, there are $c_1, \ldots, c_m \in C$ with $C \subseteq U_{c_1} \cup \cdots \cup U_{c_m} = U$. Then U has compact closure $\overline{U}$. Since $\overline{U}$ is normal, there exists a continuous map $\varphi : \overline{U} \longrightarrow [0,1]$ with $\varphi(C) \subseteq \{1\}$ and $\varphi(\overline{U} - U) \subseteq \{0\}$. If we extend φ to all of X by putting $\varphi(x) = 0$ for $x \in X - U$, then φ is continuous and has the desired properties. □

We need also the fact that a continuous function with compact support on a locally compact group is uniformly continuous.

Lemma 2.17. *Let G be a locally compact group, and suppose that $\varphi \in C_c(G)$. For every $\varepsilon > 0$ there is a symmetric identity neighborhood V such that $|\varphi(g) - \varphi(h)| < \varepsilon$ holds for all $g, h \in G$ with $g^{-1}h \in V$.*

Proof. We choose for every $a \in G$ an identity neighborhood W_a such that $|\varphi(a) - \varphi(g)| < \frac{\varepsilon}{2}$ holds for all $g \in aW_a$, and then an identity neighborhood U_a such that $U_aU_a \subseteq W_a$. Since $C = \operatorname{supp}(\varphi)$ is compact, there exist $a_1, \ldots, a_m \in G$ such that $C \subseteq a_1U_{a_1} \cup \cdots \cup a_mU_{a_m}$. We choose a symmetric identity neighborhood V such that $V^{-1}V \subseteq U_{a_1} \cap \cdots \cap U_{a_m}$ and we claim that V has the desired property. Suppose that $h \in gV$. If $gV \cap a_kU_{a_k} = \varnothing$ holds for all $k = 1, \ldots, m$, then $\varphi(g) = \varphi(h) = 0$. On the other hand, if $gV \cap a_kU_{a_k} \neq \varnothing$, then $g \in a_kU_{a_k}V^{-1}$ and therefore $gV \subseteq a_kU_{a_k}V^{-1}V \subseteq a_kU_{a_k}U_{a_k} \subseteq a_kW_{a_k}$. Hence if $h \in gV$, then $|\varphi(a_k) - \varphi(h)| < \frac{\varepsilon}{2}$, and therefore $|\varphi(g) - \varphi(h)| < \varepsilon$. □

At this stage it is convenient to introduce the group ring.

Definition 2.18. Let G be a group (without any topology) and let R be a commutative ring. The *group ring* $R[G]$ is the free R-module with basis G. The elements of $R[G]$ are thus formal linear combinations $\sum_{g\in G} c_g g$ with coefficients $c_g \in R$, where only finitely many coefficients c_g are nonzero. The group multiplication extends to a bilinear multiplication $R[G] \times R[G] \longrightarrow R[G]$ which turns $R[G]$ into an associative R-algebra. Explicitly, the multiplication is given by

$$\Big(\sum_{x\in G} a_x x\Big)\Big(\sum_{y\in G} b_y y\Big) = \sum_{x\in G}\sum_{y\in G} a_y b_{y^{-1}x} x.$$

Whenever G acts linearly on an R-module M, this action extends to $R[G]$ and turns M into an $R[G]$-module. The map

$$\epsilon : \sum_{g\in G} c_g g \longmapsto \sum_{g\in G} c_g$$

is called the *augmentation map*. This map is an algebra homomorphism $\epsilon : R[G] \longrightarrow R$.

Observation 2.19. Suppose that G is a locally compact group. Then G acts from the left on the vector space $C_c(G)$ via

$$(a\varphi)(x) = \varphi(a^{-1}x) = (\varphi \circ \lambda_{a^{-1}})(x).$$

Indeed, we have

$$b(a\varphi) = b(\varphi \circ \lambda_{a^{-1}}) = \varphi \circ \lambda_{a^{-1}} \circ \lambda_{b^{-1}} = \varphi \circ \lambda_{(ba)^{-1}} = (ba)\varphi$$

for all $a, b \in G$. For an element $a = \sum_{g\in G} a_g g$ in the real group ring $\mathbb{R}[G]$ and $\varphi \in C_c(G)$ we have thus

$$a\varphi = \sum_{g\in G} a_g \varphi \circ \lambda_{g^{-1}}.$$

We note that the sum on the right-hand side is finite.

Definition 2.20. We call a linear functional

$$I : C_c(G) \longrightarrow \mathbb{R}$$

an *invariant integral* or *Haar integral* if the following hold.

(i) If $\varphi \in C_c^+(G)$, then $I(\varphi) \geq 0$.

(ii) If $g \in G$ and if $\varphi \in C_c(G)$, then $I(g\varphi) = I(\varphi)$.

(iii) There exists a function $\varphi \in C_c^+(G)$ with $I(\varphi) > 0$.

Axiom (i) implies that $I(\varphi) \leq I(\psi)$ if $\varphi \leq \psi$. We note that if I is an invariant integral and if $s > 0$, then sI is again an invariant integral.

Example 2.21. Let G be any group, endowed with the discrete topology. Then G is locally compact and the compact subsets are the finite subsets of G. Then $C_c(G)$ can be identified with the real group ring $\mathbb{R}[G]$ as follows. An element $a = \sum_{g \in G} a_g g$ is viewed as the map $g \longmapsto a_g$. In this case $I(a) = \epsilon(a)$ is an invariant integral.

Our next aim is to prove the existence of an invariant integral on every locally compact group. We define $\mathbb{R}[G]^+$ to be the set of all elements $\sum_{g \in G} c_g g$ in the real group ring $\mathbb{R}[G]$ whose coefficients c_g are nonnegative. We note that $\mathbb{R}[G]^+$ is closed under multiplication, addition, and multiplication by nonnegative reals. If $a \in \mathbb{R}[G]^+$ and if $\varphi \in C_c^+(G)$, then $a\varphi \in C_c^+(G)$.

Lemma 2.22. *Suppose that G is a locally compact group and that $\varphi, \alpha \in C_c^+(G)$. If $\alpha \neq 0$, then there exists an element $a \in \mathbb{R}[G]^+$ with $\varphi \leq a\alpha$.*

Proof. We put $t = \|\alpha\|_\infty$, $s = \|\varphi\|_\infty$ and $U = \{x \in G \mid \alpha(x) > \frac{t}{2}\}$. Then U is a nonempty open set. Since $C = \operatorname{supp}(\varphi)$ is compact, there exist elements $g_1, \ldots, g_m \in G$ with $C \subseteq g_1 U \cup \cdots \cup g_m U$. For $x \in g_k U$ we have $g_k \alpha(x) = \alpha(g_k^{-1} x) > \frac{t}{2}$. Thus $a = \frac{2s}{t}(g_1 + \cdots + g_n)$ has the desired property. □

Let G be a locally compact group. For functions $\varphi, \alpha \in C_c^+(G)$ with $\alpha \neq 0$ we define their ratio as

$$(\varphi : \alpha) = \inf\{\epsilon(a) \mid a \in \mathbb{R}[G]^+ \text{ and } \varphi \leq a\alpha\}.$$

By Lemma 2.22, this number is finite and nonnegative.

Lemma 2.23. *Suppose that G is a locally compact group and that $\varphi, \psi, \alpha, \beta \in C_c^+(G)$, with $\alpha, \beta \neq 0$. Then we have the following.*

(i) $(g\varphi : \alpha) = (\varphi : \alpha) = (\varphi : g\alpha)$ *for all group elements $g \in G$.*

(ii) $(s\varphi : \alpha) = s(\varphi : \alpha)$ *for all real numbers $s \geq 0$.*

(iii) *If $\varphi \leq \psi$, then $(\varphi : \alpha) \leq (\psi : \alpha)$.*

(iv) $(\varphi + \psi : \alpha) \leq (\varphi : \alpha) + (\psi : \alpha)$.

(v) $(\varphi : \beta) \leq (\varphi : \alpha)(\alpha : \beta)$.

(vi) $\|\varphi\|_\infty \leq (\varphi : \alpha)\|\alpha\|_\infty$.

Proof. In what follows, a and b are elements in $\mathbb{R}[G]^+$. Claims (i) and (ii) follow directly form the definition, since for $g \in G$, the relation $\varphi \leq a\alpha$ holds if and only if $g\varphi \leq ga\alpha$. Likewise, for $s > 0$ the relation $s\varphi \leq a\alpha$ holds for $s > 0$ if and only $\varphi \leq \frac{1}{s}a\alpha$, and $(0 : \alpha) = 0$ holds in any case. For claim (iii), suppose that $\psi \leq a\alpha$. Then $\varphi \leq \psi \leq a\alpha$. For claim (iv), suppose that $\varphi \leq a\alpha$ and that $\psi \leq b\alpha$. Then $\varphi + \psi \leq a\alpha + b\alpha = (a+b)\alpha$. For claim (v), suppose that we have $a, b \in \mathbb{R}[G]^+$ with $\varphi \leq a\alpha$ and $\alpha \leq b\beta$. Then $\varphi \leq ab\beta$ and thus $(\varphi : \beta) \leq \epsilon(ab) = \epsilon(a)\epsilon(b)$. For claim (vi), suppose that $a \in \mathbb{R}[G]$ with $\varphi \leq a\alpha$. Then $\varphi(x) \leq (a\alpha)(x) \leq \epsilon(a)\|\alpha\|_\infty$ holds for all $x \in G$. □

Lemma 2.24. *Let G be a locally compact group. Given functions $\varphi, \psi \in C_c^+(G)$, there exists a function $\eta \in C_c(G)$ with the following property. For every $\varepsilon > 0$ there exists a symmetric identity neighborhood $V \subseteq G$ such that*

$$(\varphi : \alpha) + (\psi : \alpha) \leq (1+\varepsilon)(\varphi + \psi : \alpha) + \varepsilon(1+\varepsilon)(\eta : \alpha)$$

holds for all $\alpha \in C_c^+(G)$ with $\alpha \neq 0$ and $\operatorname{supp}(\alpha) \subseteq V$.

Proof. By Lemma 2.16 there exists a continuous function $\eta : G \longrightarrow [0,1]$ with compact support, such that $\eta(x) = 1$ for all $x \in \operatorname{supp}(\varphi) \cup \operatorname{supp}(\psi)$. We put

$$\xi(x) = \varphi(x) + \psi(x) + \varepsilon\eta(x)$$

and we define functions $\hat{\varphi}, \hat{\psi} \in C_c^+(G)$ as follows. On the closed set $\{x \in G \mid \varphi(x) = 0\}$ we put $\hat{\varphi}(x) = 0$. On the closed set $\operatorname{supp}(\varphi)$ we put $\hat{\varphi}(x) = \frac{\varphi(x)}{\xi(x)}$. This is possible since the zero-set of ξ is a closed set which is disjoint from $\operatorname{supp}(\varphi) \cup \operatorname{supp}(\psi)$. The function $\hat{\psi}$ is defined similarly as $\hat{\psi}(x) = \frac{\psi(x)}{\xi(x)}$ on $\operatorname{supp}(\psi)$, and $\hat{\psi}(x) = 0$ elsewhere. We note that $\hat{\varphi} + \hat{\psi} \leq 1$. We choose $\delta > 0$ so that $6\|\xi\|_\infty \delta \leq \varepsilon^3$ and $2\delta < \varepsilon^2$. By Lemma 2.17 there is a symmetric identity neighborhood V such that

$$|\varphi(x) - \varphi(y)|, |\psi(x) - \psi(y)| < \delta$$

holds whenever $x^{-1}y \in V$. It follows for such $x, y \in \operatorname{supp}(\varphi)$ that

$$\begin{aligned}
|\hat{\varphi}(x) - \hat{\varphi}(y)| = \left|\frac{\varphi(x)\xi(y) - \varphi(y)\xi(x)}{\xi(x)\xi(y)}\right| &\leq \frac{1}{\varepsilon^2}\left|\varphi(x)\xi(y) - \varphi(y)\xi(x)\right| \\
&\leq \frac{1}{\varepsilon^2}\left|\varphi(x)\xi(y) - \varphi(x)\xi(x)\right| + \frac{1}{\varepsilon^2}\left|\varphi(x)\xi(x) - \varphi(y)\xi(x)\right| \\
&\leq \frac{2\delta}{\varepsilon^2}\|\varphi\|_\infty + \frac{\delta}{\varepsilon^2}\|\xi\|_\infty \leq \frac{3\delta}{\varepsilon^2}\|\xi\|_\infty \leq \frac{\varepsilon}{2}.
\end{aligned}$$

If $x^{-1}y \in V$ and if $\hat{\varphi}(x) = 0 \neq \hat{\varphi}(y)$, then also $|\hat{\varphi}(x) - \hat{\varphi}(y)| = \frac{|\varphi(y)|}{|\xi(y)|} \leq \frac{\delta}{\varepsilon} \leq \frac{\varepsilon}{2}$.

We claim that V has the desired properties. Let $\alpha \in C_c^+(G)$ be a nonzero function with $\operatorname{supp}(\alpha) \subseteq V$ and suppose that $a \in \mathbb{R}[G]^+$ with

$$\xi \leq a\alpha.$$

Then $\varphi = \xi\hat{\varphi} \leq (a\alpha)\hat{\varphi}$. If $\alpha(g^{-1}x) \neq 0$, then $g^{-1}x \in V$ and therefore $\hat{\varphi}(x) \leq \hat{\varphi}(g) + \frac{\varepsilon}{2}$. Hence if $a = \sum_{g \in G} a_g g$, then

$$\varphi(x) \leq \sum_{g \in G} a_g \alpha(g^{-1}x)\hat{\varphi}(x) \leq \sum_{g \in G} a_g\big(\hat{\varphi}(g) + \tfrac{\varepsilon}{2}\big)\alpha(g^{-1}x).$$

and hence $(\varphi : \alpha) \leq \sum_{g \in G} a_g(\hat{\varphi}(g) + \frac{\varepsilon}{2})$. Similarly, $(\psi : \alpha) \leq \sum_{g \in G} a_g(\hat{\psi}(g) + \frac{\varepsilon}{2})$ and therefore

$$(\varphi : \alpha) + (\psi : \alpha) \leq \sum_{g \in G} a_g(\hat{\varphi}(g) + \hat{\psi}(g) + \varepsilon) \leq \sum_{g \in G} a_g(1 + \varepsilon) = \epsilon(a)(1 + \varepsilon).$$

Thus

$$(\varphi : \alpha) + (\psi : \alpha) \leq (\varphi + \psi + \varepsilon\eta : \alpha)(1 + \varepsilon) \leq (\varphi + \psi : \alpha)(1 + \varepsilon) + \varepsilon(1 + \varepsilon)(\eta : \alpha)$$

as claimed. □

Construction 2.25. We now fix a nonzero function $\varphi_0 \in C_c^+(G)$ and study the quantity

$$I(\varphi, \alpha) = \frac{(\varphi : \alpha)}{(\varphi_0 : \alpha)},$$

for $\varphi, \alpha \in C_c^+(G)$ with $\alpha \neq 0 \neq \varphi$. Note that $(\varphi_0 : \alpha) > 0$ by Lemma 2.23(vi). From the definition of $I(\varphi, \alpha)$ and Lemma 2.23 we conclude that

$$\begin{aligned} I(\varphi, \alpha) &\leq (\varphi : \varphi_0), \\ \tfrac{1}{(\varphi_0 : \varphi)} &\leq I(\varphi, \alpha), \\ I(g\varphi, \alpha) &= I(\varphi, \alpha), \\ I(s\varphi, \alpha) &= sI(\varphi, \alpha), \\ I(\varphi + \psi, \alpha) &\leq I(\varphi, \alpha) + I(\psi, \alpha), \end{aligned}$$

for all $g \in G$, $\psi \in C_c^+(G) - \{0\}$ and all $s \geq 0$. We note also that for all $\varphi, \psi \in C_c^+(G) - \{0\}$ there exists by Lemma 2.24 function $\eta \in C_c^+(G)$ such for every $\varepsilon > 0$ there is symmetric identity neighborhood V such that

$$I(\varphi, \alpha) + I(\psi, \alpha) \leq I(\varphi + \psi, \alpha)(1 + \varepsilon) + \varepsilon(1 + \varepsilon)I(\eta, \alpha) \leq I(\varphi + \psi, \alpha)(1 + \varepsilon) + \varepsilon(1 + \varepsilon)(\eta : \varphi_0).$$

holds for all nonzero $\alpha \in C_c^+(G)$ with $\operatorname{supp}(\alpha) \subseteq V$. In order to obtain an additive functional, we wish to pass to the limit as ε approaches 0. The problem is that V and α depend on ε. We circumvent this by a compactness argument. Let

$$P = C_c^+(G) - \{0\}.$$

We consider the compact infinite cube

$$Q = \prod_{\varphi \in P} \left[\tfrac{1}{(\varphi_0 : \varphi)}, (\varphi : \varphi_0)\right].$$

Given a symmetric identity neighborhood $V \subseteq G$ and a nonzero $\alpha \in C_c^+(G)$ with $\operatorname{supp}(\alpha) \subseteq V$, we may consider the element $(I(\varphi, \alpha))_{\varphi \in P} \in Q$, and the set $Q_V \subseteq Q$ consisting of all such elements. These sets Q_V have the finite intersection property. Hence

$$\bigcap \{\overline{Q_V} \mid V \subseteq G \text{ is a symmetric identity neighborhood}\} \subseteq Q$$

is nonempty. We pick an element I in this intersection. Thus I is a map $I : P \longrightarrow \mathbb{R}$ which assigns to every $\varphi \in P$ the real number

$$I(\varphi) = \operatorname{pr}_\varphi(I) \in \left[\tfrac{1}{(\varphi_0 : \varphi)}, (\varphi : \varphi_0)\right] \subseteq \mathbb{R},$$

and $I \in \overline{Q_V}$ holds for every identity neighborhood $V \subseteq G$.

Theorem 2.26. *Every locally compact group admits an invariant integral.*

Proof. Suppose that $\varphi, \psi \in P$ and that $s > 0$. For every identity neighborhood V and and every $\alpha \in P$ with $\operatorname{supp}(\alpha) \subseteq V$ we have $I(\varphi + \psi, \alpha) \leq I(\varphi, \alpha) + I(\psi, \alpha)$. Since I is contained in the closure of P_V, the relation $I(\varphi + \psi) \leq I(\varphi) + I(\psi)$ holds as well. Similarly, the relation $I(s\varphi, \alpha) = sI(\varphi, \alpha)$ implies that $I(s\varphi) = sI(\varphi)$.

There is a constant $c \geq 0$ such that for every $\varepsilon > 0$ there exists a symmetric identity neighborhood V such that $I(\varphi, \alpha) + I(\psi, \alpha) \leq I(\varphi + \psi, \alpha)(1 + \varepsilon) + \varepsilon(1 + \varepsilon)c$ holds for all $\alpha \in P$ with $\operatorname{supp}(\alpha) \subseteq V$. Since I is contained in $\overline{P_V}$, we have $I(\varphi) + I(\psi) \leq I(\varphi + \psi)(1 + \varepsilon) + \varepsilon(1 + \varepsilon)c$. This is true for every $\varepsilon > 0$, hence $I(\varphi) + I(\psi) = I(\varphi + \psi)$.

Every function $\varphi \in C_c(G) - \{0\}$ can be written as a difference $\varphi = \varphi_1 - \varphi_2$, with $\varphi_1, \varphi_2 \in P$ as defined above. We put $I(\varphi) = I(\varphi_1) - I(\varphi_2)$ and we have to check that this expression is well-defined. If $\varphi = \varphi_3 - \varphi_4$ is another decomposition of this type, then $\varphi_1 + \varphi_4 = \varphi_3 + \varphi_2$, and thus $I(\varphi_1) + I(\varphi_4) = I(\varphi_3) + I(\varphi 2)$, whence $I(\varphi_1) - I(\varphi_2) = I(\varphi_3) - I(\varphi_4)$. Conversely, every linear extension of I to $C_c(G)$ has to satisfy these formulas.

It follows readily that I is additive and that $I(s\varphi) = sI(\varphi)$ holds for all $s \in \mathbb{R}$. Hence I is a linear functional. By construction, I is invariant under the G-action. We have $I(\varphi_0, \alpha) = 1$ no matter what α and V are. Hence $I(\varphi_0) = 1$. □

Thus we have established the existence of a Haar integral on every locally compact group. We now show that such an integral is unique up to a positive constant. We first make an observation about invariant integrals.

Lemma 2.27. *Let J be an invariant integral on the locally compact group G and suppose that $\varphi \in C_c(G)$. For every $a \in \mathbb{R}[G]$ we have*

$$J(a\varphi) = \epsilon(a)J(\varphi).$$

If $\varphi, \alpha \in C_c^+(G)$ and if $\alpha \neq 0$, then $J(\alpha) \neq 0$ and

$$J(\varphi) \leq (\varphi : \alpha)J(\alpha).$$

Proof. If $a = \sum_{g\in G} a_g g$, then $J(a\varphi) = \sum_{g\in G} J(a_g g\varphi) = \sum_{g\in G} a_g J(\varphi)$ by the G-invariance of the integral. This proves the first claim. Now suppose that $\varphi, \alpha \in C_c^+(G) - \{0\}$ and that $\alpha \neq 0$. If $a \in \mathbb{R}[G]^+$ and if $\varphi \leq a\alpha$, then $J(\varphi) \leq \epsilon(a)J(\alpha)$, which proves the last claim. By assumption, there exists an element $\varphi \in C_c^+(G)$ with $J(\varphi) > 0$. By Lemma 2.23 there exists an element $a \in \mathbb{R}[G]^+$ with $\varphi \leq a\alpha$ and thus $0 < J(\varphi) \leq J(a\alpha) = \epsilon(a)J(\alpha)$. ☐

Now we show that an invariant integral is determined uniquely up to a positive scaling factor.

Theorem 2.28. *Let G be a locally compact group and suppose that I, J are two invariant integrals. Then there exists a positive real $s > 0$ such that $J = sI$.*

Proof. It suffices to prove the following. Given two functions $\varphi_1, \varphi_2 \in C_c^+(G) - \{0\}$ and two invariant integrals I, J, we have

$$\frac{J(\varphi_1)}{J(\varphi_2)} = \frac{I(\varphi_1)}{I(\varphi_2)}.$$

This, in turn, will be a consequence of the following claim.
Claim. Given $\varphi_1, \varphi_2 \in C_c^+(G) - \{0\}$ and a real number $\varepsilon > 0$, there exists a function $\alpha \in C_c^+(G) - \{0\}$ such that

$$(1-\varepsilon)(\varphi_j : \alpha)J(\alpha) \leq J(\varphi_j) \quad j = 1, 2,$$

holds for every invariant integral J.

Let us assume for the moment that the claim is correct. Since $J(\varphi_2) \leq (\varphi_2 : \alpha)J(\alpha)$ holds by Lemma 2.27, we have

$$(1-\varepsilon)\frac{(\varphi_1 : \alpha)}{(\varphi_2 : \alpha)} \leq \frac{1}{(\varphi_2 : \alpha)}\frac{J(\varphi_1)}{J(\alpha)} \leq \frac{J(\varphi_1)}{J(\varphi_2)}.$$

By symmetry, the same inequality holds if we exchange φ_1 and φ_2. Moreover, we may replace J by I. If we then take inverses on both sides, we obtain the inequality

$$\frac{I(\varphi_1)}{I(\varphi_2)} \leq \frac{1}{(1-\varepsilon)}\frac{(\varphi_1 : \alpha)}{(\varphi_2 : \alpha)}.$$

Hence we have

$$\frac{I(\varphi_1)}{I(\varphi_2)} \leq \frac{1}{(1-\varepsilon)^2}\frac{J(\varphi_1)}{J(\varphi_2)}.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\frac{I(\varphi_1)}{I(\varphi_2)} \leq \frac{J(\varphi_1)}{J(\varphi_2)}.$$

By symmetry we get also the reverse inequality. and hence the two fractions are equal. It remains to justify the claim.

Proof of the claim. We put $C = \operatorname{supp}(\varphi_1) \cup \operatorname{supp}(\varphi_2)$ and we choose a continuous function with compact support $\eta : G \longrightarrow [0, 1]$, such that $\eta(C) \subseteq \{1\}$, using Lemma 2.16. We choose $s > 0$ in such a way that $\varepsilon > 2s(\eta : \varphi_j)$ holds for $j = 1, 2$. Then we choose a symmetric identity neighborhood V such that $|\varphi_j(x) - \varphi_j(y)| < s$ holds whenever $x^{-1}y \in V$, for $j = 1, 2$. We also choose a function $\beta \in C_c^+(G) - \{0\}$ with $\operatorname{supp}(\beta) \subseteq V$ and we put $\alpha(x) = \beta(x) + \beta(x^{-1})$. Hence $\alpha(x) = \alpha(x^{-1})$. We have to show that

$$(1-\varepsilon)(\varphi_j : \alpha)J(\alpha) \leq J(\varphi_j).$$

We choose $t > 0$ such that $tJ(\varphi_j) < sJ(\alpha)$ holds, for $j = 1, 2$. Then we choose a symmetric identity neighborhood W with compact closure $\overline{W}$ such that $|\alpha(x) - \alpha(y)| < t$ holds whenever $x^{-1}y \in W$. Since C is compact, there exist elements $g_1, \ldots, g_m \in G$ such that $C \subseteq g_1W \cup \cdots \cup g_mW$. We put $U_0 = G - C$ and $U_k = g_kW$, for $k = 1, \ldots, m$. This is a finite open covering of G. Hence there exists partition of unity $\psi_0, \ldots, \psi_m$ subordinate to this covering, that is, $\psi_k : G \longrightarrow [0, 1]$ is continuous, $\operatorname{supp}(\psi_k) \subseteq U_k$, and $\sum_{k=0}^m \psi_k = 1$. It follows that $\sum_{k=1}^m \psi_k(c) = 1$ for all $c \in C$, and that $\psi_1, \ldots, \psi_m \in C_c^+(G)$. In particular we have

(a) $$\varphi_j = \sum_{k=1}^m \psi_k \varphi_j$$

for $j = 1, 2$. In what follows, x and y denote arbitrary elements of G. For $y \in xV$ we have $\varphi_j(x) - s \leq \varphi_j(y)$, and therefore $(\varphi_j(x) - s) \cdot (x\alpha) \leq \varphi_j \cdot (x\alpha)$ holds for all x. Integrating this inequality, we obtain

(b) $$(\varphi_j(x) - s)J(\alpha) \leq J(\varphi_j \cdot x\alpha).$$

For $y \in g_k W$ we have $g_k^{-1} y = (g_k^{-1} x)(x^{-1} y) \in W$ and therefore $\alpha(x^{-1}y) = \alpha((x^{-1}y)^{-1}) \leq \alpha(g_k^{-1}x) + t$. Thus $\psi_k \cdot (x\alpha) \leq \psi_k \cdot (g_k\alpha)(x) + t$. Multiplying both sides by φ_j, summing over k, and integrating we arrive at the inequality

$$\text{(c)} \qquad J(\varphi_j \cdot (x\alpha)) \leq \sum_{k=1}^{m} J(\varphi_j \psi_k)(g_k\alpha)(x) + tJ(\varphi_j).$$

Since $tJ(\varphi_j) < sJ(\alpha)$, we conclude that

$$\text{(d)} \qquad (\varphi_j(x) - 2s)J(\alpha) \leq \sum_{k=1}^{m} J(\varphi_k \cdot \psi_k)(g_k\alpha)(x).$$

We put $\varphi_j' = \max\{0, \varphi_j - 2s\}$. The right-hand side in (d) is nonnegative, hence

$$(\varphi_j' : \alpha)J(\alpha) \leq \sum_{k=1}^{m} J(\varphi_k \cdot \psi_k) = J(\varphi_j).$$

By Lemma 2.27, $J(\varphi_j) \leq (\varphi_j : \alpha)J(\alpha)$. Also, we have $\varphi_j \leq \varphi_j' + 2s\eta$. Combining (b) and (c), we obtain

$$(\varphi_j : \alpha) \leq (\varphi_j' + 2s\eta : \alpha) \leq (\varphi_j' : \alpha) + 2s(\eta : \alpha) \leq (\varphi_j' : \alpha) + 2s(\eta : \varphi_j)(\varphi_j : \alpha)$$

and thus

$$(1 - \varepsilon)(\varphi_j : \alpha)J(\alpha) \leq (\varphi_j' : \alpha) \leq J(\varphi_j).$$

This proves the claim and finishes the proof of the theorem. □

Theorems 2.26 and 2.28 show the existence and uniqueness of an invariant integral I on a locally compact group, up to a positive real scaling factor. From now on we write

$$I(\varphi) = \int_G \varphi$$

for this integral.

Example 2.29. There will be very few occasions where we have to evaluate a Haar integral explicitly. Nevertheless it might be instructive so see some examples relating the Haar integral to the integrals arising in real analysis.

(a) $G = \mathbb{R}$ is the additive group of the reals, in the usual topology. Let $\varphi \in C_c(G)$. Then there exists an interval $[u, v] \subseteq \mathbb{R}$ containing $\mathrm{supp}(\varphi)$, and $\int_G \varphi = \int_u^v \varphi(t)dt$, where the right-hand side denotes the Riemann integral. If we choose a larger interval $[u', v']$ containing $[u, v]$, then $\int_u^v \varphi(t)dt = \int_{u'}^{v'} \varphi(t)dt$, so the left-hand side is well-defined. It remains to show that the right-hand side is translation invariant. But this is true, since $\int_{u+s}^{v+s} \varphi(t - s)dt = \int_u^v \varphi(t)dt$ holds for all $s \in \mathbb{R}$.

(b) $G = \mathrm{U}(1) = \{z \in \mathbb{C}^* \mid |z| = 1\}$ is the circle group. We put $\int_G \varphi = \int_0^1 \varphi(\exp(2\pi \boldsymbol{i} t))dt$, where the right-hand side denotes again the Riemann integral. Since $\int_0^1 \varphi(\exp(2\pi \boldsymbol{i} t))dt = \int_G \varphi = \int_0^1 \varphi(\exp(2\pi \boldsymbol{i}(t-s)))dt$ holds for all $s \in \mathbb{R}$, this is the Haar integral.

(c) Let λ denote the Lebesgue measure on $\mathbb{R}^m$. For the additive group $G = \mathbb{R}^m$, the Haar integral is given by $\int_G \varphi = \int_{\mathbb{R}^m} \varphi(v)d\lambda(v)$. This follows from the fact that λ is translation invariant.

The modular function

Construction 2.30. Let G be a locally compact group. The real vector space $C_c(G)$ is a left $\mathbb{R}[G]$-module with respect to the left G-action $g\varphi = \varphi \circ \lambda_{g^{-1}}$ and for $a \in \mathbb{R}[G]$, the diagram

$$\begin{array}{ccc} C_c(G) & \xrightarrow{\ a\ } & C_c(G) \\ \big\downarrow \int_G & & \big\downarrow \int_G \\ \mathbb{R} & \xrightarrow{\ \epsilon(a)\ } & \mathbb{R} \end{array}$$

commutes. In short,

$$\int_G : C_c(G) \longrightarrow \mathbb{R}$$

is a homomorphism of $\mathbb{R}[G]$-modules, where $\mathbb{R}[G]$ acts on $\mathbb{R}$ via the augmentation map ϵ. We now consider a different $\mathbb{R}[G]$-module structure on $C_c(G)$, by putting

$$G \times C_c(G) \longrightarrow C_c(G), \qquad (g, \varphi) \longmapsto \varphi_\circ \gamma_{g^{-1}}.$$

This G-action turns $C_c(G)$ into a left $\mathbb{R}[G]$-module in a different way. We put

$$I_g(\varphi) = \int_G \varphi \circ \gamma_{g^{-1}}.$$

Since $\gamma_g = \lambda_g \circ \rho_g = \rho_g \circ \lambda_g$, we conclude that $I_g(\varphi) = \int_G \varphi \circ \rho_{g^{-1}}$ and hence that $I_g(\varphi \circ \lambda_a) = I_g(\varphi)$. Therefore I_g is an invariant integral. By Theorem 2.28 there exists a positive real number s such that $sI = I_g$. We put $\mathrm{mod}(g) = s$. Thus

$$\mathrm{mod}(g) \int_G \varphi = \int_G \varphi \circ \gamma_{g^{-1}} = \int_G \varphi \circ \rho_{g^{-1}}.$$

For $g, h \in G$ we have

$$I_{gh}(\varphi) = \int_G \varphi \circ \rho_{(gh)^{-1}} = \int_G \varphi \circ \rho_{h^{-1}} \circ \rho_{g^{-1}} = \mathrm{mod}(g)\,\mathrm{mod}(h) \int_G \varphi,$$

which shows that

$$\mathrm{mod}(gh) = \mathrm{mod}(g)\,\mathrm{mod}(h).$$

Hence $\mathrm{mod} : G \longrightarrow \mathbb{R}_{>0}$ is a group homomorphism, the *modular function*. We will show next that mod is a morphism of topological groups. But first we note the following algebraic fact. If we extend mod in the natural way to the group ring $\mathbb{R}[G]$ by putting $\mathrm{mod}(\sum_{g\in G} a_g g) = \sum_{g\in G} a_g \,\mathrm{mod}(g)$, then $\mathbb{R}$ becomes an $\mathbb{R}[G]$-module and

$$\int_G : C_c(G) \longrightarrow \mathbb{R}$$

is again a homomorphism of $\mathbb{R}[G]$-modules, but for a different module structure on $C_c(G)$ and on $\mathbb{R}$.

Proposition 2.31. *Let G be a locally compact group. Then the modular function*

$$\mathrm{mod} : G \longrightarrow \mathbb{R}$$

is a morphism of topological groups, which is independent of the Haar integral.

Proof. We choose a function $\varphi \in C_c^+(G) - \{0\}$. If I, J are two invariant integrals, then there exists by Theorem 2.28 a real number $s > 0$ with $J = sI$. Hence $0 \neq J(\varphi \circ \gamma_{g^{-1}}) = sI(\varphi \circ \gamma_{g^{-1}}) = s\,\mathrm{mod}(g)I(\varphi) = \mathrm{mod}(g)J(\varphi)$, which shows that mod is independent of the chosen Haar integral.

We now prove that mod is continuous at the identity. By Lemma 1.5 this will imply that mod is continuous. Let $C = \mathrm{supp}(\varphi)$ and suppose that $\varepsilon > 0$. Let $U \subseteq G$ be an open symmetric identity neighborhood with compact closure $\overline{U}$. By Lemma 2.16 there exists a continuous function with compact support $\eta : G \longrightarrow [0, 1]$, with $\eta(C\overline{U}) \subseteq \{1\}$. Let $V \subseteq U$ be a symmetric identity neighborhood such that

$$|\varphi(x) - \varphi(y)| \int_G \eta \leq \varepsilon \int_G \varphi$$

holds for all $x, y \in G$ with $x^{-1}y \in V$. We claim that for $x \in V$ and $y \in G$ we have

$$(1) \qquad |\varphi(yx) - \varphi(y)| \int_G \eta \leq \varepsilon\eta(y) \int_G \varphi.$$

If $y \in CU$, then $\eta(y) = 1$ and $y^{-1}(yx) = x \in V$, so (1) is true. If $y \notin CU$, then $y \notin C$ and $y \notin Cx^{-1}$, whence $\varphi(y) = 0 = \varphi(yx)$, so (1) holds also. Now we integrate (1) with respect to the variable y and obtain

$$\left|\int_G (\varphi \circ \rho_{x^{-1}} - \varphi)\right| \int_G \eta \leq \varepsilon \int_G \eta \int_G \varphi \neq 0.$$

It follows that $\left|\int_G \varphi \circ \rho_{x^{-1}} - \int_G \varphi\right| \leq \varepsilon \int_G \varphi$. Since $\int_G \varphi \circ \rho_{x^{-1}} = \operatorname{mod}(x) \int_G \varphi$, this implies that

$$|\operatorname{mod}(x) - 1| \leq \varepsilon$$

holds for all $x \in V$. ☐

Definition 2.32. A locally compact group is called *unimodular* if mod is constant on G. This holds if and only if the Haar integral on G is bi-invariant,

$$\int_G \varphi \circ \rho_a = \int_G \varphi = \int_G \varphi \circ \lambda_a$$

for all $a \in G$.

Proposition 2.33. *Let G be a locally compact group. Then G is unimodular if G is abelian, or compact, or if $\overline{[G,G]} = G$.*

Proof. If G is abelian, then $\rho_a = \lambda_{a^{-1}}$ and the claim follows from the left invariance of the Haar integral. If G is compact, then $\operatorname{mod}(G) \subseteq \mathbb{R}_{>0}$ is compact. Since every nontrivial subgroup of $(\mathbb{R}_{>0}, \cdot)$ is unbounded, $\operatorname{mod}(G) = \{1\}$. The last claim follows from the fact that $(\mathbb{R}_{>0}, \cdot)$ is abelian and that mod is continuous. ☐

3 | The Peter–Weyl Theorem

In this chapter we work over the complex field $\mathbb{C}$. The real part of a complex number z is denoted by $\mathrm{Re}(z)$, and the imaginary unit is written as $\boldsymbol{i} = \sqrt{-1}$. We denote complex conjugation by $z \longmapsto \bar{z}$. This should not be confused with the closure operation for sets in a topological space.

The support of a complex valued continuous function φ on a topological space X is

$$\mathrm{supp}(\varphi) = \overline{\varphi^{-1}(\mathbb{C} - \{0\})}.$$

We denote by $C_c(X, \mathbb{C})$ the complex vector space of all continuous complex functions on X with compact support. It follows that $C_c(X, \mathbb{C})$ splits as a real vector space as

$$C_c(X, \mathbb{C}) = C_c(X) \oplus \boldsymbol{i} C_c(X)$$

We put

$$\overline{\varphi}(x) = \overline{\varphi(x)},$$

where $z \longmapsto \overline{z}$ denotes complex conjugation. If G is a locally compact group, then G acts from the left on $C_c(G, \mathbb{C})$ via

$$G \times C_c(G, \mathbb{C}) \longrightarrow C_c(G, \mathbb{C}), \qquad (g, \varphi) \longmapsto g\varphi = \varphi \circ \lambda_{g^{-1}}.$$

Definition 3.1. Let G be a locally compact group. A linear map

$$I : C_c(G, \mathbb{C}) \longrightarrow \mathbb{C}$$

is called a *complex Haar integral* if the following hold for all $\varphi \in C_c(G, \mathbb{C})$ and all $g \in G$.

(i) $I(\varphi \circ \lambda_{g^{-1}}) = I(\varphi)$.

(ii) $I(\overline{\varphi}\varphi)$ is real and non-negative.

(iii) $I(\overline{\varphi}\varphi) \neq 0$ if $\varphi \neq 0$.

Proposition 3.2. *On every locally compact group G there exists a complex Haar integral I. If J is another complex Haar integral, then there exists a real number $s > 0$ such that $J = sI$.*

Proof. Let I be a Haar integral on G. For $\varphi \in C_c(G, \mathbb{C})$ we have $\varphi = \varphi_1 + \boldsymbol{i}\varphi_2$, with $\varphi_1, \varphi_2 \in C_c(G)$, and we extend I to $C_c(G, \mathbb{C})$ by putting $I(\varphi) = I(\varphi_1) + \boldsymbol{i}I(\varphi_2)$. Then I is a complex linear map satisfying (i) and (ii). Property (iii) follows from Lemma 2.27.

Suppose that J is another complex Haar integral. For $\varphi \in C_c^+(G) - \{0\}$ we have $\psi = \sqrt{\varphi} \in C_c(G) - \{0\}$, and $J(\varphi) = J(\psi\bar{\psi}) > 0$ by property (iii) of a complex Haar integral. Therefore the restriction of J to $C_c(G)$ is a Haar integral. Hence there exists $s > 0$ such that $J(\alpha) = sI(\alpha)$ holds for all $\alpha \in C_c(G)$. Since $C_c(G)$ generates $C_c(G, \mathbb{C})$ as a complex vector space, $J = sI$. $\square$

A complex Haar integral I on a locally compact group will again be denoted by

$$I(\varphi) = \int_G \varphi = \int_G \varphi(g)dg.$$

For an element a in the complex group ring $\mathbb{C}[G]$ we have then

$$\int a\varphi = \epsilon(a) \int_G \varphi,$$

where $\epsilon : \mathbb{C}[G] \longrightarrow \mathbb{C}$ is the augmentation homomorphism.

Definition 3.3. Suppose that E is a complex vector space. A *sesquilinear form* on E is a map

$$b : E \times E \longrightarrow \mathbb{C}$$

which is bi-additive and which satisfies the identities

$$b(zu, v) = \bar{z}b(u, v) \quad \text{and} \quad zb(u, v) = b(u, zv)$$

for all $u, v \in E$ and $z \in \mathbb{C}$. A sesquilinear form b is called *hermitian* if $b(u, v) = \overline{b(v, u)}$ holds for all $u, v \in E$. A hermitian form b is *positive definite* if $b(u, v) > 0$ holds for all nonzero $u \in E$.

Construction 3.4. Let G be a locally compact group. For $\varphi, \psi \in C_c(G, \mathbb{C})$ we put

$$\langle \varphi | \psi \rangle = \int_G \overline{\varphi}\psi.$$

By the properties of a complex Haar integral, $\langle - | - \rangle$ is a positive definite hermitian form on $C_c(G, \mathbb{C})$. Moreover, we have

$$\langle g\varphi | g\psi \rangle = \langle \varphi | \psi \rangle$$

for all $\varphi, \psi \in C_c(G, \mathbb{C})$. In order to carry on, we need results from operator theory.

Convex sets in Hilbert spaces

We recall that a *norm* on a complex vector space E is a map $\lVert - \rVert : E \longrightarrow \mathbb{R}_{\geq 0}$ with the properties

$$\lVert u+v \rVert \leq \lVert u \rVert + \lVert v \rVert, \qquad \lVert zu \rVert = |z| \lVert u \rVert, \qquad \lVert u \rVert = 0 \text{ if and only if } u = 0,$$

for all $u, v \in E$ and $z \in \mathbb{C}$. Hence a norm is a length function. A norm determines in particular a metric and a topology on E.

Definition 3.5. Suppose that $\langle - | - \rangle : E \times E \longrightarrow \mathbb{C}$ is a positive definite hermitian form on a complex vector space E. The associated *norm* on E is

$$\lVert u \rVert = \sqrt{\langle u | u \rangle}.$$

We note that

$$\lVert u+v \rVert^2 = \lVert u \rVert^2 + 2\,\mathrm{Re}\langle u|v\rangle + \lVert v \rVert^2$$

and that

$$\langle u|v\rangle = \tfrac{1}{4}\sum_{k=0}^{3} \boldsymbol{i}^k \lVert u + \boldsymbol{i}^k v \rVert^2.$$

On several occasions we will use the *parallelogram identity*

$$\lVert x+y \rVert^2 + \lVert x-y \rVert^2 = 2\big(\lVert x \rVert^2 + \lVert y \rVert^2\big),$$

which is a direct consequence of the definition of the norm. In order to see that $\lVert - \rVert$ satisfies the triangle inequality, we first prove the Cauchy–Schwarz inequality,

$$|\langle u|v\rangle| \leq \lVert u \rVert \lVert v \rVert.$$

For this we expand the nonnegative expression

$$\begin{aligned} \big\lVert \langle v|u\rangle u - \lVert u \rVert^2 v \big\rVert^2 &= |\langle v|u\rangle|^2 \lVert u \rVert^2 - 2\lVert u \rVert^2\,\mathrm{Re}(\langle v|u\rangle\langle u|v\rangle) + \lVert u \rVert^4 \lVert v \rVert^2 \\ &= \lVert u \rVert^4 \lVert v \rVert^2 - |\langle u|v\rangle|^2 \lVert u \rVert^2 = \lVert u \rVert^2\big(\lVert u \rVert^2 \lVert v \rVert^2 - |\langle u|v\rangle|^2\big). \end{aligned}$$

It follows from the Cauchy–Schwarz inequality that

$$\lVert u+v \rVert^2 \leq (\lVert u \rVert + \lVert v \rVert)^2,$$

hence $\lVert - \rVert$ is indeed a norm. The norm $\lVert - \rVert$ turns E into a metric space, and the Cauchy–Schwarz inequality shows that $\langle - | - \rangle$ is continuous. This follows of course also from the explicit formula for $\langle - | - \rangle$ in terms of $\lVert - \rVert^2$. The pair $(E, \langle - | - \rangle)$ is called a *pre-Hilbert* *space*. A linear subspace F of a pre-Hilbert space E is again a pre-Hilbert space if we restrict the hermitian form to F. If the pre-Hilbert space E is complete with respect to $\|-\|$, then E is called a *Hilbert space*. Hence a closed linear subspace of a Hilbert space is again a Hilbert space. For a locally compact group G, the complex vector space $C_c(G, \mathbb{C})$ is an example of a pre-Hilbert space.

Example 3.6. On the complex vector space $\mathbb{C}^m$ we put

$$\langle u|v\rangle = \sum_{k=1}^{m} \bar{u}_k v_k.$$

We call this the *standard hermitian form*. Then $(\mathbb{C}^m, \langle -|-\rangle)$ is a Hilbert space.

The previous example is universal, as the following lemma shows.

Lemma 3.7. *Let $(E, \langle -|-\rangle)$ be a Hilbert space of finite dimension m. Then there exists a linear isomorphism $f : \mathbb{C}^m \longrightarrow E$ such that $\langle f(u)|f(v)\rangle = \langle u|v\rangle$ holds for all $u, v \in \mathbb{C}^m$.*

Proof. Let $v_1, \dots, v_m$ be a basis for E. We use the Gram–Schmidt algorithm to transform this basis into an orthonormal basis $\hat{v}_1, \dots, \hat{v}_m$. Then

$$f(z_, \dots, z_m) = \sum z_1\hat{v}_1 + \cdots + z_m\hat{v}_m$$

is a linear isomorphism which preserves the hermitian forms. □

We call a nonempty subset K of a real or complex vector space E *convex* if for all $u, v \in K$ and all $s \in [0, 1]$ we have $su + (1 - s)v \in K$. Arbitrary intersections of convex sets are again convex. If K and L are convex, then $K + L$ is also convex.

Lemma 3.8. *Let E be a pre-Hilbert space and let $K \subseteq E$ be a nonempty complete subset which is convex. For every $u \in E$ there exists a unique point $p(u) \in K$ which minimizes the distance to u.*

Proof. We put $d = \inf\{\|u - v\| \mid v \in K\}$. For $v, w \in K$ we have by the parallelogram identity

$$\|(v - u) - (w - u)\|^2 + \|(v - u) + (w - u)\|^2 = 2\|v - u\|^2 + 2\|w - u\|^2,$$

whence

$$\|v - w\|^2 = 2\|v - u\|^2 + 2\|w - u\|^2 - 4\|u - \tfrac{1}{2}(v + w)\| \leq 2\|v - u\|^2 + 2\|w - u\|^2 - 4d^2.$$

Hence if $(v_n)_{n\geq 0}$ is a sequence in K with $\lim_n \|u - v_n\| = d$, then $(v_n)_{n\geq 0}$ is a Cauchy sequence. Its limit $v = \lim_n v_v \in K$ then satisfies $\|u - v\| = d$. For $v, w \in K$ with $\|u - v\| = \|u - w\| = d$ the inequality shows that $v = w$. □

The previous lemma has several important consequences for convex sets. If E is a normed vector space, we define the *closed convex hull* of a subset $X \subseteq E$ as

$$\overline{\operatorname{conv}}(X) = \bigcap \{K \subseteq E \mid K \text{ is closed, convex and } X \subseteq K\}.$$

This set is in particular closed and convex.

Let w be a nonzero vector in a pre-Hilbert space E and let s be a real number. A *half-space* $H \subseteq E$ is a set of the form

$$H = H(w, s) = \{u \in E \mid \operatorname{Re}\langle w|u\rangle \leq s\}.$$

Half-spaces are closed and convex. The next result is a special case of the Banach–Hahn–Mazur Theorem.

Proposition 3.9. *Let K be a closed convex set in a Hilbert space. Then*

$$K = \bigcap \{H \mid H \subseteq E \textit{ is a half-space containing } K\}.$$

Proof. The right-hand side is closed and convex and contains the left-hand side. It suffices therefore to prove that for every $u \in E - K$, there exists a halfspace $H \subseteq E$ containing K with $u \notin H$. We put $v = p(u)$, where p is the map defined in Lemma 3.8, and $w = u - v$. Thus $w \neq 0$. Let $s = \operatorname{Re}\langle v|w\rangle$. Then

$$0 < \operatorname{Re}\langle w|w\rangle = \operatorname{Re}\langle w|u\rangle - s,$$

hence $u \notin H(w, s)$. Suppose that $y \in E - H(w, s)$. We have to show that $y \notin K$. We consider the map

$$t \longmapsto \|(1-t)v + ty - u\|^2 = \|w\|^2 + 2t \operatorname{Re}\langle v - y|w\rangle + t^2\|y - v\|^2.$$

Its derivative at time $t = 0$ is

$$2s - 2\operatorname{Re}\langle y|w\rangle < 0.$$

Hence there exists $t \in [0, 1]$ such that $\|(1-t)v + ty - u\| < \|v - u\|$, and thus $y \notin K$. ☐

Proposition 3.10. *Let C be a compact set in a Hilbert space E. Then*

$$K = \overline{\operatorname{conv}}(C)$$

is compact.

Proof. Given $\varepsilon > 0$, we put $D = \{u \in E \mid \|u\| \leq \frac{\varepsilon}{3}\}$ and $U = \{u \in E \mid \|u\| < \varepsilon\}$. Since C is compact, there exists a finite set $A \subseteq C$ such that $C \subseteq D + A$. The convex hull L of the finite set A is the continuous image of a simplex, and hence compact. Hence there exists a finite set $B \subseteq L$ such that $L \subseteq D + B$. Since both L and D are closed and convex and since L is compact, the set $D + L$ is closed by Lemma 1.15 and convex. We have $C \subseteq D + A \subseteq D + L \subseteq D + D + B$. Since $D + L$ is closed, convex and contains C, we have $K \subseteq D + L \subseteq D + D + B \subseteq U + B$. Hence K is totally bounded. Since K is also complete, K is compact. ☐

For a subset X in a pre-Hilbert space E we put

$$X^{\perp} = \{v \in E \mid \langle x|v\rangle = 0 \text{ for all } x \in C\}.$$

Since $x^{\perp} = \ker[u \longmapsto \langle x|u\rangle]$ is closed, $X^{\perp}$ is a closed complex subspace of E.

Lemma 3.11. *Suppose that $F \subseteq E$ is a complete complex subspace in a pre-Hilbert space E. Then $E = F \oplus F^{\perp}$.*

Proof. For $u \in E$ we put $u_1 = p(u)$ and $u_2 = u - u_1$, where $p : E \longrightarrow F$ is the map defined in Lemma 3.8. We claim that $u_2 \in F^{\perp}$. For every $w \in F - \{0\}$ and every $z \in \mathbb{C}$ we have

$$\|u_2\|^2 \leq \|u_2 + zw\|^2 = \|u_2\|^2 + 2\,\mathrm{Re}(\langle u_2|w\rangle z) + |z|^2\|w\|^2.$$

Putting $z = \frac{-1}{\|w\|^2}\langle w|u_2\rangle$, we see that

$$0 \leq \frac{-1}{\|w\|^2}|\langle w|u_2\rangle|^2.$$

Thus $u_2 \in F^{\perp}$ and therefore $E = F + F^{\perp}$. Since $F \cap F^{\perp} = \{0\}$, we have a direct sum. ☐

Normed vector spaces and bounded operators

We need to recall some notions and facts from functional analysis. In what follows, we consider only complex vector spaces. Most of the results have analogues for real vector spaces, with the obvious modifications. If $(E, \|-\|_E)$ is a normed vector space, we put

$$B_r^E(v) = \{u \in E \mid \|u - v\|_E \leq r\},$$

for $r > 0$. For $r, s > 0$ we have

$$sB_r^E(0) = B_{rs}^E(0).$$

As customary in functional analysis we will call linear maps *operators*.

Lemma 3.12. *Let F be a linear subspace of a normed vector space E. Then its closure $\overline{F}$ is also a linear subspace.*

Proof. The closure of F is a subgroup of E by Proposition 1.7. For every $z \in \mathbb{C}$ we have $zF \subseteq F$, whence $z\overline{F} \subseteq \overline{F}$. This shows that $\overline{F}$ is a linear subspace. ☐

Definition 3.13. Let $(E, \|-\|_E)$ and $(F, \|-\|_F)$ be normed vector spaces. A *bounded operator*

$$T : E \longrightarrow F$$

is a linear map which maps bounded sets to bounded sets.

Lemma 3.14. *Let $(E, \|-\|_E)$ and $(F, \|-\|_F)$ be normed vector spaces and let $T : E \longrightarrow F$ be an operator. The following are equivalent.*

(i) *T is Lipschitz continuous.*

(ii) *T is continuous.*

(iii) *T is bounded.*

(iv) *$T(B^E_r(0))$ is bounded for some $r > 0$.*

Proof. It is clear that (i) implies (ii) and that (iii) implies (iv). Suppose that (ii) holds. Then there exists $\varepsilon > 0$ such that $T(B^E_\varepsilon(0)) \subseteq B^F_1(0)$. Therefore $T(B^E_r(0)) \subseteq B^F_{r/\varepsilon}(0)$ holds for all $r > 0$, and hence T is bounded. Hence (ii) implies (iii). Finally, suppose that (iv) holds. Then there exist $r, s > 0$ such that $T(B^E_r(0)) \subseteq B^F_s(0)$ holds, which implies that $T(B^E_t(0)) \subseteq B^F_{ts/r}(0)$ holds for all $t > 0$. Thus $\|T(u)\|_F \leq \frac{s}{r}\|u\|_E$ holds for all $u \in E$, and hence $\|T(u) - T(v)\|_F \leq \frac{s}{r}\|u - v\|_E$ holds for all $u, v \in E$. Thus (iv) implies (i). ☐

For a bounded operator $T : E \longrightarrow F$ we define its *operator norm*

$$\|T\| = \sup\{\|Tu\|_F \mid u \in B^E_1(0)\}.$$

It follows that

$$\|Tu\| \leq \|T\|\|u\|_E.$$

In particular, T is a $\|T\|$-Lipschitz map.

Lemma 3.15. *Suppose that E and F are normed vector spaces. Then $(\mathfrak{B}(E, F), \|-\|)$ is a normed vector space. If D is another normed vector space and if $S \in \mathfrak{B}(D, E)$ and $T \in \mathfrak{B}(E, F)$, then $\|TS\| \leq \|T\|\|S\|$.*

Proof. For $S, T \in \mathfrak{B}(E,F)$, $z \in \mathbb{C}$ and $u \in E$ we have

$$\|(S - zT)u\|_F \leq (\|S\| + |z|\|T\|)\|u\|_E.$$

It follows that $\mathfrak{B}(E,F)$ is a vector space and that the operator norm satisfies the triangle inequality. Moreover, $\|T\| = 0$ holds if and only if $T = 0$. Suppose that $z \in \mathbb{C} - \{0\}$. Then $\|zTu\|_F = |z|\|Tu\| \leq |z|\|T\|\|u\|_E$, whence $\|zT\| \leq |z|\|T\|$. The same reasoning shows that $\|T\| = \|\frac{1}{z}(zT)\| \leq \frac{1}{|z|}\|zT\|$, and thus $\|zT\| = |z|\|T\|$. Hence the operator norm is indeed a norm. For $v \in D$ we have $\|TSv\|_F \leq \|T\|\|Sv\|_E \leq \|T\|\|S\|\|v\|_D$, which proves the last claim. ☐

We recall that a complete normed vector space is called a *Banach space.*

Proposition 3.16. *Suppose that $(E, \|-\|_E)$ and $(F, \|-\|_F)$ are normed vector spaces.*

If F is a Banach space, then $(\mathfrak{B}(E,F), \|-\|)$ is a Banach space. In particular, the dual space

$$E^* = \mathfrak{B}(E, \mathbb{C})$$

of a normed vector space is always a Banach space.

If E has finite dimension, then every operator $T : E \longrightarrow F$ is bounded.

Proof. Suppose that F is complete and that $(T_n)_{n\geq 0}$ is a Cauchy sequence in $\mathfrak{B}(E,F)$. For every $u \in E$, the sequence $(T_n u)_{n \geq n}$ is a Cauchy sequence in F because

$$\|T_n u - T_m u\| \leq \|T_n - T_m\|\|u\|_E.$$

We put $Tu = \lim_n T_n u$. Since each T_n is linear, $T_n(u + zv) - T_n u - zT_n v = 0$ holds for all vectors u, v and scalars z. Therefore $T(u + zv) - Tu - zTv = 0$ holds as well, which shows that $T : E \longrightarrow F$ is linear. There exists $k > 0$ such that $\|T_m - T_k\| \leq 1$ for all $m \geq k$. Therefore $\|Tu - T_k u\|_F \leq \|u\|_E$, and thus $T - T_k$ is continuous. Hence T is also continuous, and we have proved that $\mathfrak{B}(E,F)$ is complete.

If E has finite dimension m, then E is as a topological vector space isomorphic to $\mathbb{C}^m$ by Theorem 1.44. We fix such an isomorphism of topological vector spaces $f : \mathbb{C}^m \longrightarrow E$. Let $u_1, \ldots, u_m$ denote the standard basis of $\mathbb{C}^m$. We put $v_k = T(f(u_k))$. The map $h : \mathbb{C}^m \longrightarrow F$ that maps $(z, \ldots, z_m)$ to $z_1 v_1 + \cdots + z_m v_m$ is continuous, and $h = T \circ f$. Hence T is continuous. ☐

It follows that every bijective linear map between finite dimensional normed vector spaces is Lipschitz continuous. In particular, such a vector space is always a locally compact Banach space.

Proposition 3.17. *Let $(E, \|-\|_E)$ be a normed vector space. Then there exists a Banach space $(\hat{E}, \|-\|_{\hat{E}})$ and a linear isometric injection $j : E \longrightarrow \hat{E}$ with dense image. If $(F, \|-\|_F)$ is a Banach space and if $T : E \longrightarrow F$ is a bounded operator, then there is a unique bounded operator $\hat{T} : \hat{E} \longrightarrow F$ such that $\hat{T} \circ j = T$,*

$$\begin{array}{ccc} E & \xrightarrow{T} & F \\ {\scriptstyle j}\downarrow & \nearrow_{\hat{T}} & \\ \hat{E}. & & \end{array}$$

Proof. Let $C \subseteq E^{\mathbb{N}}$ denote the complex vector space of all Cauchy sequences in E, and let $N \subseteq C$ denote the complex vector space of all sequences in E converging to 0. For a sequence $\mathbf{u} = (u_n)_{n\geq 0}$ in C, the sequence $(\|u_n\|_E)_{n\geq 0}$ is a real Cauchy sequence by the triangle inequality. We put $\|\mathbf{u}\| = \lim_n \|u_n\|_E$. Then $N = \{\mathbf{u} \in C \mid \|\mathbf{u}\| = 0\}$ and $\hat{E} = C/N$ is a normed vector space, with norm $\|\mathbf{u} + N\|_{\hat{E}} = \|\mathbf{u}\|$. For $u \in E$ we put $j(u) = (u)_{n\geq 0} + N \in \hat{E}$. Then j is an isometric linear injection. For a Cauchy sequence $\mathbf{u} = (u_n)_{n\geq 0} \in C$, we claim that $\mathbf{u} + N = \lim_n j(u_n)$. Given $\varepsilon > 0$, there exists $k \geq 0$ such that $\|u_n - u_m\| \leq \varepsilon$ for all $m, n \geq k$. Hence we have for all $n \geq k$ that $\lim_m \|u_n - u_m\| \leq \varepsilon$, which shows that $\|j(u_n) - \mathbf{u} + N\|_{\hat{E}} \leq \varepsilon$. This proves the claim, and it also shows that $j(E) \subseteq \hat{E}$ is dense.

We have proved so far that every Cauchy sequence in $j(E)$ has a limit in $\hat{E}$, and that $j(E)$ is dense in $\hat{E}$. Hence every Cauchy sequence in $\hat{E}$ has a limit, which shows that $\hat{E}$ is a Banach space.

If $T \in \mathfrak{B}(E, F)$ and if F is complete, then we may define $\hat{T}(\mathbf{u}+N) = \lim_n Tu_n$. This is a linear operator which extends T. For $\mathbf{u} + N \in \hat{E}$ we have $\lim_n \|Tu_n\|_F \leq \|T\| \lim_n \|u_n\|$, hence $\hat{T}$ is bounded. Since $j(E) \subseteq \hat{E}$ is dense, $\hat{T}$ is the unique continuous extension of T. ☐

Corollary 3.18. *If $(E, \langle -|-\rangle)$ is a pre-Hilbert space, then $(\hat{E}, \langle -|-\rangle)$ is a Hilbert space in such a way that $\langle j(u)|j(v)\rangle = \langle u|v\rangle$ holds for all $u, v \in E$.*

Proof. We use the notions introduced in the previous proof and we put

$$\langle \mathbf{u} + N | \mathbf{v} + N\rangle = \frac{1}{4}\sum_{k=0}^{3} \boldsymbol{i}^k \|\mathbf{u} + \boldsymbol{i}^k\mathbf{v} + N\|^2,$$

for $\mathbf{u}, \mathbf{v} \in C$. This map is continuous on $\hat{E} \times \hat{E}$. We have $\langle j(u)|j(v)\rangle = \langle u|v\rangle$ for all $u, v \in E$. Since $j(E) \times j(E) \subseteq \hat{E} \times \hat{E}$ is dense, $\langle -|-\rangle$ is a positive definite hermitian form on $\hat{E}$. ☐

The following results are not needed for the proof of the Peter–Weyl theorem, but they fit well in this section on bounded operators.

Lemma 3.19. *Let $T : E \longrightarrow F$ be a bounded operator between normed vector spaces E, F. Suppose that E is a Banach space, and that $r, s > 0$. If $B_r^F(0) \subseteq \overline{T(B_s^E(0))}$ holds, then $B_r^F(0) \subseteq T(B_{s(1+\varepsilon)}^E(0))$ holds for every $\varepsilon > 0$.*

Proof. We put $\delta = \frac{\varepsilon}{1+\varepsilon}$. Suppose that $w \in B_r^F(0)$. We claim that we can find elements $u_n \in \delta^{n-1} B_s^E(0)$, for $n = 1, 2, \ldots$, such that

$$\|w - T(u_1 + u_2 + \cdots + u_n)\|_F \leq r\delta^n.$$

For $n = 1$, this is clear: we choose $u_1 \in B_s^E(0)$ in such a way that $\|w - Tu_1\| \leq r\delta$. Now we proceed by induction. Given $u_1, \ldots, u_n$, we have

$$v = w - T(u_1 + u_2 + \cdots + u_n) \in \delta^n B_r^F(0) \subseteq \delta^n \overline{T(B_s^E(0))}.$$

Hence we can find $u_{n+1} \in \delta^n B_s^E(0)$ such that

$$\|w - T(u_1 + u_2 + \cdots + u_n) - Tu_{n+1}\| \leq r\delta^{n+1}.$$

Now $\|u_n\| \leq s\delta^{n-1}$. Since E is complete and since $0 < \delta < 1$, the limit $u = \sum_{n=1}^\infty u_n$ exists in E, and $T(u) = w$. Moreover,

$$\|u\| \leq s \sum_{n=1}^{\infty} \delta^{n-1} = s\tfrac{1}{1-\delta} = s(1+\varepsilon).$$

□

Theorem 3.20. *Let E be a Banach space, let F be a normed vector space and suppose that $T : E \longrightarrow F$ is a bounded operator. Then either $T(E)$ is meager or T is surjective and open.*

Proof. Suppose that $T(E)$ is not meager. Since $T(E) = \bigcup_{n\geq 1} T(B_n^E(0))$, there exists some $n \geq 1$ such that $\overline{T(B_n^E(0))}$ has nonempty interior. Hence there exists $v \in F$ and $r > 0$ such that $v + B_r(0) \subseteq \overline{T(B_n(0))}$. Thus

$$B_r(0) \subseteq \overline{T(B_n(0))} + \overline{T(B_n(0))} \subseteq \overline{T(B_n(0)) + T(B_n(0))} \subseteq \overline{T(B_{2n}^E(0))}.$$

By Lemma 3.19, $B_r(0) \subseteq T(B_{2n+1}^E(0))$. Hence $T(B_s^E(u))$ contains for every $s > 0$ an open neighborhood of Tu. Thus T is an open map, and $T(F)$ contains an identity neighborhood $B_\varepsilon^F(0)$. Since $F = \bigcup_{n\geq 1} nB_\varepsilon^F(0)$, we have $F = T(E)$. □

Corollary 3.21 (Open mapping theorem). *Suppose that E, F are Banach spaces and that $T : E \longrightarrow F$ is a bounded operator. If T is surjective, then T is open.*

Corollary 3.22 (Closed graph theorem). *Suppose that E, F are Banach spaces and that $T : E \longrightarrow F$ is an operator. Then T is continuous if and only its graph*

$$X = \{(u, Tu) \mid u \in E\} \subseteq E \times F$$

is closed.

Proof. If T is continuous, then its graph X is closed. Conversely, suppose that X is closed. Then X is a closed linear subspace of $E \times F$ and hence a Banach space with respect to the norm $\|(u, v)\| = \|u\|_E + \|v\|_F$. The projection map $\mathrm{pr} : E \times F \longrightarrow E$ is continuous and linear. Its restriction to X is continuous, bijective and, by Corollary 3.21, open. Hence the map $u \longmapsto (u, Tu)$ is continuous, and thus T is continuous. □

Corollary 3.23. *Let T be an operator on a Hilbert space E which satisfies $\langle Tu|v\rangle = \langle u|Tv\rangle$. Then T is bounded.*

Proof. Let $X \subseteq E \times E$ denote the graph of T. We show that X is closed. If $(u_n)_{n\geq 0}$ is a sequence in E_n such that $(u_n, Tu_n)_{n\geq 0}$ converges to (u, v), then we have for $w \in E$ that

$$\langle v|w\rangle = \lim\langle Tu_n|w\rangle = \lim_n\langle u_n|Tw\rangle = \langle u|Tw\rangle = \langle Tu|w\rangle.$$

It follows that $Tu = v$, and thus $(u, v) \in X$. Hence X is closed, and T is continuous by Corollary 3.22. □

Adjoint operators in Hilbert spaces

The next result is a special case of Riesz' Representation Theorem. We recall that $E^* = \mathfrak{B}(E, \mathbb{C})$.

Proposition 3.24. *Let E be a Hilbert space. The map*

$$L : E \longrightarrow E^*, \quad u \longmapsto \langle u|-\rangle$$

is a semilinear isometric bijection.

Proof. It is clear that L is semi-linear, $L(zu) = \bar{z}L(u)$. For $u \neq 0$ we have $\langle u|u\rangle > 0$, hence $L(u) \neq 0$. This shows that L is injective, and we claim that L is also surjective. Suppose that $\xi \in E^*$ is a nonzero bounded linear form, with kernel $F \subseteq E$. Since $\mathbb{C} \cong E/F$, the kernel F is a closed hyperplane. By Lemma 3.11, $E = F \oplus F^\perp$, and $F^\perp$ has dimension 1. Let $v \in F^\perp$ be a nonzero vector. Then $a = \xi(v) \neq 0$ and the vector $u = \frac{\bar{a}}{\|v\|^2}v$ has the property that $\xi(u) = \frac{|a|^2}{\|v\|^2} = \langle u|u\rangle = L(u)(u)$. Since $\ker(\xi) = F = \ker(L(u))$, we conclude that $\xi = L(u)$. Hence L is surjective.

For $v \in B_1^E(0)$ and $u \in E$ we have $|L(u)v| = |\langle u|v\rangle| \leq \|u\|$, hence $\|L(u)\| \leq \|u\|$. If $u \neq 0$, then $\langle u|\frac{1}{\|u\|}u\rangle = \|u\|$, hence $\|L(u)\| \geq \|u\|$. Thus $\|L(u)\| = \|u\|$ holds for all $u \in E$. □

Proposition 3.25. *Let E be a Hilbert space and suppose that $b : E \times E \longmapsto \mathbb{C}$ is a sesquilinear form. The following are equivalent.*

(i) *b is continuous.*

(ii) *There exists a real number $r \geq 0$ such that $|b(u,v)| \leq r\|u\|\|v\|$ holds for all $u, v \in E$*

(iii) *There exists a bounded operator $T \in \mathfrak{B}(E)$ such that $b(u,v) = \langle Tu|v\rangle$ holds for all $u, v \in E$.*

If these equivalent conditions are satisfied, then $\|T\| \leq r$ and T is uniquely determined.

Proof. Suppose that b is continuous. Then there exists $\varepsilon > 0$ such that $|b(u,v)| \leq 1$ holds for all $u, v \in B_\varepsilon^E(0)$. Then $b(u,v)| \leq \frac{1}{\varepsilon}\|u\|\|v\|$ holds for all $u, v \in E$, so (ii) implies (ii).

Suppose that (ii) holds. Then for every $u \in E$, the map $b(u,-)$ is linear and bounded. We define a semilinear map $B : E \longrightarrow E^*$ by $B(u) = b(u,-)$. By Proposition 3.24 there is a unique operator $T : E \longrightarrow E$ with $L \circ T = B$. Thus $\langle Tu|v\rangle = b(u,v)$ holds for all u, v. For $v \in B_1^E(0)$ we have

$$\|Tv\|^2 = \langle Tv|Tv\rangle = b(v, Tv) \leq r\|Tv\|$$

and thus $\|Tv\| \leq r$. Hence T is bounded, with $\|T\| \leq r$.

It is clear that (iii) implies (i). ☐

Corollary 3.26. *If E is a Hilbert space, then there exists for every $T \in \mathfrak{B}(E)$ a unique $T^* \in \mathfrak{B}(E)$ such that*

$$\langle Tu|v\rangle = \langle u|T^*v\rangle$$

holds for all $u, v \in E$. The operator T^* is called the adjoint *of T. We have*

$$T^{**} = T \quad \text{and} \quad \|T\| = \|T^*\|.$$

Proof. We put $b(u,v) = \langle u|Tv\rangle$. By Proposition 3.25 there exists a unique $T^* \in \mathfrak{B}(E)$ such that $\langle T^*v|u\rangle = \langle v|Tu\rangle$ holds for all $u, v \in E$. Thus $\langle u|T^*v\rangle = \langle Tu|v\rangle$ and $T^{**} = T$. Since $|\langle u|Tv\rangle| \leq \|T\|\|u\|\|v\|$, we have $\|T^*\| \leq \|T\|$. But this implies that $\|T\| = \|T^{**}\| \leq \|T^*\|$, and therefore $\|T\| = \|T^*\|$. ☐

A linear operator $T : E \longrightarrow E$ on a Hilbert space E is called *self-adjoint* if

$$\langle Tu|v\rangle = \langle u|Tv\rangle$$

holds for all $u, v \in E$. It follows that $\langle Tu|u\rangle \in \mathbb{R}$, and T is bounded by Corollary 3.23. Conversely, a bounded operation $T \in \mathfrak{B}(E)$ is self-adjoint if and only if

$$T = T^*.$$

An example of a bounded self-adjoint operator is the projection operator $p : E = F \oplus F^\perp \longrightarrow F$ associated to a closed linear subspace $F \subseteq E$ as in Lemma 3.11.

Lemma 3.27. *Let T be a self-adjoint bounded operator on a Hilbert space E. Then*

$$\lVert T \rVert = \sup\{|\langle Tu|u\rangle| \mid u \in B_1^E(0)\}.$$

Proof. We put $\nu(T) = \sup\{|\langle Tu|u\rangle| \mid u \in B_1^E(0)\}$. Thus

$$|\langle Tu|u\rangle| \leq \nu(T)\lVert u \rVert^2 \tag{1}$$

holds for all $u \in E$. By the Cauchy–Schwarz inequality we have $|\langle Tu|u\rangle| \leq \lVert T \rVert \lVert u \rVert^2$, whence

$$\nu(T) \leq \lVert T \rVert.$$

In order to show that $\lVert T \rVert \leq \nu(T)$ we use the identity

$$\langle T(x+y)|x+y\rangle - \langle T(x-y)|x-y\rangle = 4\,\mathrm{Re}\langle Tx|y\rangle$$

For $s > 0$ we substitute $x = su$ and $y = \frac{1}{s}Tu$. By (1) and by the parallelogram identity we obtain

$$4\lVert Tu \rVert^2 \leq \nu(T)\big(\lVert su + \tfrac{1}{s}Tu \rVert^2 + \lVert su - \tfrac{1}{s}Tu \rVert^2\big) = 2\nu(T)\big(s^2\lVert u \rVert^2 + \frac{1}{s^2}\lVert Tu \rVert^2\big).$$

For $Tu \neq 0$ we may put $s = \sqrt{\frac{\lVert Tu \rVert}{\lVert u \rVert}}$ and we obtain $4\lVert Tu \rVert^2 \leq 2\nu(T)2\lVert Tu \rVert \lVert u \rVert$ and thus

$$\lVert Tu \rVert \leq \nu(T)\lVert u \rVert.$$

This inequality is also valid if $Tu = 0$. Hence

$$\lVert T \rVert \leq \nu(T).$$ □

Definition 3.28. Let E, F be normed vector spaces and assume that F is complete. An operator $T : E \longrightarrow F$ is called *compact* if for every bounded set $B \subseteq E$, the set $T(B)$ has compact closure. It follows that every compact operator is bounded. An example of a compact operator is the projection operator $p : E = F \oplus F^\perp \longrightarrow F$ associated to a finite dimensional linear subspace $F \subseteq E$.

The following is the Spectral Theorem for compact self-adjoint operators. We denote the set of all complex eigenvalues of an operator T by $\sigma_P(T)$. This set is called the *point spectrum* of T.

Theorem 3.29. *Let T be a compact self-adjoint operator on a Hilbert space E. For an eigenvalue $z \in \sigma_P(T)$, we let $E_z = \ker(T - z\mathrm{id}_E)$ denote the corresponding eigenspace. Then we have the following.*

(i) *$\sigma_P(T) \subseteq \mathbb{R}$ and for distinct eigenvalues s, t, the eigenspaces E_s, E_t are orthogonal.*

(ii) *$\|T\| \in \sigma_P(T)$ or $-\|T\| \in \sigma_P(T)$.*

(iii) *We have $\overline{\sum_{t\in\sigma_P(T)} E_t} = E$.*

(iv) *The set $\sigma_P(T)$ is finite or countable. If $\sigma_P(T)$ has an accumulation point z, then $z = 0$.*

(v) *For every $0 \neq t \in \sigma_P(T)$, the eigenspace E_t has finite dimension.*

Proof. We may assume that $T \neq 0$, since otherwise all claims are trivially true.

If $u \in E$ is an eigenvector for the eigenvalue $z \in \mathbb{C}$, then $\langle Tu|u\rangle = \bar{z}\langle u|u\rangle$ is real, which shows that $z \in \mathbb{R}$. If s, t are different eigenvalues and if $u \in E_t$ and $v \in E_s$, then $t\langle u|v\rangle = \langle Tu|v\rangle = \langle u|Tv\rangle = s\langle u|v\rangle$, which shows that $\langle u|v\rangle = 0$. Hence (i) is true.

For (ii) we use Lemma 3.27. Let $u_n \in B_1^E(0)$ be sequence of vectors with

$$\lim_n |\langle Tu_n|u_n\rangle| = \|T\| > 0.$$

Passing to a subsequence, we may assume in that both $t = \lim_n\langle Tu_n|u_n\rangle$ and $v = \lim_n Tu_n$ exist. We note that $\langle Tu_n|tu_n\rangle = t\langle Tu_n|u_n\rangle$ is real. Thus

$$0 \leq \|Tu_n - tu_n\|^2 = \|Tu_n\|^2 - 2\langle Tu_n|tu_n\rangle + t^2\|u_n\|^2 \leq 2t^2 - 2t\langle Tu_n|u_n\rangle.$$

The limit over the right-hand side is 0, hence $v = \lim_n Tu_n = \lim_n tu_n$. Since $|t| = \|T\| \neq 0$, we conclude that $\lim_n u_n = u = \frac{1}{t}v$ and thus $v = Tu = tu$. Since $t = \langle Tu|u\rangle \neq 0$, we have $u \neq 0$. So $E_t \neq 0$, and $|t| = \|T\|$. This proves (ii)

For (iii) we first note the following. If $D \subseteq E$ is a T-invariant subspace, then $D^\perp$ is also T-invariant. For if $x \in D$ and $y \in D^\perp$, then $0 = \langle Tx|y\rangle = \langle x|Ty\rangle$. We put $F = \sum_{t\in\sigma_P(T)} E_t$. Then $T(F) \subseteq F$ and hence $T(\overline{F}) \subseteq \overline{F}$. It follows that $T(\overline{F}^\perp) \subseteq \overline{F}^\perp$. Then the restriction-corestriction of T to the Hilbert space $\overline{F}^\perp$ is again a compact self-adjoint operator, hence either $\overline{F}^\perp = 0$ or $\overline{F}^\perp$ contains a nontrivial eigenspace of T by (ii). Therefore $\overline{F}^\perp = 0$. By Lemma 3.11 and Lemma 3.12, $E = \overline{F} \oplus \overline{F}^\perp$, and thus (iii) is proved.

Now we prove (iv) and (v). Given $\varepsilon > 0$, let $S = \{t \in \sigma_P(T) \mid |t| \geq \varepsilon\}$. We claim that $D = \sum_{t\in S} E_t$ has finite dimension. If D would have infinite dimension, we could find in D a sequence of pairwise orthogonal eigenvectors u_n of length $\|u_n\| = 1$, with eigenvalues $t_n \in S$. But then we would have $\|Tu_n - Tu_m\|^2 \geq 2\varepsilon$ for $m \neq n$, contradicting the compactness of T. It follows that S is finite and that E_t has finite dimension for all $t \in S$. Therefore $\sigma_P(T)$ is finite or countable, and 0 is the only possible accumulation point of $\sigma_P(T)$. This proves (iv) and (v). □

Hilbert modules for locally compact groups

Suppose that E is a pre-Hilbert space. An operator $T : E \longrightarrow E$ is called *unitary* if T is bijective and if

$$\langle Tu|Tv\rangle = \langle u|v\rangle$$

holds for all $u, v \in E$. A unitary operator is bounded, and so is its inverse.

Definition 3.30. Let G be a topological group. A *Hilbert G-module* E consists of a Hilbert space E and a continuous linear action

$$G \times E \longrightarrow E,$$

such that every $g \in G$ acts as a unitary operator on E.

Example 3.31. Let $E = \mathbb{C}^m$ with the standard hermitian form as in Example 3.6. A matrix $g \in \mathbb{C}^{m\times m}$ is unitary if and only if $g^*g = \mathbb{1}$ holds, where g^* denotes the conjugate-transpose matrix, $(g^*)_{i,j} = \overline{g_{j,i}}$. The unitary group

$$\mathrm{U}(m) = \{g \in \mathbb{C}^{m\times m} \mid g^*g = \mathbb{1}\}.$$

is a compact matrix group. If E is a finite dimensional Hilbert space, then there exists by Lemma 3.7 an isomorphism $f : \mathbb{C}^m \longrightarrow E$ which preserves the hermitian forms. If E is a Hilbert G-module, then $\mathbb{C}^m$ becomes a Hilbert G-module $G \times \mathbb{C}^m \longrightarrow \mathbb{C}^m$ via $gu = (f^{-1} \circ g \circ f)(u)$. The associated homomorphism $G \longrightarrow \mathrm{U}(m)$ is continuous.

For a locally compact group G with complex Haar integral $\int_G$ we denote by $L^2(G)$ the completion of the pre-Hilbert space $C_c(G, \mathbb{C})$. We view $C_c(G, \mathbb{C})$ as a dense subspace of $L^2(G)$. Since every $g \in G$ acts as a unitary operator on $C_c(G, \mathbb{C})$ via $g\varphi = \varphi \circ \lambda_{g^{-1}}$, it acts also as a unitary operator on the completion $L^2(G)$ by Lemma 3.17. In particular, G acts linearly on $L^2(G)$. If $\int_G$ is replaced by another complex Haar integral, then the norm and the hermitian form on $L^2(G)$ are scaled by a positive real number. This does not change the topology and geometry of $L^2(G)$.

Lemma 3.32. *The action of G on $L^2(G)$ is faithful.*

Proof. Let $g \in G - \{e\}$, and let U be an identity neighborhood with $UU^{-1} \subseteq G - \{g\}$. Let $\varphi \in C_c(G) - \{0\}$ be a function with $\mathrm{supp}(\varphi) \subseteq U$. Then $\mathrm{supp}(g\varphi) = g\,\mathrm{supp}(\varphi)$. Since $gU \cap U = \varnothing$, we have $g\varphi \neq \varphi$. □

Proposition 3.33. *Let G be a locally compact group. Then the action*

$$G \times L^2(G) \longrightarrow L^2(G)$$

is continuous and thus $L^2(G)$ is a faithful Hilbert module for G.

We first prove a lemma. An action satisfying the assumptions of the following lemma is sometimes called *strongly continuous*.

Lemma 3.34. *Suppose that G is a topological group acting as an abstract group linearly on a Hilbert space E. Assume that each $g \in G$ acts as a unitary operator on E, and that there is a dense subset $X \subseteq E$ with the property that for each $x \in X$, the there is an identity neighborhood $V \subseteq G$ such that the map $V \longrightarrow E$, $a \longmapsto ax$ is continuous. Then the action $G \times E \longrightarrow E$ is continuous and thus E is a Hilbert G-module.*

Proof. Suppose that $\varepsilon > 0$ and that $g \in G$ and $u \in E$. We choose $x \in X$ with $\|x - u\| \leq \varepsilon/4$, and we choose an identity neighborhood $V \subseteq G$ such that $\|x - ax\| \leq \varepsilon/4$ holds for all $a \in V$. For $h \in gV$ and $w \in B^E_{\varepsilon/4}(u)$ we have then

$$\begin{aligned} \|gu - hw\| &\leq \|gu - gx\| + \|gx - hx\| + \|hx - hu\| + \|hu - hw\| \\ &= \|u - x\| + \|x - g^{-1}hx\| + \|x - u\| + \|u - w\| \leq \varepsilon. \end{aligned}$$ ☐

Proof of Proposition 3.33. We put $X = C_c(G, \mathbb{C}) \subseteq L^2(G)$ and we apply Lemma 3.34. We choose an open identity neighborhood $U \subseteq G$ with compact closure. By Lemma 2.16 there exists a continuous function $\eta : G \longrightarrow [0, 1]$ with compact support and with $\eta(\overline{U}C) \subseteq \{1\}$. Given $\varepsilon > 0$ and $\varphi \in C_c(G, \mathbb{C})$, there exists by Lemma 2.17 a symmetric open identity neighborhood $V \subseteq U$ such that $|\varphi(x) - \varphi(a^{-1}x)|\|\eta\| \leq \varepsilon/4$ holds for all $a \in V$ and all $x \in G$. We claim that

$$|\varphi(x) - \varphi(a^{-1}x)|\|\eta\| \leq (\varepsilon/4)\eta(x) \tag{1}$$

holds for all $a \in V$ and all $x \in X$. This is certainly true if $x \in UC$, because then $\eta(x) = 1$. If $x \notin UC$, then $x \notin C$ and $a^{-1}x \notin C$, so the left-hand side is 0, and the inequality is also valid. We integrate the squares of both sides of (1) over x. We obtain

$$\|\varphi - a\varphi\|^2 \|\eta\|^2 \leq (\varepsilon/4)^2 \|\eta\|^2.$$

Therefore $\|\varphi - a\varphi\| \leq \varepsilon/4$. By Lemma 3.34, E is a Hilbert G-module. ☐

Now we prove an important results which use an averaging process over a compact group. We call a complex Haar integral on a compact group G *normalized* if

$$\int_G dg = 1.$$

Theorem 3.35. *Let G be a compact group with normalized complex Haar integral $\int_G$ and let E be a Hilbert module for G. Let $T \in \mathfrak{B}(E)$ be a bounded operator. Then there exists a unique bounded operator $\widetilde{T} \in \mathfrak{B}(E)$ such that*

$$\langle \widetilde{T}u | v \rangle = \int_G \langle Tg^{-1}u | g^{-1}v \rangle dg$$

holds. The operator $\widetilde{T}$ *commutes with all elements of* G*, and* $\|\widetilde{T}\| \leq \|T\|$*. If* T *is self-adjoint, the* $\widetilde{T}$ *is also self-adjoint. If* T *is compact, then* $\widetilde{T}$ *is compact.*

Proof. Given $u, v \in E$, the map $g \longmapsto \langle Tg^{-1}u|g^{-1}v\rangle$ is continuous and therefore contained in $C_c(G, \mathbb{C})$, since G is compact. Also, $b(u,v) = \int_G \langle Tg^{-1}u|g^{-1}v\rangle dg$ is a sesquilinear form on E. We have $|\langle Tg^{-1}u|g^{-1}v\rangle| \leq \|T\|\|g^{-1}u\|\|g^{-1}v\| = \|T\|\|u\|\|v\|$, whence

$$|b(u,v)| \leq \int_G \|T\|\|u\|\|v\| dg = \|T\|\|u\|\|v\|.$$

By Proposition 3.25, the sesquilinear form b is continuous and the existence and uniqueness of $\widetilde{T}$ follows. The left invariance of the complex Haar integral shows that $\langle h\widetilde{T}h^{-1}u|v\rangle = \langle \widetilde{T}h^{-1}u|h^{-1}v\rangle = \langle \widetilde{T}u|v\rangle$ holds for all $u, v \in E$ and $h \in G$, whence $h\widetilde{T}h^{-1} = \widetilde{T}$.

If T is self-adjoint, then $\langle T - |-\rangle$ is a hermitian form, and then $\langle \widetilde{T} - |-\rangle$ is also hermitian. Thus $\widetilde{T}$ is self-adjoint.

Suppose that T is compact. Then $A = \overline{T(B_1^E(0))}$ is compact, and so is the set

$$GA = \{ga \mid g \in G \text{ and } a \in A\} \subseteq E.$$

The closed convex hull $K = \overline{\mathrm{conv}}(GA)$ is compact by Proposition 3.10. Suppose that $H(w,s) \subseteq E$ is a half-space containing K. For $u \in B_1^E(0)$ and $g \in G$ we have then $gTg^{-1}u \in GA$, whence

$$\mathrm{Re}\langle \widetilde{T}u|w\rangle = \int_G \mathrm{Re}\langle Tg^{-1}u|g^{-1}w\rangle dg = \int_G \mathrm{Re}\langle gTg^{-1}u|w\rangle dg \leq \int_G s dg = s.$$

By Proposition 3.9, $\widetilde{T}(B_1^E(0)) \subseteq \overline{\mathrm{conv}}(GA)$, and therefore $\widetilde{T}$ is compact. □

A nonzero Hilbert G-module is called *irreducible* if it contains no proper nontrivial Hilbert G-submodule. It follows that every nonzero finite dimensional Hilbert G-module contains a nonzero irreducible Hilbert G-module.

Lemma 3.36. *Let* G *be a compact group with normalized complex Haar integral* $\int_G$*. Suppose that* E *is a nonzero Hilbert* G*-module. Then there exists a finite dimensional irreducible Hilbert* G*-module* $F \subseteq E$*.*

Proof. Let $w \in W$ be a nonzero vector. The operator $T = \langle w|-\rangle w$ is continuous and self-adjoint. The image of T is the 1-dimensional space $\mathbb{C}w$, and thus T is compact. Therefore $\widetilde{T}$ is compact and self-adjoint by Theorem 3.35. We claim that $\langle \widetilde{T}w|w\rangle \neq 0$. The map $\varphi : g \longmapsto \langle Tg^{-1}w|g^{-1}w\rangle = |\langle w|g^{-1}w\rangle|^2$ is non-negative and not constant. Hence its integral $\langle w|\widetilde{T}w\rangle$ is positive. In particular, $\widetilde{T} \neq 0$. By Theorem 3.29, $\widetilde{T}$ has a finite dimensional eigenspace E_t. Since $\widetilde{T}$ is G-invariant, its eigenspace E_t is also G-invariant. In the finite dimensional vector space E_t we choose a nonzero G-invariant subspace F of minimal dimension. Then F is an irreducible Hilbert G-module. □

Corollary 3.37. *Let G be a compact group. Then every irreducible Hilbert G-module E has finite dimension.*

The next result is the main theorem for Hilbert G-modules for compact groups.

Theorem 3.38. *Let G be a compact group with normalized complex Haar integral $\int_G$. Suppose that E is a Hilbert module for G. Then there exists a family $(F_i)_{i\in I}$ of finite dimensional pairwise orthogonal irreducible Hilbert G-modules F_i such that*

$$E = \overline{\sum_{i\in I} F_i}.$$

Proof. We may assume that $E \neq 0$. Let $\mathcal{F}$ denote the set of all nonzero finite dimensional irreducible Hilbert G-modules contained in E. We fix a set K whose cardinality is strictly bigger than the cardinality of $\mathcal{F}$. Let $\mathcal{P}$ denote the collection of all maps $F : I \longrightarrow \mathcal{F}$ which are defined on subsets I of K, and which have the property that $F_i \perp F_j$ holds for $i \neq j$. Such a map is injective and therefore $I \subsetneq K$. We define a partial order $\leq$ on $\mathcal{P}$ by putting $F \leq F'$ if F' extends F. Then $(\mathcal{P}, \leq)$ is inductive and we can choose a maximal element F in $\mathcal{P}$. We put $X = \overline{\sum_{i\in I} F_i}$ and we claim that $X = E$. Otherwise, $X^\perp$ would be a nontrivial Hilbert G-module. By Lemma 3.36 there exists then a nonzero finite dimensional irreducible Hilbert G-module $D \subseteq X^\perp$. We may choose an element $k \in K - I$ because the cardinality of K is big enough, and we may extend F to $I \cup \{k\}$ by putting $F_k = D$. Then we have constructed a larger element than F, a contradiction. Thus $X = E$, and the claim is proved. □

Theorem 3.39 (Peter–Weyl Theorem). *Let G be a compact group, and suppose that $g \in G - \{e\}$. Then there exists a finite dimensional irreducible Hilbert G-module E for G such that g acts nontrivially on E.*

Proof. We decompose the Hilbert G-module $L^2(G)$ as in Theorem 3.38 as $L^2(G) = \overline{\sum_{i\in I} F_i}$. Since G acts faithfully on $L^2(G)$, there exists an index i such that g acts nontrivially on F_i. □

Theorem 3.40. *Let G be a compact group. Then there exists an index set I, a family $(m_i)_{i\in I}$ of natural numbers $m_i \geq 1$ and an injective closed morphism of topological groups*

$$G \longrightarrow \prod_{i\in I} \mathrm{U}(m_i).$$

Proof. We choose a faithful Hilbert G-module E for G, for example $E = L^2(G)$. Let $(F_i)_{i\in I}$ be a family of irreducible Hilbert G-modules as in Theorem 3.38. For each $i \in I$ we fix a Hilbert space isomorphism $F_i \cong \mathbb{C}^{m_i}$. In this way we obtain a morphism $\rho : G \longrightarrow \prod_{i\in I} \mathrm{U}(m_i)$. Since E is faithful, ρ has trivial kernel. Since G is compact, ρ is closed. □

In short, every compact group is isomorphic as a topological group to a closed subgroup of a (possibly infinite) product of unitary matrix groups.

Theorem 3.41. *Let G be a compact group. For every identity neighborhood $U \subseteq G$, there exists a closed normal subgroup $N \trianglelefteq G$ with $N \subseteq U$ such that G/N is isomorphic as a topological group to a closed subgroup of $\mathrm{U}(m)$, for some m.*

Proof. We choose an injective morphism

$$\rho : G \longrightarrow \prod_{i \in I} \mathrm{U}(m_i) = H.$$

as in Theorem 3.40. There exists a basic open set $W = \prod_{i \in I} V_i \subseteq \prod_{i \in I} \mathrm{U}(m_i)$ with $\rho^{-1}(W) \subseteq U$. Thus there is a finite set $I_0 \subseteq I$ such that $V_i = \mathrm{U}(m_i)$ holds for all $i \in I - I_0$. Let $M = \prod_{i \in I} M_i$ with $M_i = \{\mathbb{1}\}$ for all $i \in I - I_0$, and $M_i = \mathrm{U}(m_i)$ for $i \in I_0$. Then M is a closed normal subgroup of H, and the quotient H/M is isomorphic to $\prod_{i \in I_0} \mathrm{U}(m_i)$. We put $N = \rho^{-1}(M)$. Then the induced morphism $G/N \longrightarrow H/M$ is an injective morphism, and $N \subseteq U$. □

Corollary 3.42. *Let G be a compact group. If G has no small subgroups, then there exists $m \geq 1$ and an injective morphism*

$$G \longrightarrow \mathrm{U}(m).$$

Lemma 3.43. *Let Γ be an abelian group group and let E be a nonzero finite dimensional complex vector space. If Γ acts linearly and irreducibly on E, then $\dim(E) = 1$.*

Proof. Suppose that $\gamma \in \Gamma$. If $F \subseteq E$ is an eigenspace of γ, then F is Γ-invariant, because Γ is commutative. Hence $F = E$. It follows that every $\gamma \in \Gamma$ acts as a multiple of the identity. Hence every subspace of E is Γ-invariant, and thus $\dim(E) = 1$. □

Corollary 3.44. *Let G be a compact abelian group. Then there exists a set I and a closed injective morphism*

$$G \longrightarrow \prod_I \mathrm{U}(1).$$

For a locally compact group G, we let $\widehat{G}$ denote a set of representatives for the class of all irreducible Hilbert G-modules. This is called the *unitary dual* of G.

Proposition 3.45. *Suppose that G is a compact group. Then for every $D \in \widehat{G}$, there exists a submodule $F \subseteq C_c(G, \mathbb{C}) \subseteq L^2(G)$ with $F \cong D$.*

Proof. We may assume that $D \cong \mathbb{C}^m$ as a Hilbert space. Thus we are given a continuous homomorphism $f : G \longrightarrow \mathrm{U}(m) \subseteq \mathbb{C}^{m\times m}$. For $k = 1, \dots, m$ we put

$$\varphi_k(x) = \overline{f_{k,1}(x)},$$

and we define a linear map $h : \mathbb{C}^m \longrightarrow C_c(G, \mathbb{C})$ via $h(z_1v_1 + \cdots + z_mv_m) = z_1\varphi_1 + \cdots + z{+}m\varphi_m$, where $v_1, \dots, v_m$ is the standard basis for $\mathbb{C}^m$. We claim that h is G-equivariant. For $g \in G$ we have

$$gv_k = \sum_{j=1}^{m} f_{j,k}(g)v_j$$

and

$$g\varphi_k(x) = \overline{f_{k,1}(g^{-1}x)} = \sum_{j=1}^{m} \overline{f_{k,j}(g^{-1})}\,\overline{f_{j,1}(x)} = \sum_{j=1}^{m} f_{j,k}(g)\varphi_j(x),$$

which shows that h is equivariant. By Proposition 3.25 there exists a self-adjoint operator $T : \mathbb{C}^m \longrightarrow \mathbb{C}^m$ such that

$$\langle Tu|v\rangle = \langle h(u)|h(v)\rangle$$

holds for all $u, v \in \mathbb{C}^m$. Since the hermitian form on the right is G-invariant, T commutes with all elements of G. Since T is a compact operator and since every eigenspace of T is G-invariant, $T = t\mathbb{1}$ for some $t \geq 0$. Since $h \neq 0$, we conclude that $t > 0$. This implies that h is injective and therefore $\frac{1}{t}h$ is an isomorphism of Hilbert G-modules. ☐

Bibliography

[1] S. Banach, Théorème sur les ensembles de première catégorie, Fundamenta Mathematicae **16** (1930), 395–398.

[2] Y. Cornulier and P. de la Harpe, *Metric geometry of locally compact groups*, EMS Tracts in Mathematics, 25, European Mathematical Society (EMS), Zürich, 2016. MR3561300

[3] A. Deitmar and S. Echterhoff, *Principles of harmonic analysis*, second edition, Universitext, Springer, Cham, 2014. MR3289059

[4] J. Dugundji, *Topology*, Allyn and Bacon, Inc., Boston, MA, 1978. MR0478089

[5] E. Hewitt and K. A. Ross, *Abstract harmonic analysis. Vol. I*, second edition, Grundlehren der Mathematischen Wissenschaften, 115, Springer-Verlag, Berlin, 1979. MR0551496

[6] K. H. Hofmann and S. A. Morris, *The structure of compact groups*, third edition, revised and augmented., De Gruyter Studies in Mathematics, 25, De Gruyter, Berlin, 2013. MR3114697

[7] K. H. Hofmann and S. A. Morris, Open mapping theorems for topological groups, preprint 2009.

[8] I. Kaplansky, *Lie algebras and locally compact groups*, reprint of the 1974 edition, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1995. MR1324106

[9] C. Kuratowski, *Topologie. I et II*, reprint of the fourth (Part I) and third (Part II) editions, Éditions Jacques Gabay, Sceaux, 1992. MR1296876

[10] J. C. Oxtoby, *Measure and category*, second edition, Graduate Texts in Mathematics, 2, Springer-Verlag, New York, 1980. MR0584443

[11] B. J. Pettis, On continuity and openness of homomorphisms in topological groups, Ann. of Math. (2) **52** (1950), 293–308. MR0038358

[12] M. Stroppel, *Locally compact groups*, EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2006. MR2226087

[13] T. Tao, *Hilbert's fifth problem and related topics*, Graduate Studies in Mathematics, 153, American Mathematical Society, Providence, RI, 2014. MR3237440