Lectures on
Groups of Transformations

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Contents

1 **1**
0 . . . 1
1 Proper groups of transformations . . . 2
2 Some properties of proper transformation... . . . 3
3 A characterisation of proper transformation groups . . . 5
4 Existence of invariant metrics . . . 6

2 **11**
1 Slices . . . 11
2 General Lemmas . . . 13
3 Lie groups acting with compact isotropy groups . . . 14
4 Proper differentiable action . . . 15
5 The discrete case . . . 20
6 . . . 21

3 **25**
1 Finite presentations for discrete proper groups.... . . . 25
2 Finite presentations for groups of automorphisms.... . . . 31
3 Groups generated by reflexions . . . 37

4 **43**
1 Criterion for proper action for groups of isometries . . . 43
2 The rigidity of proper actions with compact quotients . . . 46
3 Discrete subgroup of Lie group. Witt's Theorem . . . 49

5 **55**
1 55
2 60

6 **63**
1 63
2 64
3 65
4 68
5 70
6 72
7 74

Chapter 1

This chapter collects some basic facts about proper actions of topolog- 1
ical groups on topological spaces; the existence of invariant metrics is discussed in. §4 (Bourbaki [1], Palais [1]).

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Let G ba a topological group, acting continuously on a topological space X. We shall always suppose that the action is on the left, and if $m : G \times X \to X$ defines the action, we shall write, for $s \in G$ and $x \in X, m(s, x) = sx$.

Notation. For $A, B \subset X$, we set

$$G(A/B) = \left\{ s \in G \middle| sB \cap A \neq \phi \right\}.$$

Clearly, we have, for any $A, B, C \subset X$,
$G(A|B) = G(B|A)^{-1}, G(A \cup B|C) = G(A|C) \cup G(B|C), G(\{A \cap B\}|C) \subset G(A|C) \cap G(B|C)$ and for any $s, t, \in G$,

$$G(sA|tB) = sG(A|B)t^{-1}.$$

We shall denote the orbit of $x \in X$ (i. e. the set $\{sx|s \in G\}$) by Gx, and the space of all orbits by G/X. We shall denote by $G(x)$ the isotropy group at $x \in X$; thus $G(x) = G(\{x\}|\{x\})$.

In what follows, we shall suppose that G is locally compact, and that X is a Hausdorff space.

1 Proper groups of transformations

2 **Definition.** *A locally compact transformation group G of a Hausdorff topological space X is proper if the following condition is satisfied.* (P) *For any $x, y \in X$, there exist neighbourhoods U of x and V of y such that $G(U|V)$ is relatively compact.*

Clearly (P) implies
(P_1) For any $x \in X$, there exists a neighbourhood U of x such that $G(U|U)$ is relatively compact.

Although (P_1) implies (P) in many cases, it is not equivalent to (P), as the following example shows.

Example. Consider the action of $\mathbb{Z}$ (with the discrete topology) on $\mathbb{R}^2 - \{0\}$ defined by

$$n(x, y) = (2^n x, 2^{-n} y), (x, y) \in \mathbb{R}^2 - \{0\}, n \in \mathbb{Z}.$$

Clearly (P_1) is satisfied, but (P) fails to hold, for instance for the pair of points $(1, 0)$ and $(0, 1)$.

Also, $\{P_1\}$ implies the condition

(P_2) Let $\{s_n\}$ be any sequence in G, and suppose that for some $x \in X, \{s_n\ x\}$ converges in X, then there exists a compact set in G which contains all the s_n.

Again, (P_2) implies (P) in many cases.

Remark 1. Let G act on two spaces X and Y, and let $f : X \to Y$ be a continuous mapping which commutes with the action of G, i. e. we have $f(sx) = sf(x)$ for every $x \in X$ and $s \in G$. Then it is clear that if
3 G acts properly on Y, it acts properly on X. This applies in particular to the natural action of G on a subspace X of Y which is stable under the action of G (i. e. for which $Gx \subset X$ for all $x \in X$).

Remark 2. It is easy to see that (P_1) is equivalent to the condition: every point of X has a G-stable open neighbourhood, on which the action of G is proper. Thus (P) is not a local property. On the other hand, it is easy to see that (P_1) implies (P) if the orbit space $G \backslash X$ is Hausdorff.

2 Some properties of proper transformation groups

In this article, it is assumed that G is a proper transformation group of the space X.

(i) If $A, B \subset X$ are relatively compact (resp. Compact),$G(A|B)$ is relatively compact (resp. compact). (Note that $G(A|B)$ is closed whenever A is closed and B is compact.) The proof is immediate.

In particular, $G(x) = G(\{x\}|\{x\})$ is compact.

(ii) The orbit space $G|X$ is a Hausdorff space.

Proof. Since the equivalence relation defined on X by G is open, we have only to check that the graph

$$\Gamma = \{(x, y) \in X \times X | x \in Gy\}$$

of the relation is closed in $X \times X$. Thus let $(a, b) \in \overline{\Gamma}$. Then the family $\{G(U|V)|U$ a neighbourhood of a, V a neighbourhood of $b\}$ generates a filter on G. Since G acts properly, this filter contains a compact set. Hence there exists a $t \in G$ such that $t \in \overline{G(U|V)}$ for 4
all the U, V, and it is easily seen that $tb = a$. This proves that Γ is closed. □

In particular, each orbit is closed in X.

(iii) For every $x \in X$, the mapping $m_x : s \rightsquigarrow sx$ of G onto Gx is proper. (Since Gx is closed in X, this is equivalent to saying that $m_x : G \to X$ is proper.)

[We recall that a continuous mapping $f : X \to Y$ of Hausdorff spaces is *proper* if (a) f is closed, and (b) for every $y \in Y$, $f^{-1}(y)$ is compact.]

Proof. For any $y = sx \in Gx, m_x^{-1}(y) = sG(x)$ is compact by (i). We shall now show that m_x is closed. Let F be a closed set in G; we must show that $m_x(F) = Fx$ is closed. Let $y \in \overline{Fx}$, and let U, V be neighbourhoods of x, y respectively such that $G(V|U) \subset K, K$

compact. Then $Fx \cap V = (F \cap K)x \cap V$ is closed in V, since $(F \cap K)x$ is compact. Thus Fx is closed in a neighbourhood of every point of $\overline{Fx}$, hence Fx is closed. ◻

Thus in the canonical decomposition

$$G \to G/_{G(x)} \xrightarrow{f} Gx \to X$$

f is a closed continuous bijection, hence a homomorphism. In other words, the orbits (with the topology induced from X) are homogeneous spaces of G.

(iv) Let G' be a locally compact group, and $h : G' \to G$ a continuous homomorphism. Then G' also acts on X in a natural way if we set,
5 for $s' \in G'$ and $x \in X$, $s'x = h(s')x$. We have : G' acts properly on X if and only if the mapping h is proper.

Proof. We have, for $A, B \subset X$,

$$G'(A|B) = h^{-1}[G(A|B)];$$

hence if h is proper, G' acts properly on X. ◻

For the converse, we first note that G' also acts on G by means of h; we may set, for $s' \in G'$, $s \in G$, $s's = h(s')s$. And the mapping $m_x : G \to X$ commutes with the actions of G' on G and X. Hence if G' acts properly on X, it acts properly on G(Remark 1, 1). Hence by (iii) the mapping $s' \rightsquigarrow h(s')e_G = h(s')$ is proper.

In particular, every closed subgroup of G acts properly on X.

Example. Let G be a locally compact group, and K a compact subgroup. Then the action of G (by left multiplication) on the space G/K of left cosets of G modulo K is a proper action.

In fact, let $q : G \to G/K$ be the natural mapping, and let $q(s), q(t) \in G/K$. If U and V are compact neighbourhoods of s, t respectively in G, $q(U), q(V)$ are neighbourhoods of $q(s), q(t)$ respectively, and

$$G(q(U)|q(V)) = \left\{s \in G|(sVK) \bigcap (UK) \neq \phi\right\}$$

$$= (UK)(VK)^{-1},$$

which is compact.

Using (iv), we see that every closed subgroup of G acts properly on G/K.

3 A characterisation of proper transformation groups

Theorem 1. *Let G be a locally compact group of transformations of the Hausdorff space X. In order that G be proper, it is necessary and sufficient that the mapping $f : (s, x) \rightsquigarrow (sx, x)$ of $G \times X$ into $X \times X$ be proper.* 6

Proof. *Sufficiency :* Let $x, y \in X$ be given. □

Case 1. *If $x \notin Gy$, then $(x, y) \notin f(G \times X)$. Since f is proper, $f(G \times X)$ is closed in $X \times X$. Hence there exist neighbourhoods U of x and V of y such that $(U \times V) \cap f(G \times X) =$, i.e., $G(U|V) =$. Hence in this case, the condition (P) is trivially satisfied.*

Case 2. *Let $x \in Gy$. Then $f^{-1}((x, y)) = G(x|y) \times y$ is compact, since f is proper. Hence $G(x|y)$ is compact; let W be a compact neighbourhood of $G(x|y)$. $W \times X$ is a neighbourhood of $f^{-1}(x, y)$; since f is proper, there exists a neighbourhood $U \times V$ of (x, y) such that $f^{-1}(U \times V) \subset W \times X$. Then the projection of $f^{-1}(U \times V)$ on G is contained in W. But this projection is precisely $G(U|V)$, and W is compact, hence (P) is verified for (x, y).*

Necessity. We first prove the

Lemma 1. *Let G be a proper transformation group of the space X. Then, for every $x \in X$ and every neighbourhood W of $G(x)$ in G, there exists a neighbourhood U of x such that $G(U|U) \subset W$.*

Proof of the lemma . *W may be assumed open. Let V be a neighbourhood of x such that $\overline{G(V|V)}$ is relatively compact, and let $A = \overline{G(V|V)} - W$. Then $\overline{A} \cap G(x) = \phi$ (note that $G(x) \subset W$). Hence, for every $t \in \overline{A}$, there exist neighbourhoods W_t of t and V_t of x such that*
7 *$(W_t\, V_t) \cap V_t = \phi$. Since $\overline{A}$ is compact, we have a finite subset F of $\overline{A}$ such that $\overline{A} \subset \bigcup\limits_{t \in F} W_t$. Let $U = V \cap \bigcap\limits_{t \in F} V_t$. Then clearly $G(U|U) \subset G(V|V)$ and $G(U|U) \cap A \subset \left\{ \bigcap\limits_{t \in F} G(V_t|V_t) \right\} \cap \bigcup\limits_{t \in F} W_t = \phi$, hence $G(U|U) \subset W$.*

We now proceed with the proof of the theorem. Suppose that G acts properly on X. Then for any $(x, y) \in X \times X$, $f^{-1}((x, y)) = G(x\, y) \times y$ is compact. Hence we need only prove that f is closed.

Let $F \subset G \times X$ be closed. since $f(G \times X)$ is the graph of the relation defined by G, it is closed in $X \times X$ (§2, (ii)), so that $\overline{f(F)} \subset f(G \times X)$. Let $f(s, y) = (x, y) \in \overline{f(F)}$. We must show that $(x, y) \in f(F)$, i.e., $f^{-1}((x, y)) \cap F \neq \phi$. Suppose this is false. since $f^{-1}(x, y) = sG(y) \times y$, and $G(y)$ is compact, we then have neighbourhoods W of $G(y)$ and V of y such that $(sW \times V) \cap F = \phi$ (recall that F is closed). Now, by Lemma 1, there exists a neighbourhood U of y such that $G(U|U) \subset W$; clearly we may assume $U \subset V$. We then have

$$f^{-1}(sU \times U) \subset G(sU|U) \times U = sG(U\, U) \times U \subset sW \times V.$$

Hence $f^{-1}(sU \times U) \cap F = \phi$. It follows that $(sU \times U) \cap f(F) = \phi$, which is a contradiction since $sU \times U$ is a neighbourhood of (x, y).

4 Existence of invariant metrics

If G is a *compact* Lie group operating differentiably on a paracompact differentiable manifold X, it is well-known that there exists a Riemannian metric on X, invariant under the action of G'. We shall show now that similar results hold for proper transformation groups of locally compact spaces.

8 We begin with the

Lemma 2. *Let G be a locally compact group acting properly on a locally compact space X, and suppose that $G \backslash X$ is paracompact. Then* *there exists a closed set A in X, and an open neighbourhood B of A such that*

(i) $GA = X$,

(ii) for every compact set $K \subset X, G(B|K)$ is relatively compact.

Proof. Let $q : X \to_G \backslash X$ be the natural mapping; in the proof we use the following statement, valid for any open mapping of a locally compact space onto another; for any relatively compact open set W in $G\backslash X$ and any compact set $K \subset W$, there exists a relatively compact open set U in X and a compact set $K_1 \subset U$ such that $q(U) = W$ and $q(K_1) = K$. □

Since ${}_G\backslash X$ is paracompact (and locally compact), we can cover it by a locally finite family $(W_1)_{i\in I}$ of relatively compact open sets. Let $(V_i)_{i\in I}$ be a covering og ${}_G\backslash X$ such that $\bar{V}_i \subset W_i$ for every $i \in I$. We now choose, for every $i \in I$, a relatively compact open set U_i in X and a compact set $A_i \subset U_1$ such that $q(U_i) = W_i$ and $q(A_i) = \bar{V}_i$. Let $A = \cup A_i, B = \cup U_i$. Now $(U_i)_{i\in I}$ is a locally finite family on X. Hence A is a closed set in X, and clearly $GA = X$. Now, let K be any compact set in X. Since $G(U_i|K) = \phi$ implies $W_i \cap q(K) \neq \phi$, and $(W_i)_{i\in I}$ is locally finite, $G(U_i|K) = \phi$ for only finitely many $i \in I$. Since each $G(U_i|K)$ is relatively compact, it follows that $G(B|K)$ is relatively compact.

Remark. Suppose a group G acts on a locally compact paracompact space X, such that ${}_G\backslash X$ is Hausdorff. Then ${}_G\backslash X$ is paracompact whenever the connected components of X are open, or X is countable at infinitely, or G is connected. 9

Theorem 2. *Let G be a Lie group acting properly and differentiably on a paracompact differentiable manifold X. Then X admits a Riemannian metric invariant under G.*

Proof. Since X is paracompact, there exists a Riemannian metric g on X. Further, if A and B are as in Lemma 2, there exists a differentiable function $f \geq 0$ on X, such that $f = 1$ on A and $f = 0$ on $X - B$.

Let $x \in X$; let T_X be the tangent space of X at x, and $s^T = s^T_X$: $T_x \to T_{sx}$ the differential at x of the mapping $y \rightsquigarrow sy$. Then for any

$u, v \in T_X, s \rightsquigarrow f(sx)g(s^T u, s^T v)$ is a continuous function on G, whose support is compact since $f(sx) \neq 0$ implies $s \in G(B|\{x\})$. Let ds be a right-invariant Haar measure on G. If we set

$$g'_X(u^I, v) = \int_G f(sx)\, g\, (s^T u, s^T v) ds,$$

It is easily verified that $x \rightsquigarrow g'_X$ is a Riemannian metric on X, invariant under the action of G. □

Theorem 3. *Let G be a locally compact group acting properly on a locally compact metrisable space X such that ${}_G\backslash X$ is paracompact. Then X admits a G-invariant metric compatible with its topology.*

10 *Proof.* Let d be a metric on X, and let B be as in Lemma 2; thus B is open, $GB = X$, and for any compact set $K \subset X, G(B|K)$ is relatively compact in G. Define

$$r(x) = d(x, X - B), x \in X.$$

Clearly, for any $x, y \in X, r(x) - r(y) \leq d(x, y)$, and hence, for any $x, y, z, \in X$,

$$r(x) + r(z) \leq d(x, y) + \{r(y) + r(z)\}.$$

□

Thus, if we define

$$h(x, y) = \inf\{d(x, y), r(x) + r(y)\}, x, y \in X,$$

it is clear that h is a pseudo-metric on X; note that if $x \in B, h(x, y) > 0$ for $y \neq x$. Now the function $s \rightsquigarrow h(sx, sy)$ is continuous. Its support is compact, since $h(sx, sy) \neq 0$ implies $s \in G(B|\{x, y\})$. Set

$$D(x, y) = \int_G h(sx, sy) ds,$$

with ds a right-invariant Haar measure on X. Then clearly D is a continuous G-invariant distance function on X. We shall now verify that it defines the topology of X. Since $GB = X$, and since D as well as the topology of X is G-invariant, we have only to show that, for every $x \in B$, every neighbourhood W of X contains a D-neighbourhood of x.

We choose an r, $0 < r < r(x)$, such that

$$\mathscr{B} = \{z \in X | d(z, x) \leq r\} = \{z \in X | h(z, x) \leq r\}$$

is compact and contained in W. It is sufficient to find a compact neighbourhood V of e in G such that, for any $y \in X, h(x, y) > r$ implies $h(sx, sy) > \dfrac{r}{2}$ for every $s \in V$. For then **11**

$$\mathscr{B} \supset \{z \in X | D(x, z) < R\}, \text{ where } R = \frac{r}{2} \int\limits_V ds. \text{ In fact, if}$$

$z \in X - \mathscr{B}, h(z, x) > r$, hence

$$\begin{aligned} D(x, z) &= \int\limits_G h(sx, sz) ds \\ &\geq \int\limits_V h(sx, sz) ds \geq \frac{r}{2} \int\limits_V ds. \end{aligned}$$

We proceed to find such a V. Let U be a compact symmetric neighbourhood of e in G such that for $s \in U, h(x, sx) \leq \frac{r}{2}$. Then, since the continuous function

$$(s, y) \rightsquigarrow h(sx, sy) - h(x, y)$$

Vanishes on the compact set $\{e\} \times U\mathscr{B}$ in $G \times X$, we can find a compact neighbourhood $V \subset U$ of e such that $|h(sx, sy) - h(x, y)| \leq \frac{r}{2}$ for $(s, y) \in V \times U\mathscr{B}$. We claim that this V suffices. In fact suppose for an $s \in V$ and $y \in X$ that $h(sx, sy) \leq \frac{r}{2}$. Then $h(x, sy) \leq h(x, sx) + h(sx, sy) \leq r$, so that $sy \in \mathscr{B}$, i.e., $y \in V^{-1}\mathscr{B} \subset U\mathscr{B}$. Hence $|h(sx, sy) - h(x, y)| \leq \frac{r}{2}$, and so $h(x, y) \leq r$.

Remark 1. If G is a group of isometric transformations of a metric space X, the condition (P_1) and (P) of §1 are equivalent. In fact, let d be **12**

the metric on X, and suppose (P_1) is satisfied. Let $x, y \in X$, and let $W = \{z \in X | d(x,z) < r\}$ be a neighbourhood of x such that $G(W|W)$ is relatively compact in G. Let

$U = \{z \in X | d(z,x) < \frac{r}{3}\}, V = \{z \in X | d(z,y) < \frac{r}{3}\}$. Then $G(V|U)$ is relatively compact. For let $s, s_o \in (V|U)$. Then there exist $z, z_o \in U$ such that $sz, s_o z_o \in V$, and we have

$$\begin{aligned} d(s^{-1}s_0 z_0, x) &= d(s_0 z_0, sx) \\ &\leq d(s_0 z_0, y) + d(y, sz) + d(sz, sx) \\ &< \frac{r}{3} + \frac{r}{3} + \frac{r}{3} = r \end{aligned}$$

so that $s^{-1}s_o \in G(W|W)$. Thus $G(V|U) \subset s_o\, G(W|W)$.

Remark 2. Let G be a locally compact group of isometric transformations of a metric space. Assume that G is countable at infinite. Then the condition (P_2) of §1 implies (P_1), and hence (P) by Remark 1. In fact let $G = \bigcup_1^\infty K_n, K_n$ compact and $K_n \subset K^0_{n+1}$. Suppose that (P_1) fails at $x \in X$. Let $U_n = \{z \in X | d(z,x) < \dfrac{1}{n}\}, n = 1, 2, \ldots$ since no $G(U_n|U_n)$ is relatively compact in G, we have, for every $n, ag_n \notin K_n$ and an $x_n \in U_n$ such that $g_n x_n \in U_n$. Then

$$\begin{aligned} d(g_n x, x) &\leq d(g_n x, g_n x_n) + d(g_n x_n, x) \\ &\leq \frac{1}{n} + \frac{1}{n} \end{aligned}$$

so that $g_n x$ converges to x. However, for every $n > 0, g_n \notin K_n$, and every compact set in G is contained in some K_n, so that (P_2) fails.

Chapter 2

The aim of this chapter is the description of the action of a group of transformations in the neighbourhood of an orbit. For proper actions, the existence of "slices" reduces the general case to the case of a neighbourhood of a fixed point. For proper and differentiable actions, a description can be given in terms of linear representations of compact groups (Koszul [1], Mostow [1], Montgomery-Yang [1], Palais [1]) . 13

1 Slices

Let G be a topological group, and H a subgroup acting on a apace Y. We can then construct in a natural manner a topological space X on which G acts. In fact, we let H operate on $G \times Y$ (on the right) by setting

$$(s, y)t = (st, t^{-1}y);\ s \in G, y \in Y, t \in H,$$

and take $X = (G \times Y)/H$. If $q : G \times Y \to X$ is the natural mapping, then the left action of G on X is defined by $sq(r, y) = q(sr, y)$.

Note that in the above situation, if we set $A = q(e \times Y)$, we have (i) $G(A|A)A = A$, (ii) $G(A|A) = H$, (iii) the mapping $(s, a) \rightsquigarrow sa$ of $G \times A$ into X is open. The property (iii) follows trivially from the fact that the mapping $y \rightsquigarrow q(e, y)$ of y onto A is a homeomorphism.

Conversely, let G be a transformation group of a space X, and A a subset of X such that $G(A|A)A = A$. Then it is clear that $H = G(A|A)$ is a subgroup of G. By the above considerations, G acts on $(G \times A)/G(A|A)$. Let $F : G \times A \to X$ be the map $F(s, a) = sa$, and $q : G \times A \to G \times$

$A/G(A|A)$ the natural map. Then there is a map $f : G \times A/G(A|A) \to X$ such that $F = f0q$. It is easy to verify that the mapping f is injective, and commutes with the actions of G on X and $(G \times A)/G(A|A)$.

14 **Definition.** *Let G be a group of transformations of a space X. A slice is a subset A of X such that* (i) $G(A|A)A = A$, (ii) *the mapping* $(s, a) \rightsquigarrow sa$ *of* $G \times A$ *into X is open.*

Condition (ii) means that the mapping $f : (G \times A)/_{G(A|A)} \to X$ defined above is a homeomorphism onto the G-stable open set GA in X.

Definition. *Let G be a transformation group of a space X. A slice A at a point* $x \in X$ *is a slice such that* $(i) x \in A, (ii)$ $G(A|A) = G(x)$.

Note that a slice need not be a slice at any of its points.

Definition. *Let G be a transformation group of a space X. A normal slice is a slice A such that* $G(y) = G(A|A)$ *for every* $y \in A$. *A regular point of X is a point at which a normal slice exists*

A normal slice is characterised by the property that it is a slice at each of its points. It is clear that if A is a normal slice, the orbit of each $s\varepsilon A$ is naturally homeomorphic to $G/_{G(x)} = G/_{G(A|A)}$, and the G-stable open set GA is naturally homeomorphic to $A \times G/_{G(A|A)}$. Since, for every $s\varepsilon G$, sA is also a normal slice, it is clear that the set of regular points is a G-stable open subset of X.

Examples. 1) Let G be a topological group, and H a subgroup acting on a space Y, and q the natural mapping $G \times Y \to (G \times Y)/H$. Then $q(e \times Y)$ is a slice for the natural action of G on $G \times Y/H$. In fact, this motivated our definition of slices.

2) Let G act without fixed points on a space X. Then for any $x \in X$, any slice at x is a normal slice. If $X \to G/X$ is a locally trivial
15 principal fibre space, normal slices in X are precisely the images of open sets in $G \backslash X$ by continuous sections.

2 General Lemmas

Lemma 1. *Let G be a topological group, and H a subgroup of G acting continuously on a space Y. Let $X = (G \times Y)/H$; we suppose that G acts on X in the natural way. Let $q : G \times Y \to X$ be the natural mapping. Then we have;*

(i) *for any $B \subset Y, G(q(e \times B)|q(e \times B)) = H(B|B)$,*

(ii) *for any $y \in Y, G(q(e \times y)) = H(y)$*

(iii) *$B \subset Y$ is a slice in Y if and only if $q(e \times B)$ is a slice in X.*

(iv) *$B \subset Y$ is a normal slice if and only if $q(e \times B)$ is a normal slice.*

(v) *if $y \in Y$ is regular, then $q(e \times y)$ is regular;*

(vi) *if G is locally compact and H is closed, and if H acts properly on Y, then G acts properly on X.*

Proof. It is easy to verify (i), and (ii) is a special case. Also, once (iii) is proved, (iv) and (v) follows from (i) and (ii). We shall prove (iii) and (vi). ◻

Proof of (iii). Let $B \subset Y$ be a slice for the action of H. We shall prove that the natural mapping $(G \times B)/G(B|B) \to X$, which is clearly one-one and commutes with the action of G, is actually an open mapping; since B is a slice for $(G \times B)/G(B|B)$, it will follow that $q(e \times B)$ is a slice for X.

To prove that the mapping $(G \times B)/G(B|B) \to X$ is open, it is plainly sufficient to prove that for any neighbourhood V of e in G, and any neighbourhood W in B of any $b \in B$, the saturation by H of $V \times W$ is a **16**
neighbourhood of $e \times b$ in $G \times Y$. Now, if U is a symmetric neighbourhood of e in G such that $U^2 \subset V$, it is clear that $(V \times W)H$ contains the neighbourhood $UX\{(H \cap U)W\}$.

The converse assertion in (iii) is easy to verify.

Proof of (vi). Suppose that H acts properly on Y. Let $q(s, y), q(s', y') \in X$. Let V, V' be neighbourhoods of y, y' respectively in Y such that

$H(V|V')$ is relatively compact. For any compact neighbourhoods U, U' of s, s' respectively in $G, q(U \times V)$, $q(U' \times V')$ are neighbourhoods of $q(s, y)$ $q(s', y')$ in X. We assert that $G(q(U \times V) \mid q(U' \times V'))$ is relatively compact in G. In fact, it is easily verified that $G(q(U \times V) \mid q(U' \times V')) < U' \, H(V' \mid V)U^{-1}$.

Lemma 2. *Let the topological topological group G act on two spaces X and Y, and let $f : X \to Y$ be a continuous mapping commuting with the actions of G. Then for any slice B in Y, $f^{-1}(B)$ is a slice in X.*

Proof. We may assume that $f^{-1}(B)$ is non- empty. Since f commutes with the actions of G, we have

$$G(f^{-1}(B) \mid (f^{-1}(B)) = G(B \mid B), G(B \mid B) f^{-1}(B) = f^{-1}(B),$$

hence we need only prove that the mapping $G \times f^{-1}(B) \to X$ is open. For this it is sufficient to prove that for any $x \in f^{-1}(B)$, and for any neighbourhoods U of e in G and V of x in $X, U(V \cap f^{-1}(B))$ is a neighbourhood of x in X. To do this, we choose a neighbourhood U' of e in G, and neighbourhood V' of x in X, such that $U'V' \subset V$. Since B is
17 slice in $Y, U'B$ is a neighbourhood of $f(x)$. Since f commutes with the
action of $G, U' f^{-1}(B) \supset f^{-1}(U'B)$ and hence is neighbourhood of x in X. It is easily verified that $U(V \cap f^{-1}(B))$ contains the neighbourhood $V' \cap (U' f^{-1}(B))$. □

Remark. If B is a slice at $y = f(x) \in X$, $f^{-1}(B)$ need not be a slice at x.

3 Lie groups acting with compact isotropy groups

We now consider the case of a *Lie group* G acting on a space X such that *the isotropy groups are all compact*. We wish to study the function associating to any $x \in X$ the conjugacy class of $G(x)$.

We denote by $\mathscr{C} = \mathscr{C}(G)$ the set of all conjugacy classes of compact subgroups of G. For $T, T' \in \mathscr{C}$, we write $T < T'$ if there exist $H \in T, H' \in T'$ such that $H \subset H'$. Since a compact Lie group cannot have proper Lie subgroups isomorphic to it, we see that $T < T' < T$ implies $T = T'$. For any $x \in X$, we denote by $\tau(x)$ the conjugacy class of $G(x)$.

Since, for any $x \in X$ and $s \in G$, $G(sx) = sG(x)s^{-1}, \tau$ can in fact be regarded as a mapping of $G^{/X}$ into $\mathscr{C}(G)$.

Lemma 3. *Let G be a Lie group acting on a topological space X such that all the isotropy groups are compact. Let $x \in X$, and suppose there exists a slice at x. Then,* (1) *there exists a neighbourhood V of x such that $\tau(y) < \tau(x)$ for every $y \in V$;* (2)*x is regular if and only if τ is constant in a neighbourhood of x.*

Proof. Let A be a slice (resp. normal slice) at x. Then for any $y \in A$, $G(y) \subset G(A \mid A) = G(x)(resp. G(y) = G(x))$. Hence it is clear that $\tau(y) < \tau(x)(resp. \tau(y) = \tau(x))$ for all y belonging to the neighbourhood **18**
GA of x. □

Now suppose that τ is constant in an open neighbourhood V of x. If A is any slice at x, we have, for any $y \in V \cap A$, $G(y) \subset G(x)$ and $\tau(y) = \tau(x)$, which implies $G(y) = G(x)$. Thus $V \cap A$ is a normal slice at x, hence x is regular.

Remark. We have also proved that if $x \in X$ is regular, there exists a neighbourhood V of x such that *every slice at x* contained in V is normal.

4 Proper differentiable action

In this article, we study the case of a Lie Group G acting differentiably and properly on a paracompact differentiable manifold X of dimension n. Note that, in this case the orbits Gx are closed submanifolds of X, naturally diffeomorphic with the $G/_{G(x)}$.

Lemma 4. *Let G be a Lie group acting properly and differentiably on a paracompact differentiable manifold of dimension n. Then for any $x \in X$, there exist a representation of $G(x)$ in a finite-dimensional real vector space N, and a differentiable mapping f of a $G(x)$ -stable neighbourhood B of $0 \in N$ in to X such that*

(i) $f(0) = x$

(ii) *f commutes with the actions of $G(x)$.*

(iii) $\dim N + \dim Gx = \dim X$

(iv) *$Gf(B)$ is open in X, and the mapping $h : (s, b) \rightsquigarrow sf(b)$ of $G \times B$ into X passes down to a diffeomorphism ψ of $(G \times B)_{/G(x)}$ onto $Gf(B)$.*

19 *Proof.* By Theorem 2, Chapter 1, we can choose a G-invariant Riemannian metric on X. Let $T(x)$ be the tangent bundle of X, and let Ω be an open neighbourhood of the zero section of $T(x)$ on which the exponential mapping $\exp : \Omega \to X$ is defined (Nomizu [1]). Since G acts isometrically on X, we may assume that Ω is stable for the induced action of G on $T(X)$; we denote this action by $(s, u) \rightsquigarrow s^T u$, $s \in G, u \in T(X)$. We have the relations

$$s \exp u = \exp(s^T u); s \leq G, u \in T(X)$$

and

$$d(x, \exp u) \leq u;\ \ x \in X, u \in T_x(X),$$

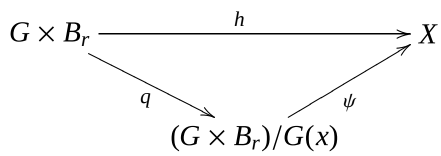
where d is the distance on X induced by the Riemannian metric, and $||u||$ is the length of u. □

Now let $x \in X$, and let $T_x(Gx)$ denote the subspace of $T_x(X)$ tangential to Gx. $G(X)$ leaves $T_x(X)$ invariant, and clearly $T_x(Gx)$ is stable under this action. Since G acts isometrically on X, the orthogonal complement N of $T_x(Gx)$ in $T_x(X)$ is also stable under G_x:

$$N = \{u \in T_x(X) \,|< u, v > \ = 0 \text{ for all } v \in T_x(G_x)\}.$$

Clearly this N has property (iii). Now for any $r > 0$, let $B_r = \{u \in N \big| ||u|| < r\}$. Then B_r is $G(x)$-stable, and is contained in Ω if r is small. We set $f = \exp\big|B_r$. Clearly f has the properties (i) and (ii) of the lemma. We shall now show that if r is small enough, (iv) is also valid with $B = B_r$.

20 We have as usual the commutative diagram



Here, $(G\times B_r, q, (G\times B_r)_{/G(x)})$ is a (locally trivial) differentiable principal bundle, so that ψ is differentiable. Since h is obviously of maximal rank at $(e, 0)$, ψ is of maximal rank at $q(e, 0)$. Since $\dim(G\times B_r)/_{G(x)} = \dim X$, it follows that t ψ is a diffeomorphism in a neighbourhood of $q(e, 0)$. Hence if W is a suitable neighbourhood of $G(x)$ in G, and r is small enough, we have that ψ is a diffeomorphism of $q(W \times B_r)$ onto an open set in X and, if $U = \{z \in X \big| d(z, x) < 2r\}$, $G(U \mid U) \subset W$ (Lemma 1, Chapter 1). We set $B_r = B$, and assert that ψ is a diffeomorphism of $(G \times B)/_{G(x)}$ onto an open subset of X. First, since ψ commutes with the actions of G, and $G(q(W\times B)) = q(G\times B)$, it is clear that ψ is everywhere of maximal rank. We shall now show that it is injective. Equivalently we shall show that for $s, s' \in G$ and $u, u' \in B, h(s, u) = h(s', u')$ implies $q(s, u) = q(s', u')$. In fact, let $h(s, u) = h(s', u')$, i.e., $s\ \exp u = s' \exp u'$, or $s^{-1}s'\ \exp u' = \exp u$.

Then

$$\begin{aligned} d(x, s^{-1}s'x) &\leq d(x, \exp u) + d(s^{-1}s'x, \exp u) \\ &< 2r, \end{aligned}$$

since $d(s^{-1}s'x, \exp u) = d(s^{-1}s'x,\ \theta^{-1}s' \exp u') = d(x, \exp u')$.

Hence $s^{-1}s' \in W$. Since ψ is one - one on $q(W\times B)$, it follows easily that $q(s, u) = q(s', u')$. 21

In what follows, the hypothesis and notation of Lemma 4 are retained.

Theorem 1. *For every $x \in X$, there exists a slice at x.*

Proof. With the notation of Lemma 4, $f(B)$ is a slice at x. In fact ψ : $(G \times B)/_{G(x)} \to Gf(B)$ is a diffeomorphism of $(G \times B)/_{G(x)}$ onto the G- stable open set $Gf(B)$ in X, commuting with the action of G. Since $q(e \times B)$ is a slice in $(G \times B)/_{G(x)}$, it follows that $h(e \times B) = f(B)$ is a slice in X. ◻

Theorem 2. *A point $x \in X$ is regular if and only the action of $G(x)$ in $T_x(X)/_{T_x(G_x)}$ is trivial.*

Proof. If we choose a G-invariant Riemann metric on X, and use the notation of Lemma 4, we have to prove that x is regular if and only if the action of $G(x)$ on N is trivial. Now we know, by (v) of Lemma 1, and the remark after Lemma 3, that X is a regular point of X if and only if, for sufficiently small $\rho, B_\rho = \{u \in B \big| \|u\| < \rho\}$ is a normal slice for the action of $G(X)$ in N, *i.e.* if and only if $G(x)$ acts trivially on N. □

Theorem 3. *The set of regular points is dense in X.*

Proof. We proceed by induction on $\dim X$. If $\dim X = 0$, every point of X is regular. Now let $\dim X = n > 0$, and assume the theorem proved for all manifolds of dimension $< n$. Take any $x \in X$. Since the theorem
22 is of a local nature, we may assume, with the notation of Lemma 4, that $X = (G \times B)/_{G(x)}$. Then, by (v) of Lemma 1, it is sufficient to prove that the set of regular points in B for the action of $H = G(x)$ on B is dense at $0 \in B$. □

For any $\rho, 0 < \rho < r$(= radius of B), let S_ρ be the sphere $\{v \in B \big| \|V\| = \rho\}$. Clearly S_ρ is H-stable. It is clear from Theorem 2 that a $V \in S_\rho$ is regular for the action of H on B if and only if it is regular for the action of H on S_ρ. Since $\dim S_\rho < \dim B \leq \dim X$, it follows by the induction hypothesis that the set of H- regular points of B is dense in S_ρ. Since this is true for all $\rho > 0$, our assertion follows. (We also note that if a $v \in N$ is regular for the action of H, so is λv, for every $\lambda > 0$.)

Theorem 4. *Let G be a Lie group acting properly and differentiably on a paracompact differentiable manifold of dimension n. Let $\tau : X \to \mathscr{C}(G)$ be the function assigning to any x in X the conjugacy class of $G(x)$ in G, and let $\mathcal{R}$ be the set of regular points of X. Then,*

(i) *every $x \in X$ has a neighbourhood V such that $\tau(V)$ is a finite set;*

(ii) *if ${}_G\backslash^X$ is connected, ${}_G\backslash^{\mathcal{R}}$ is connected;*

(iii) *if ${}_G\backslash^X$ is connected, τ is constant on $\mathcal{R}$;*

(iv) *if ${}_G\backslash^X$ is connected, a point $x \in X$ is regular if and only if $\tau(x)$ is minimal (i.e. $\tau(x) < \tau(y)$ for every $y \in X$.*

Proof of (i). We use induction on $\dim X$; if $\dim X = 0$, the statement is
23 trivial. Let $\dim X > 0$, and assume that (i) is proved for all manifolds of dimension $< n$. On account of the local nature of (i), we may assume, with the notation of Lemma 4, that $X = (G \times B)/_{G(x)}$. We assert now that $\tau(X)$ is a finite set. In fact let $0 < \rho < r$ (= radius of B), and let $S = \{u \in B \big| \|u\| = \rho\}$. S is stable for the action of $G(x)$ on B, and $\dim S < \dim X$. By the induction hypothesis and the compactness of S, we conclude that $\tau(S)$ is a finite set. However, since $G(x)$ operates linearly on N, we have, for any $u \in N$ and any $\lambda \in \mathbb{R} - \{0\}$, $\tau(u) = \tau(\lambda u)$. Hence $\tau(B) = \{\tau(0)\} \bigcup \tau(S)$. Thus $\tau(B)$ is finite. By (*ii*) of Lemma 1, $\tau(q(e \times B)) = \tau(B)$. Finally, since $Gq(e \times B) = X$, $\tau(X) = \tau(q(e \times B))$, hence $\tau(X)$ is finite as asserted.

Proof of (ii). Again, we use induction on $\dim X$; if $\dim = 0$, $\mathcal{R} = X$, and (ii) holds trivially. Let $\dim X > 0$, and assume (ii) proved for manifolds of dimension $< n$. We shall prove that every point of ${}_G\backslash^X$ has a neighbourhood V such that $V \cap_G \backslash^{\mathcal{R}}$ is connected. Since ${}_G\backslash^{\mathcal{R}}$ is dense in ${}_G\backslash^X$, it follows easily that if ${}_G\backslash^X$ is connected ${}_G\backslash^{\mathcal{R}}$ is also connected. Again, we may assume, with the notation of Lemma 4, that $X = (G \times B)\backslash_{G(x)}$; and we shall prove that ${}_G\backslash^{\mathcal{R}}$ is connected.

Let $\mathcal{R}'$ be the set of regular points of B for the action of $H = G(x)$. We assert that ${}_H\backslash\mathcal{R}'$ is connected. If $\dim B = 1$, or if x is a regular point, this is trivially verified. Thus let $\dim B > 1$, and x be not regular. Let r be the radius of B, and let $S = \{u \in B \big| \|u\| = r/2\}$. S is H- stable and connected. Hence, by induction, ${}_H\backslash\mathcal{R}$" is connected, where $\mathcal{R}$" is the set of regular points of S. Since $\mathcal{R}' = \bigcup\limits_{0<\lambda<2} \lambda\mathcal{R}''$, it follows easily that ${}_H\backslash\mathcal{R}'$ 24
is connected.

Now, $q(e \times \mathcal{R}')$ is a dense set of regular points in the slice $q(e \times B)$,hence its in ${}_G\backslash^X$ is dense in ${}_G\backslash^{\mathcal{R}}$. On the other hand, since $q(e \times^{\mathcal{R}})'$ is contained in the slice $q(e \times \mathcal{B})'$ at x, it is easy to verify that the mapping $\mathcal{R}' \to_G \backslash^X$ obtained by composing the mappings $\mathcal{R}' \to q(e \times \mathcal{R}')$ and $q(e \times \mathcal{R}'_G \backslash^X$, passes down to a mapping ${}_H\backslash^{\mathcal{R}'} \to_G \backslash^X$. Since ${}_G\backslash^{\mathcal{R}'}$ is connected, ${}_G\backslash^{\mathcal{R}}$ thus contains a dense connected subset, hence is connected.

Proof of (iii). Use (ii), and (ii) of Lemma 3.

Proof of (iv). Let ${}_G\backslash^X$ be connected, and let $y \in X$. By (i) of Lemma

3, there exists a neighbourhood V of y such that $\tau(z) < \tau(y)$ for every $z \in V$. Now $\mathcal{R}$ is dense in X, and τ, is constant on $\mathcal{R}$, hence we have $\tau(x) < \tau(y)$ for every $x \in \mathcal{R}$. The converse assertion (even without any assumption on ${}_G\backslash X$) follows from Lemma 3, since we know that there exists a slice at every $x \in X$.

Remark 1. We see from (iv) of Theorem 4 that for the proper differentiable action of a Lie group on a connected paracompact manifold, the orbits of regular points are of maximal dimension. The converse is not true, even if the Lie group is connected. For instance, consider the group $G =$ SO $(3, \mathbb{R})$ of rotations of the two - sphere, acting on it self by inner automorphisms. Then the regular points are the rotations of angle $\neq 0$ or π; the isotropy group at such point is the one parameter group
25 through that point, consisting of rotations about the same axis, and the orbit is a two sphere. For rotation of angle π the isotropy group has *two* connected components (the identity component being the one parameter group through that point), and the orbit is a projective plane.

Remark 2. Let G be a *connected* Lie group. For any compact subgroup H of G, let $[\ H\ ]$ denote its conjugacy class. Now suppose we are given two conjugacy classes $T_1, T_2 \in \mathscr{C}(G)$. Then in the set $\{[H_1 \cap H_2], H_1 \in T_1, H_2 \in T_2\}$, there exists a (unique) minimal class for the relation $<$. In fact G acts in the obvious manner on the connected space $G\backslash_{H_1} \times G\backslash_{H_2}, H_1 \in T_1, H_2 \in T_2$, and the class we are looking for is the conjugacy class of the isotropy groups at regular points. Thus, given $T_1, T_2 \in \mathscr{C}(G)$ we are able to associate with them an element $T_1 \circ T_2$ of $\mathscr{C}(G)$ characterised by the minimality property.

5 The discrete case

Let G be a discrete group, acting properly on a Hausdorff space X. Then there exists a slice at every point of X. In fact for any $x \in X$, there exists an open neighbourhood U of x such that $G(U \mid U) = G(x)$ (Lemma 1, Chapter 1). Since $G(x)$ is finite, $V = \bigcap\limits_{g \in G(x)} gU$ is an open neighbourhood of x; clearly $G(V|V) = G(x)$ and V is $G(x)$- stable. Since V is an open neighbourhood of x, it follows that it is a slice at x.

Remark. In the classical constructions of fundamental domains for a group G acting isometrically on a metric space X, one defines, for any $x \in X$, the set

$$A = \{z \in X \mid d(z, x) < d(z, sx) \text{ for every } s \in G - G(x)\}.$$

A has the properties. 26

(i) $G(A \mid A) = G(x)$,

(ii) $G(x)A = A$.

In fact let $t \in G(A|A)$, and let $z \in A$ be such that $tz \in A$. If $t \notin G(x)$, we have

$$d(tz, tx) > d(tz, x) = d(z, t^{-1}x) > d(z, x),$$

which is impossible since t is an isometry. Thus (i) is proved, and (ii) is easily verified. But A is in general not a slice. However, a slightly different construction produces a slice at x.

Let A be defined as above. Since G is discrete and acts properly, Gx is discrete, hence $\lambda = \inf_{s \in G - G(x)} d(sx, x) > 0$. Set $V = \{z \in X \mid d(z, x) < \lambda/2\}$. Clearly V is stable under $G(x)$. On the other hand $V \subset A$, hence $G(V \mid V) <\subset G(A|A) = G(x)$. Since V is open, it follows that V is a slice at x.

Our construction of a slice in the differentiable case (Lemma 4) is somewhat similar to the construction given above, namely, the slice $q(e \times B)$ in Theorem 1 is the intersection of a neighbourhood of x with $\{y \in X \mid d(y, sx) > d(y, x) \text{ for every } s \in G - G(x)\}$.

6

Let G now be a *compact lie group*, action continuously on a *completely regular* topological space. The following lemmas reduce the problem of constructing a slice at a point of X to that of the differentiable case. 27

Lemma 5. *Let G be a compact Lie group. For any closed subgroup H of G, there exists a representation of G in a finite dimensional real vector space E, and a $u \in E$, such that $G(u) = H$.*

Proof. We consider the left regular representation of G in $L^2(G)$. We know then, by the Peter-Weyl theorem, that $L^2(G) = \sum\limits_{i \in I} E_i$, where the E_i are finite dimensional, G-invariant, and pairwise orthogonal. □

Let $q : G \to G/_H$ be the natural mapping, and let f be a continuous function on $G/_H$ such that $f(z) = 0$ if and only if $z = q(H)$. Let $g = f \; o \; q$, and consider the decomposition $g = \sum g_i, g_i \in E_i$, of g in $L^2(G)$. Since $g(y) = 0$ if and only of $y \in H$, it is clear that $H = G(g)$, the isotropy group of G at g. On the other hand we have $G(g) = \bigcap\limits_{i \in I} G(g_i)$. Since the $G(g_i)$ are compact Lie groups, we can find a finite subset J of I such that $H = \bigcap\limits_{i \in I} G(g_i)$. For the E and u of the lemma, we can take $E = \sum\limits_{i \in J} E_i, u = \sum\limits_{i \in J} g_i$.

Lemma 6. *Let G be a compact Lie group acting on a completely regular space X. Then for any $x_o \in X$, there exists a finite dimensional representation of G in a real vector-space E, and a mapping $f : X \to E$ commuting with the action of G, such that $G(f(x_o)) = G(x_o)$.*

Proof. By Lemma 5, we have a finite dimensional representation of G in a real vector space E, and a $u \in E$, such that $G(u) = G(x_o)$. Hence the continuous mapping $s \rightsquigarrow su$ of G into E passes down to a mapping of $G/_{G(x_o)}$ into E. Since G is compact, $G/_{G(x_o)}$ is canonically homeomorphic to Gx_o and hence we get a continuous mapping $f : Gx_o \to E$
28 with the property $f(sx_o) = su = sf(x_o)$. Since X is completely regular, and Gx_o is compact, f cab be extended to a continuous mapping $f^* : X \to E$. The required f is now given by $f(x) = \int\limits_G sf^*(s^{-1}x)ds$, where ds, is the Haar measure on G with $\int\limits_G ds = 1$. □

Theorem 5 (Mostow [1])**.** *Let G be a compact Lie group operating on a completely regular space X. Then there exists a slice at every $x \in X$.*

Proof. Let $f : X \to E$ be as in Lemma 6; thus f commutes with G, and $G(f(x)) = G(x)$. By theorem 1, there exists a slice B at $f(x)$. Then by Lemma 2, $f^{-1}(B)$ is a slice. Since $G(f^{-1}(B)|f^{-1}(B)) = G(f(x)) = G(x)$, $f^{-1}(B)$ is a slice at x. □

Remark 1. Because of theorem 5, the considerations of §3 are valid in the case of a compact Lie group acting on a completely regular space.

Remark 2. By similar methods, Palais [1] has proved Theorem 5 for arbitrary Lie groups acting properly on completely regular spaces.

Chapter 3

This chapter is devoted to the following problem: given a discrete group G acting properly on a topological space, determine a presentation of G (by generators and relations) and if possible a finite one. The treatment given here is due to Behr [1], [2]. The classical presentation of groups generated by reflections (Coxeter [1]) is discussed in section §3 by a similar method. 29

1 Finite presentations for discrete proper groups of transformations

Let G be a group operating on a connected topological space X. Assume that each $s \in G$ acts continuously on X. Let $A \subset X$ be such that

1) $GA = X$

2) $G(A|A)A$ is a neighbourhood of A.

Proposition. $S = G(A|A)$ *generates* G.

Proof. Let G' be the subgroup of G generated by S. We first assert that $G'A = X$. In fact, it is clear that $G'A$ is open in X. $G'A$ is also closed in X. For, let $x = sa \in X, s \in G, a \in A$. Then $V = sSA$ is a neighbourhood of x. If $V \cap G'A \neq \emptyset$, we have $s \in G(G'A|SA) \subset G'G(A|A)S \subset G'$, so that $x \in G'A$. Since X is connected, we have $G'A = X$, Now, let $a \in A$. For any $s \in G$, we have $s' \in G', a' \in A$ such that $sa = s'a'$. Hence $s^{-1}s' \in S \subset G'$. Hence $s \in G'$. □

Remark. If G is a locally compact group operating on a space X, and $A \subset X$ has properties 1) and 2) above, and if further $G(A|A) = S$
30 is relatively compact in G, then G *acts properly* on X. In fact, for $x = sa, x' = s'a' (s, s' \in G, a, a' \in A), U = sSA$ and $U' = s'SA$ are respectively neighbourhoods of x and x', and we see easily that $G(U|U') = sS^3 s'^{-1}$.

In particular, if S is finite, we are in the case of a discrete group acting property.

Lemma. *Let G be a discrete group acting continuously on a connected topological space X. Let A be a closed subset of X such that*

(1) *$GA = X$,*

(2) *for each $x \in A$, there exists a finite subset S_x of $G(A|A)$ such that $S_x A$ is a neighbourhood of x,*

(3) *A is connected,*

(4) *any connected covering of X which admits a section over A is trivial.*

Let $L(S)$ be the free group generated by $S = G(A|A)$; for each $s \in G$, let s_L be the generator of $L(S)$ corresponding to s. Then G is isomorphic to the quotient group $L(S)/_K$, where K is the normal sub group to $L(S)$ generated by the elements $s_L s'_L s''_L$ with $s, s', s'' \in S$ and $ss's'' = e$.

Proof. Let us set $G = L(S)/_K$. For $s \in S$, let $s \in G$ denote $s_L \mod K$, and let $S = \{s | s \in S\}$. Then we have:

(i) $e \in S$

(ii) $S = (S)^{-1}$; in fact, for $s \in S$, $(s)^{-1} = (s^{-1})$;

(iii) if $s, s' inS$ are such that $ss' \in S$, then $ss' \in S$; in fact $ss' = ss'$.

□

31 It is clear that $\underline{e} \in \underline{S}$. Also, since $S^{-1} = S$ (iii) implies (ii). To prove (iii) note that $\underline{s}\,\underline{s}'(\underline{ss}')^{-1} = \underline{e}$, since $s_L s'_L((ss')_L^{-1}) \in K$.

We put the discrete topology on $\underline{G}$ and consider the product space $\underline{G} \times A$. Let $\varphi : \underline{G} \to G$ be the homomorphism induced by the mapping $s_L \rightsquigarrow s$; clearly, φ is surjective. We define a relation $\mathcal{R}$ on $\underline{G} \times A$ by setting $(t, a)\mathcal{R}(t', a')$ if $\varphi(t)a = \varphi(t')a'$ and $t^{-1}t' \in \underline{S}$. From $(i), (ii)$and (iii) above, it follows that $\mathcal{R}$ is an equivalence relation on $\underline{G} \times A$. Let Y be the quotient space $(G \times \underline{A})_{/\mathcal{R}}$, and $q : \underline{G} \times A \to Y$ be the natural mapping. The mapping $(t, a) \rightsquigarrow \varphi(t)$ a of $\underline{G} \times A$ into X induces a mapping $f : Y \to X$. We make $\underline{G}$ act on Y by setting $rq(t, a) = q(rt, a); r, t \in \underline{G}, a \in A.\underline{G}$ also acts on X through φ, and it is clear that f commutes with the action of $\underline{G}$.

We now wish to prove that $f : Y \to X$ is a connected covering, with $H = \ker \varphi$ as the group of covering transformations. We do this in several steps.

(i) *f is surjective*. In fact, $f(Y) = \varphi(\underline{G})A = GA = X$.

(ii) *f is locally injective*. We remark first that f is injective on $q(\underline{S} \times A)$. In fact, let $s, s' \in S$ and $fq(\underline{s}, a) = fq(\underline{s'}, a')$. Then $sa = s'a'$, hence $s^{-1}s' \in S$, implying $\underline{s}^{-1}\underline{s}' \in \underline{S}$. This means that $q(\underline{s}, a) = q(\underline{s'}, a')$. We shall prove now that $q(\underline{S} \times A)$ is a neighbourhood of $q(\underline{e} \times A)$. It will follow that f is locally injective.

Let B= interior of SA; and for $s \in S$, let $B_s = A \cap (s^{-1}B)$. Clearly, $B = \bigcup\limits_{s \in S} sB_s$. Let $L = \bigcup\limits_{s \in S} (\underline{s} \times B_s)$. Clearly, $q(\underline{S} \times A) \supset q(L) \supset q(\underline{e} \times A)$. We now assert that $q(L)$ is open in Y, i.e., that $L' = q^{-1}(q((L)$ 32
is open $\underline{G} \times A$. In fact let $(t, a) \in L'$.

Then, for some $b \in B_s, \varphi(t)a = sb$ and $t^{-1}\underline{s} \in \underline{S}$. Let $S' = \{s' inS | sb \in s'A\}$, and $W = \bigcup\limits_{s' \in S'} s'B_{s'}$. Then $W \supset B \cap \bigcup\limits_{s' \in S' \cap S_{sb}} s'A$, hence W is a neighbourhood of sb in X. Then $\varphi(t^{-1})W$ is a neighbourhood of a in X, and it is easily seen that the neighbourhood $\{t\} \times \{A \cap \varphi(t^{-1})W\}$ of (t, a) is contained in L'.

(iii) *For every $x \in X$, there are local sections for f at x.*

It is sufficient to give a section on SA. To each $h \in H = \ker \varphi$, we associate the section $\sigma_h : SA \to Y$ defined by $sa \rightsquigarrow q(h\underline{s}, a)$.$\sigma_h$ is well-defined: if $sa = s'a'$ s, s' in $(S; a, a' \in A), s^{-1}s' \in S$, so that $q(h\underline{s}, a) = q(h\underline{s}'a')$. Clearly, $fo\sigma_h$ = identity, and for every $s \in S, \sigma_h|sA$ is continuous. Since (2) holds, it follows that σ_h is a section of f.

(iv) $\underline{f^{-1}(SA)} = \bigcup_{h\in H} \underline{\sigma_h(SA)}$. In fact, let $f(q(t, a)) \in SA$.

Then $\varphi(t)a = sa'$ with $s \in S$ and $a' \in A$. Hence $\varphi(t^{-1})s \in S$, i.e $t^{-1}h\underline{s} \in \underline{S}$ for suitable $h \in H$. Then clearly $\sigma_h(s, a') = q(t, a)$.

(v) *If* $h \neq h'\sigma_h(SA) \cap \sigma'_h(SA) = \phi$. Suppose, for $h, h' \in H$, that $\sigma_h(s, a) = \sigma_{h'}(s', a')(s, s' \in S$ and $a, a' \in A)$; i.e., $q(h\underline{s}, a) = q(h'\underline{s}', a')$. We have then $\underline{s}^{-1}h^{-1}h'\underline{s}' = \underline{s}''$, with $s'' \in S$.

Hence $h^{-1}h' = \underline{ss}''\underline{s}'^{-1}$. Since $\varphi(h^{-1}h') = e = ss''s'^{-1}$ in G, it follows that $h^{-1}h' = \underline{e}$ in $\underline{G}$.

33 (vi) *Y is connected*. In fact, since $\underline{G}q(\underline{e}\times A) = Y$, we have only to verify that connected component Y_o of Y which contains $q(\underline{e} \times A)$ is $\underline{G}$-stable. But this is clear, since, for any $s \in S, q(\underline{e}\times A) \cap \underline{s}q(\underline{e}\times A) \neq \phi$, and $\underline{S}$ generates $\underline{G}$

Thus (Y, f) is a connected covering of X, with H = kernel φ as the group of covering transformations. Since (4) holds it follows that $H = (\underline{e})$, and this proves the lemma.

Theorem 1. *Let G be a discrete group, acting continuously on a connected topological space X. Suppose that there exists a connected subset A of X such that*

(1) $GA = X$,

(2) $G(A|A)$ *is finite* ,

(3) $G(A|A)A$ *is a neighbourhood of A.*

Suppose further that there exists a compact subset C of X such that any connected covering of X which admits a section over C is trivial. Then G is finitely presentable.

Proof. We first remark that A may be assumed to be closed. In fact we shall verify the conditions (2) to (3) for $\bar{A}$. Now, we note that $\bar{A} \subset S^2A$; for, if $x = sa \in \bar{A}(s \in G, a \in A)$. the neighbourhood sSA of a meets A, hence $s \in S^2$. Hence $G(\bar{A}|\bar{A}) \cap S^5$, and so is finite. Also, S^3Ais clearly a neighbourhood of $S^2A \supset \bar{A}$, hence $G(\bar{A}|\bar{A})A$ is a neighbourhood of $\bar{A}$. □

Let now $S = G(A|A)$. For every n, S^nA satisfies conditions (1), (2), (3) of the lemma. If n is large enough, $S^nA \supset C$, and therefore satisfies condition (4). Hence there exists a finite presentation of G with **34**
$G(S^nA|S^nA) \subset S^{2n+1}$ as set of generators.

Remark 1. For a locally simply connected space X,the existence of a compact set C satisfying the condition of the theorem means that $\prod_1(X)$ is finitely generated.

Remark 2. Suppose that, in Theorem 1, we drop the assumption that A is connected. We can still assert that G is finitely presentable, if X is locally connected. We may assume that A satisfies conditions (1), (2) and (4) of the lemma. The space Y constructed above need not now be connected, so that we will have to enlarge K suitably.

We retain the notation of the proof of Theorem 1. Let B = interior of SA, and let B_o be a connected component of B. We first prove that

$$X = \cup G(B_o|B_1)G(B_1|B_2)\ldots G(B_{n-1}|B_n)B_n, \qquad (*)$$

where the union is over all finite sequences $B_1, \ldots, B_n$ connected components of B. In fact, since X is locally connected, the connected components of B are open, hence the right side X' of $(*)$ is open in X.

Now let x be any point of X. Since $GB = X$, we have $x \in tB'$, for $t \in G$ and some connected component B' of B. Now suppose the neighbourhood tB' of x meets X', say

$$tB' \cap \{G(B_o|B_1)\cdots G(B_{n-1}|B_n)B_n\} \neq \phi.$$

Then $$t \in G(B_o|B_1)\cdots G(B_{n-1}|B_n)G(B_n|B'),$$

hence $$x \in G(B_o|B_1)\cdots G(B_{n-1}|B_n)G(B_n|B')B' \subset X'.$$

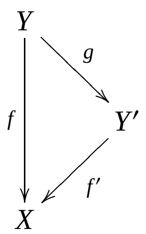
Hence X' is also closed in X. Hence $X' = X$. 35

We also have:

(**) For any $B_1, B_2 \subset B$ and any $t \in G(B_1|B_2)$, there exists a $\underline{t} \in \underline{G}$ such that $\underline{t}\sigma_e(B_2) \cap \sigma_{\underline{e}}(B_1) \neq \phi$

This is clear since φ is surjective, and $\underline{G}$ is transitive on the fibres of f.

Now let $s \in S$. By$(*)$, there exist connected components $B_1, \ldots, B_n$ of B such that $sB_\circ \cap \{G(B_\circ B_1) \ldots G(B_{n-1}B_n)B_n\} \neq \phi$. We thus have $t_i \in G(B_{i-1}B_i), i = 1, \ldots, n$, such that $t_{n+1}^{-1} = s^{-1}t_1 \cdots t_n \in G(B_o B_n)$. For each $t_i, i = 1, \ldots, n+1$, we choose $\underline{t}_i \in \underline{G}$ as in $(**)$, and consider the normal subgroup K', of $\underline{G}$ generated by all the $\underline{s}^{-1}\underline{t1} \cdots t_{\underline{n}+1}, s \in S$. Obviously, $K' \subset H$. Hence $f : Y \to X$ induces a mapping $f' : Y' = K'/Y \to X$ such that the diagram



is commutative; here $g : Y \to Y'$ is the natural mapping. Clearly (Y', f') is a covering of X and $\underline{G}' = \underline{G}_{/K}$ operates on Y', transitively on the fibres of f'. We now assert that Y' is connected. In fact let Y'_o (resp. Y_o) denote the connected component of Y' (resp. Y) which contains $g\sigma_{\underline{e}}(B_o)$ (resp. $\sigma_e(B_o)$). Since $f'(Y'_o) = X$, we need only prove that Y'_o is stable under $\underline{G}'$. For this again it is sufficient to check that for any $\underline{s} \in \underline{S}$, we have a $\underline{t} \in K'$ such that $\underline{s}^{-1}\sigma_{\underline{e}}(B_o) \cap \underline{t}Y_o \neq \phi$. In fact, we can choose $\underline{t} = \underline{s}^{-1}t_{\underline{1}} \cdots t_{n+1} \in K'$.

36 Since A satisfies condition (4), it follows that $H_{/K'}$, the group of covering transformations of (Y', f), is trivial. Hence $G \approx \underline{G}/'_K$ is finitely presentable .

Remark 3. Let G be a discrete group, acting properly on a locally compact connected space X such $G^{/X}$ is compact. If $\prod_1(X)$ is finitely generated, then G is finitely presentable.

In fact, we can find a compact subset A of X containing a set of loops which generate $\prod_1(X)$, such that $GA = X$, and this A satisfies the conditions of Theorem 1.

In particular, since connected Lie groups have finitely generated fundamental groups, we see that, in a connected Lie group, any discrete subgroup with compact quotient is finitely presentable.

2 Finite presentations for groups of automorphisms of graphs

For the next result, we need some elementary notions about graphs.

A *graph* is a set X in which there is associated to each $x \in X$ a subset $\sum(x)$ of X such that (i) for every $x \in X, x \in \sum(x)$, and (ii) for any $x, y \in X, x \in \sum(y)$ implies $y \in \sum(x)$. A graph X is *finite at* $x \in X$ if $\sum(x)$ is finite.

A *path* in a graph X is a sequence $(a_o, a_1, \ldots, a_n)$ of elements of X such that $a_{i+1} \in \sum(a_i), 0 \leq i \leq n-1$; a_o and a_n are respectively the *initial* and *end points* of the path, and if $a_o = a_n$, the path is called a *loop at* a_o. A graph is said to be *connected* if any two of its points can be joined by a path.

Consider the operations which respectively associate to any path 37
$(a_o, \ldots, a_n)$ in the graph the path $(a_o, , \ldots, a_i, a_{i+1}, \ldots a_n)$ and the path $(a_o, , \ldots, a_i, b, a_i, \ldots, a_n)$ with $b \in \sum(a_i)$. Two paths in a graph are *homotopic* if we can obtain one from the other by means of a finite number of the above operations and their inverses. The product of paths is defined in the usual way.

A loop $(a_o, \ldots, a_n = a_o)$ is said to be of *length* $\leq m$ if $a_i = a_{m-i}$ for $0 \leq i \leq \dfrac{n-m}{2}$. A graph X is of *breadth* $\leq m$ if every loop in X is homotopic to a product of loops of length $\leq m$.

Let X and Y be graphs. A *homomorphism* $f : X \to Y$ is a mapping such that for every $x \in X$, $f(\sum(x)) \subset \sum(f(x))$.

Let Y be a connected graph. A homomorphism $f : X \to Y$ is a *covering* if, for every $y \in Y$ and $y' \in \sum(y)$, and every $x \in f^{-1}(y)$, there exists a unique $x' \in \sum(x)$ such that $f(x') = y'$. If f is a covering, it is easily seen that any path in Y can be lifted to a path in X with any given initial point.

If X is connected, and $f : X \to Y$ is a covering such that every lift of any loop in Y is a loop in X, then f is bijective. In fact, it is sufficient to assume that for a point $y_o \in Y$ and an $x_o \in f^{-1}(y_o)$, the lift through x_o of any loop at y_o in Y of length $\leq$ breadth Y is a loop.

Theorem 2. *Let X be a connected graph of finite breadth, finite at each point. Let G be a transitive group of automorphisms of X. If the isotropy*
38 *group is finitely presentable, then G is finitely presentable.*

Proof. Let $x_o \in X$. For each $x \in \sum(x_o)$,choose an $s_x \in G$ such that $s_x x_o = x$, and let $S = \{s_x | x \in \sum(x_o)\}$. Since X is finite at x_o, S is a finite set. ◻

Let $L(S)$ be the free group on the set, and let $H = G(x_o)$. Let $L(S)\mathbb{X}H$ be the free product of $L(S)$ and H. We have a homomorphism

$$\psi : L(S)\sharp H \to G$$

induced by the obvious maps of $L(S)$ and H into G. Since X is connected, we have $\psi(L(S))x_o = X$, and hence $G = \psi(L(S))H$. In particular, ψ is surjective.

Let T be a finite set of generators of H. Let $s \in S, t \in T$. We have $tsx_o \in \sum(x_o)$. Hence there exists a unique $s' \in S$ such that $tsx_o = s'x_o$. Clearly $s'^{-1}ts \in H$. Denoting by s_L the element of $L(S)$ corresponding to $s \in S$, we consider the normal subgroup K of $L(S)\sharp H$ generated by the (finitely many)elements of the following type

(i) $(s'_L)^{-1}t(s_L).(s'^{-1}ts)^{-1}; s \in S, t \in T$

(ii) $(s_1)_L(s_2)_L \cdots (s_n)_L(s_1 s_2 \cdots s_n)^{-1}; s_1, \dots s_n \in S, s_1 s_2 \cdots s_n \in H, n \leq$ breadth of X.

Clearly $K \subset \ker \psi$, hence ψ induces a homomorphism

$$\varphi : \underline{G} = (L(S)\sharp H)_{/K} \to G.$$

39 We shall prove now that φ is an isomorphism. Since $L(S)\sharp H$ is

finitely presentable, this will prove that G is finitely presentable.

Let $\underline{\mathrm{H}}$ be the image of H in $\underline{\mathrm{G}}$. Since, for any $s \in S$ and $t \in T, K$ contains an element of the type $(s'_L)^{-1} t(s_L)h$ with $h \in H$, we see that $\underline{\mathrm{SH}} = \underline{\mathrm{HS}}$ where $\underline{\mathrm{S}}$ is the image of the set $\{s_L | s \in S\}$ in $\underline{\mathrm{G}}$. Let $Y = \underline{\mathrm{G}}_{/\underline{\mathrm{H}}}$, and $q : \underline{\mathrm{G}} \to \underline{\mathrm{H}}$ the natural mapping. The mapping $\underline{\mathrm{t}} \rightsquigarrow \varphi(\underline{\mathrm{t}})x_o$ of $\underline{\mathrm{G}}$ onto X induces a mapping $f : Y \to X$, and we have commutative diagram

$$\begin{array}{ccc} \underline{\mathrm{G}} & \xrightarrow{\varphi} & G \\ {\scriptstyle q}\big\downarrow & & \big\downarrow \\ Y & \xrightarrow{f} & X \end{array}$$

where $G \to X$ is the mapping $s \rightsquigarrow sx_o$.$\underline{\mathrm{G}}$ acts on Y(by left multiplication), and we have for any $\underline{\mathrm{t}} \in \underline{\mathrm{G}}$ and $y \in Y$, $f(\underline{\mathrm{t}}y) = \varphi(\underline{\mathrm{t}})f(y)$.

We define the structure of a graph on Y as follows. Set $y_o = q(\underline{\mathrm{e}})$, and for any $y = \underline{\mathrm{t}}y_o \in Y$, set $\sum(y) = \underline{\mathrm{tS}}y_o$. We check first that $\sum(y)$ is well-defined. In fact, let $y = \underline{\mathrm{t}}'y_o$. Then $\underline{\mathrm{t}}' = \underline{\mathrm{th}}$, with $\underline{\mathrm{y}} \in \underline{\mathrm{H}}$. Hence for any $\underline{\mathrm{s}} \in \underline{\mathrm{S}}, \underline{\mathrm{t}}'\underline{\mathrm{s}}y_o = \underline{\mathrm{ths}}y_o = \underline{\mathrm{ts}}'\underline{\mathrm{h}}y_o = \underline{\mathrm{ts}}'y_o$, since $\underline{\mathrm{HS}} = \underline{\mathrm{SH}}$. The verification that $y_1 \in \sum(y_2)$ implies $y_2 \in \sum(y_1)$ is similar. Since S generates $\underline{\mathrm{G}}$ modulo $\underline{\mathrm{H}}$(i.e.$\underline{\mathrm{G}} = \cup \underline{\mathrm{S}}^n \underline{\mathrm{H}}$), it is easily seen that Y is a *connected* graph. Moreover, f is a homomorphism of graphs.

We assert now that f is a covering. To prove this it is enough to lift paths starting at x_o. Let $y \in f^{-1}(x_o)$. If $y = \underline{\mathrm{t}}y_o$, we have $\varphi(\underline{\mathrm{t}})x_o =$ **40**
$\varphi(\underline{\mathrm{t}})f(y_o) = f(\underline{\mathrm{t}}y_o) = x_o$, hence $\varphi(\underline{\mathrm{t}}) \in H$. Now let $sx_o \in \sum(x_o)$. Then there exists a unique $s' \in S$ such that $\varphi(\underline{\mathrm{t}})sx_o = s'x_o$, and $y' = \underline{\mathrm{ts}}'y_o \in \sum(y_o)$ is clearly the unique lift of sx_o in $\sum(y_o)$.

We verify finally that the lift of any loop of the type $(x_o, s_1x_o, \ldots, s_1 s_2 \cdots s_n x_o)$ with $n \leq$ breadth of X is a *loop*at $y_o \in Y$. This will prove that the covering $Y \to X$ is trivial, since every loop at x_o is homotopic to a product of loops of this type. Now, it is clear that $(y_o, s_1y_o, \ldots, s_1s_2 \cdots s_ny_o)$ is a path at y_o which lifts the above loop. And since $s_1s_2 \cdots s_n = e$, we have $\underline{\mathrm{s}}_1 \ldots \underline{\mathrm{s}}_n \in \underline{\mathrm{H}}$, i.e. this path is a loop.

Hence it follows that f is bijective. Suppose now that $\underline{\mathrm{t}} \in \underline{\mathrm{G}}$, and $\varphi(\underline{\mathrm{t}}) = e$ in G. Then, $f(\underline{\mathrm{t}}y_o) = \varphi(\underline{\mathrm{t}})x_o = x_o$. Since f is bijective, we must have $\underline{\mathrm{t}} \in \underline{\mathrm{H}}$. However, $\varphi|\underline{\mathrm{H}}$ being injective, this means that $\underline{\mathrm{t}} = \underline{\mathrm{e}}$ in $\underline{\mathrm{G}}$, and hence is finitely presented.

Remark. It follows from Theorem 2 that if a group G admits of a left invariant graph structure which is (i) connected, (ii) finite at each point, and (iii) of finite breadth, then G is finitely presentable. The converse is also true, i.e. any finitely presentable group admits of a left-invariant graph structure which satisfies conditions (i), (ii) and (iii). In fact let G be a finitely presentable group, and let S be a finite set of generators of G such that $e \in S$, and $S = S^{-1}$. We define a graph structure on G by setting, for any $t \in G$,

$$\sum(t) = \{t' \in G | t^{-1}t' \in S\}.$$

41 It is easy to see that this defines a graph structure on G which is left-invariant, connected, and finite at each point. We shall now prove that the breadth of this graph is finite.

Let $L(S)$ be the free group on S; for $s \in S$, we denote be s_L the corresponding generator of $L(S)$. Let K be the kernel of the natural mapping $L(S) \to G$. Since any loop at e in G can be written in the form $(e, s_1, s_1 s_2, \ldots s_n = e)$, we see that K is naturally isomorphic to the group of homotopy classes of loops at e.

Now, since G is finitely presentable, we have by a theorem of Schreier a finite subset F of K such that K is the normal closure of F in $L(G)$. It follows easily that, if for each element of F we choose a representative loop at e, and 1 is an upper bound for the lengths of these loops, our graph structure on G has breadth ≤ 1.

Remark 2. Let G be a group which has a finitely presentable normal subgroup N such that $G_{/N}$ is finitely presentable. Then G is finitely presentable. In fact, by the above remark, there exists a G-invariant graph structure on $G_{/N}$ which is connected, finite at each point, and of finite breadth. Since G acts transitively on $G_{/N}$ with isotropy group N which is finitely presentable, it follows from Theorem 2 that G is finitely presentable.

As an application of Theorem 2, we shall prove the following

Theorem 3 (Behr [2]). *For any finite set P of primes, the group $GL(n\mathbb{Z}[P^{-1}])$ is finitely presentable.*

42 *Here $\mathbb{Z}[P^{-1}]$ is the subring of the rationals $\mathbb{Q}$ generated by $P^{-1} = \{p^{-1} | p \in P\}$. To prove Theorem 3 we need some preliminaries.*

For any prime p, let $\mathbb{Q}_p$ be the p-adic field, $\mathbb{Z}_p \subset \mathbb{Q}_p$ the ring of p-adic integers. Let $\mathcal{R}$ be the set of all lattices in $\mathbb{Q}_p^n$. (A lattice in $\mathbb{Q}_p^n$ is a $\mathbb{Z}_p$-submodule generated by a basis of $\mathbb{Q}_p^n$).

If we set, for $A, B \in \mathcal{R}$,

$$d(A, B) = \inf\{r \in \mathbb{Z}^+ | p^r A \subset B, p^r B \subset A\}$$

d is a metric on $\mathcal{R}$. We define a graph structure on $\mathcal{R}$ by setting, for any $A \in \mathcal{R}$, $\sum(A) = \{B \in \mathcal{R} | d(A, B) \leq 1\}$.

$\mathcal{R}$ is finite at every point. In fact,$d(A, B) \leq 1$ implies that $pA \subset B \subset p^{-1}A$, and this can hold (for a given $A \in \mathcal{R}$) only for finitely many B. Also, $\mathcal{R}$ is connected, in view of the following

Proposition. *Given $A, B \in \mathcal{R}, A \neq B$, there exists a $C \in \mathcal{R}$ such that:*

(i) $d(A, C) = 1$, *and* $d(C, B) = d(A, B) - 1$*;*

(ii) *for any $D \in \mathcal{R}$, we have*

$$d(D, C) \leq \sup\{d(D, A), d(D, B)\}.$$

Proof. Since $\mathbb{Z}_p$ is a principal ideal domain, there exists a basis $(a_1, \ldots, a_n)$ for A, and integers $r_1, \ldots, r_n$, such that $(p^{r_1} a_1, \ldots, p^{r_n} a_n)$ is a basis for B; clearly we have then $d(A, B) = \sup |r_i|$. Let $c_i = p_i^{\alpha(i)} a_i$, where

$$\alpha(i) = \begin{cases} +1 & \text{if } r_i > 0, \\ 0 & \text{if } r_i = 0 \\ -1 & \text{if } r_i < 0. \end{cases}$$

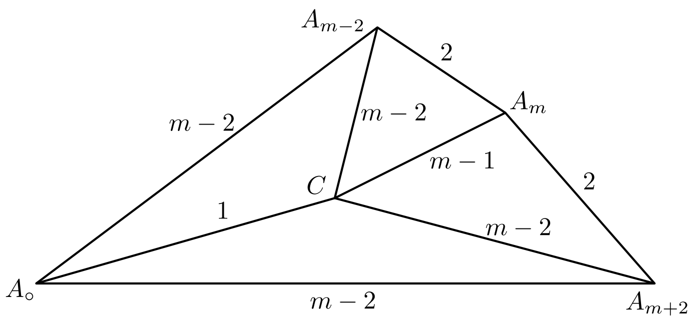
Then the lattice C with the c_i as basis obviously satisfies (i). □ 43

Now let $D \in \mathcal{R}$, and let $r = sup(d(D, A), d(D, B))$. Then clearly $p^r D \subset A \cap B \subset C$. On the other hand, for each i, $p^r a_i$ and $p^{r+r_i} \alpha_i \in D$, hence $p^{r+\alpha(i)} \alpha_i \in D$; this means that $p^r C \subset D$. This proves (*ii*).

It follows that $\mathcal{R}$ is connected: the above proposition shows that, given $A, B \in \mathcal{R}$, there exists a path of length $d(A, B)$ joining A and B.

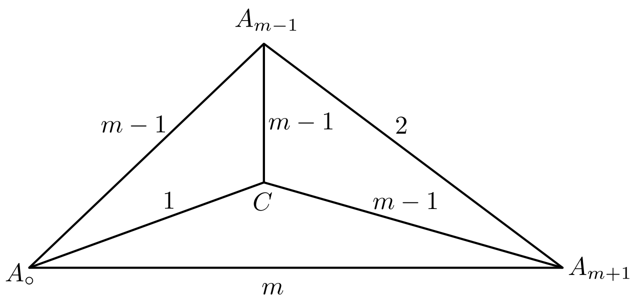
We shall now prove that $\mathcal{R}$ *has breadth* ≤ 8. We shall show that any loop $(A_o, \ldots, A_n)$ in $\mathcal{R}$ with $n > 8$ is homotopic to a product of loops of length $< n$, and this will prove our assertion.

Case 1. *n* even. Let $n = 2m$. If $d(A_o, A_m) < m$, there exists a path of length $< m$ from A_o to A_m, and it is obvious that the given loop is homotopic to the product of two loops of length $< 2m$. Let then $d(A_o, A_m) = m$. We choose $C \in \mathcal{R}$ such that $d(A_o, C) = 1$, and $d(C, A_m) = m - 1$. By the proposition above, we have then $d(A_{m\pm 2}, C) \leq m - 2$. Since there exists a path from C to A_m(resp. $A_{m\pm 2}$) of length $m - 1$(resp. $\leq m - 2$). It follows easily that the given loop is homotopic to the product of four loops, each of length $< 2m$(see the figure below).



44 (In the figure, the lengths of the paths are less than or equal to the numbers marked along them.)

Case 2. *n* odd. Let $n = 2m+1$. Then $d(A_o, A_{m+1}) \leq m$. If $d(A_o, A_{m+1}) < m$, there is a path of length $< m$ from A_o to A_{m+1}, and we are through. If $d(A_o, A_{m+1}) = m$, we proceed as in Case 1. See the figure below:



In the proof of Theorem 3, we shall use the following lemma, a proof of which can be found in M. Eichler [1], §12.

Lemma. *Suppose we are given, for each prime p, a lattice A_p in $\mathbb{Q}_p^n$ such that $A_p = E \otimes \mathbb{Z}_p$ except for finitely many p; here E is the unit lattice in $\mathbb{Q}^n$. Then there exists a lattice A in $\mathbb{Q}^n$ such that $A_p = A \otimes \mathbb{Z}_p$ for every p. (In fact $A = \bigcap_p (\mathbb{Q}^n \cap A_p)$).*

Proof of Theorem 3. Using Theorem 2, we shall now prove that $G = GL(n, \mathbb{Z}[P^{-1}])$ is finitely presentable, by induction on the cardinality of P. If $P = \phi, G = GL(n, \mathbb{Z})$, and this is finitely presentable (Remark following Lemma 11, 6). Now let $P \neq \phi$. We choose a $p \in P$, and consider G as a group acting on the set $\mathcal{R}$ of lattices in $\mathbb{Q}_p^n$. The graph structure introduced on $\mathcal{R}$ is clearly invariant under the action of G; in fact the metric on $\mathcal{R}$ which defines its graph structure is itself invariant 45
under G. Further, it is clear that the isotropy group of G at the unit lattice $E_p = E \otimes \mathbb{Z}_p$ of $\mathcal{R}$ is precisely $GL(n, \mathbb{Z}[P_1^{-1}])$, where $P_1 = P - \{p\}$. Hence if we verify that G is transitive, then all the conditions of Theorem 2 will be satisfied on account of the induction hypothesis, and Theorem 3 will be proved. We shall now show that the subgroup $GL(n, \mathbb{Z}[P_1^{-1}])$ of G is already transitive on $\mathcal{R}$.

Given any $A \in \mathcal{R}$, consider the family $\{A_q, q \text{ prime}\}$, where $A_q = E \otimes \mathbb{Z}_q$ for $q \neq p$, and $A_q = A$. By the above lemma, there exists a lattice A in $\mathbb{Q}^n$ such that, for every prime $q, A_q = A \otimes \mathbb{Z}_q$. Consider the $g \in GL(n, \mathbb{Q})$ such that $g.E = A$. Then $g(E \otimes \mathbb{Z}_p) = A$ (where g is now regarded as in $GL(n, \mathbb{Q}_p)$). But since, for every $q \neq p, g(E \otimes \mathbb{Z}_q) = (E \otimes \mathbb{Z}_p)$, we must have $g \in GL(n, \mathbb{Z}[p^{-1}])$, and our assertion is proved.

3 Groups generated by reflexions

Let M be a connected differentiable manifold. A diffeomorphism r of M onto itself is called a *reflexion* if $(i) r^2 =$ identity, $(ii) M - M(r)$ is disconnected, where $M(r) = \{x \in M | r(x) = x\}$. Since, in a suitable coordinate neighbourhood of any $x \in M(r) r$, acts as an orthogonal linear transformation (see Montogomery and Zippin [1], p.206), we see that $M - M(r)$

has exactly two connected components which are carried each into the other by r, and that $M(r)$ is a (not necessarily connected) submanifold of M of codimension one.

46 **Theorem 4.** *Let G be a discrete proper group of differentiable automorphisms of a simply connected differentiable manifold M, generated by reflexions. Then G has a presentation of the form $\{r_\alpha; (r_\alpha r_\beta)^{p_{\alpha\beta}} = e\}$, where the r_α are reflexions.*

Proof. Let $\mathcal{R}$ be the set of all reflexions belonging to G. Since $M(r))_{r\in\mathcal{R}}$ is a locally finite family, $M - \bigcup_{r\in\mathcal{R}} M(r)$ is an open set; we denote the set of its connected components by $(W_i)_{i\in I}$. G acts on the set of $M(r)'s, r \in \mathcal{R}$; in fact $gM(r) = M(r^{g-1}), r^{g-1} = grg^{-1}$ being clearly a reflexion. Hence G also acts on the $W_i' s$. ◻

Let W_o denote any one of the W_i. Let $\mathcal{R}' = \{r \in \mathcal{R}|$ there exists $x \in \bar{W}_o$ such that $r \in \mathcal{R} \cap G(x)\}$. Let $\mathscr{L}(\mathcal{R}')$ be the free group generated by $\mathcal{R}'$; we denote the natural injection $\mathcal{R}' \to \mathscr{L}(\mathcal{R}')$ by $r \rightsquigarrow r_L$. Let K be the normal closure in $\mathscr{L}(\mathcal{R}')$ of the set

$$\left\{(r_i)_L(r_j)_L^{\text{ord } r_i r_j} | r_i, r_j \in \mathbb{R}', M(r_i) \cap M(r_i) \cap \bar{W}_o \neq \phi\right\}.$$

We denote by $\varphi : \underline{G} = \mathscr{L}(\mathcal{R}')/_K \to G$ the natural homomorphism induced by $r_L \rightsquigarrow r$. Also, for any $r \in \mathbb{R}'$, we denote r_L mod K by $\underline{r}$. We shall prove by induction on the dimension of X that

(1) $\varphi : \underline{G} \to G$ is a bijection

(2) G acts freely transitively on the set $(W_i)_{i\in I}$.

For any $x \in M$ let G_x(resp. $\underline{G}_x$) be the subgroup of G (resp. $\underline{G}$) generated by $\mathcal{R}' \cap G(x)$(resp. the $\underline{r}$ such that $r \in \mathcal{R}' \cap G(x)$).

Since G is discrete and proper, we have for every $x \in M$ a coordinate
47 neighbourhood V_x such that V_x is $G(x)$-stable, and $G(V_x|V_x) = G(x)$. We may assume the coordinate system so chosen that $G(x)$ acts on V_x by orthogonal linear transformations. We assert that, for $x \in \bar{W}_o$,

(a) $\varphi : \underline{G}_x \to G_x$ is bijective,

(b) for any $y \in V_x, G_y$ is simply transitive on the set of W_i such that $y \in \overline{W}_i$, in particular, $G_x(\overline{W}_o \cap V) = V_x$. These assertions are easy to verify if $\dim X \leq 2$; if $\dim X \geq 3$, they follow from the induction hypothesis (1) and (2), when we consider the action of G_x on the spheres about x in V_x.

Now let Y be the quotient space of $\underline{G} \times \overline{W}_o$($\underline{G}$ having the discrete topology) by the equivalence relation

$$(t', x') \sim (t, x) \Longleftrightarrow x' = x \text{ and } t^{-1}t \in \underline{G}_x.$$

The mapping $(t, x) \rightsquigarrow \varphi(t)x$ of $\underline{G} \times \overline{W}_o$ to M induces a mapping $f : Y \to M$. Similarly the action $(t, x) \rightsquigarrow (st, x)$ of $\underline{G}$ on $G \times \overline{W}_o$ induces an action of $\underline{G}$ on Y. $\underline{G}$ acts on X through φ. It is clear that f commutes with the action of $\underline{G}$. We proceed to show that $f : Y \to M$ is a connected covering.

(i) *Y is connected*. This is clear since for every $r \in \mathcal{R}', \underline{r}\, q(e\overline{w}_\circ) \cap q(q, \overline{W})\circ) \neq \phi$. Here $q : \underline{G} \times \overline{W}_o \to Y$ is the natural map.

(ii) *f is locally injective* . It is sufficient to know that f is injective **48**
in a neighbourhood of any $q(e, x), x \in \overline{W}_o$. Now $\underline{G}_x \times (V_x \cap \overline{W}_o)$ is saturated with respect to q, hence $q(\underline{G}_x \times (V_x \cap \overline{W}_o))$ is a neighbourhood of $q(e, x)$. Using the inductive assertions *a*) and *b*), we see that f is injective on $q(\underline{G}_x \times (V_x \cap \overline{W}_o)$.

(iii) *f is surjective*. We must show that $\varphi(\underline{G})\overline{W}_o = M$. Now, $\varphi(\underline{G})\overline{W}_o$ is obviously closed in M, being a locally finite union of closed sets. But it is also open, since for any $x \in \overline{W}_o, \varphi(\underline{G}_x)\overline{W}_o = G_x\overline{W}_o$ is a neighbourhood of x by the inductive assertion *b*).

(iv) *f has local sections*. Since f commutes with the action of $\underline{G}$, and $\phi(\underline{G})\overline{W}_o = M$, it is sufficient to consider points of $\overline{W}_o$.

Now let N be the subgroup of $\underline{G}$ defined by

$$N = \{n \in \underline{G} | \varphi(n)\overline{W}_o = \overline{W}_o\}.$$

Clearly $N \supset \ker \varphi$. For $n \in N$ and $r \in \mathcal{R}'$, it is clear that $r^n (= r^{\varphi(n)}) \in \mathcal{R}'$. Further if $r, r' in \mathcal{R}'$ and $M(r) \cap M(r') \cap \overline{W}_o \neq \phi$, we have also $M(r^n) \cap M(r'^n) \cap \overline{W}_o \neq \phi$. Hence we can define the automorphism $h \rightsquigarrow h^n$ of $\underline{G}$ by setting $(\underline{r})^n = \overline{(r^n)}$. Clearly $\varphi(h^n) = \varphi(n^{-1})\varphi(h)\varphi(n)$.

Now, for any $n \in N$ and any $x \in \overline{W}_o$, we define the section $\sigma_n : V_x \to Y$ of f by

$$\sigma_n(\varphi(t)y) = q(nt^n, \varphi(n^{-1})y); t \in \underline{G}_x, y \in V_x \cap \overline{W}_o.$$

By (ii), σ_n is well-defined.

(v) $f^{-1}(V_x) = \bigcup_{n \in N} \sigma_n(V_x)$. Let $h \in \underline{G}, z \in \overline{W}_o$, and let $f(q(h,z)) \in V_x$. Then $\varphi(h) = \varphi(t)y$, with $t \in \overline{G}_x$, and $y \in V_x \cap \overline{W}_o$; thus $\varphi(h^{-1}t)y = z$.
49 Now, since $\underline{G}_z$ is transitive on the $\overline{W}_i$ containing z, there exists $s \in \underline{G}_z$ such that $s^{-1}h^{-1}t \in N$. Let $n = t^{-1}$hs. Then

$$q(tn, \varphi(n^{-1})y) = q(h,z).$$

Now, Since $\varphi(t\ n) = \varphi(n\ t^n)$, there exists $u \in \ker\varphi$ such that $tn = unt^n = unt^{un}$. Then

$$\begin{aligned} q(h,z) &= q(tn, \varphi(n^{-1})y) \\ &= q(unt^{un}, \varphi((un)^{-1}))y \\ &= \sigma_{un}(\varphi(t)y), \end{aligned}$$

and (V) is proved.

(vi) For $n, n' \in N, \sigma_n(V_x) \cap \sigma_{n'}(V_x) \neq \phi \Rightarrow n = n'$. Let $n, m \in N, \sigma_n(y) = \sigma_m(y)$ for some $y \in V_x$. Since f is locally injective, and $G_x W_o$ is dense in V_x, there $z \in V_x \cap G_x W_o$ such that $\sigma_n(z) = \sigma_m(z)$. Let $z = \varphi(t)z', t \in \underline{G}_x$ and $z' \in W_o \cap V_x.\sigma_n(z) = \sigma_m(z)$ gives

$$\varphi(n^{-1})z' = \varphi(m^{-1})z' \Rightarrow \varphi(nm^{-1}) \in G(z') \subset G(x),$$

and $(nt^n)^{-1}(mt^m) \in \underline{G}_{z'} = e$.

Hence $n^{-1}m = t^n(t^{-1})^m$. Now $t^n \in \underline{G}_{\varphi(n^{-1})x}$ and $t^m \in G_{\varphi(m^{-1})x'}$ since $t \in \underline{G}_x$. But, since $\varphi(nm^{-1}) \in G(x), \varphi(n^{-1})x = \varphi(m^{-1})x$, hence $n^{-1}m \in \underline{G}_{\varphi(n^{-1})x}$. Since $n^{-1}m \in N$, it follows by the induction assumption b) that $n^{-1}m = e$, and (vi) is proved.

50 Thus $f : Y \to M$ is a connected covering. Since M is simply connected, f is bijective. Since the fibres of f are parametrised by $N \supset \ker\varphi$, φ is injective. And then $N = \{e\}$ means precisely that $\varphi(\underline{G})$ is simply transitive on the W_i. It follows easily that for every $r \in \mathscr{R}$, there exist $h \in \varphi(\underline{G})$ and $r' \in \mathscr{R}'$ such that $r' = r^h$; hence $\varphi(\underline{G}) = G$, and the assertions (1) and (2) are proved.

Chapter 4

This chapter contains results related with the following kind of problem: **51**
given a discrete group of continuous transformations, use information on the behaviour of a set of generators to prove that the action of the group is proper. The solution of such a problem is based here on a Lemma (Lemma 2) related to the methods of Chapter 3 as well as to a Theorem of Weil on discrete subgroup of Lie groups (A. Weil [1], [2]).

1 Criterion for proper action for groups of isometries

Let G be a topological group acting on a *connected* space X.

Let $S \subset G$ and $A \subset X$ be such that

(i) $e \in S$

(ii) $S \subset G(A|A)$

(iii) $s, s' \in S, A \cap sA \cap s'A \neq \phi$ imply $s^{-1}s' \in S$.

Note that these conditions imply $S = S^{-1}$.

On the product space $G\chi A$, consider the relation $\mathscr{R}$ defined as follows:

$$(t, a)\mathscr{R}(t', a') \text{ if } ta = t'a' \text{ and } t^{-1}t' \in S.$$

It is easily seen that $\mathscr{R}$ is an equivalence relation. Let $y = (G\chi A)/\mathscr{R}$, and let $q : G\chi A \to Y$ be the canonical mapping. The mapping $(t, a) \rightsquigarrow ta$ of $G\chi A$ into X induces a mapping $f : Y \to X$ such that the diagram

$$\begin{array}{ccc} G \times A & \xrightarrow{\quad q \quad} & Y \\ & \searrow \quad \swarrow {\scriptstyle f} & \\ & X & \end{array}$$

52 is commutative. G acts Y in the usual manner, and f commutes with the action of G. Our object is to give sufficient conditions on S and A so that f is a homeomorphism.

Lemma 1. *If A is connected and S generates G, then Y is connected.*

Proof. Let Y_o be the connected component of Y containing $q(e\chi A)$. $\square$

Since $Gq(e \times A) = Y$, we need only verify that Y_o is G-stable . This is clear since, for any $s \in S, q(e \times A) \cap sq(e\chi A) \neq \phi$, and S generates G.

Lemma 2. *Suppose that: (i) there exists a G invariant metric d on X; (ii) S is a neighbourhood of e in G; (iii) there exists a $\varrho > 0$ such that for any $a \in A$ there is an $s \in S$ with $\left\{x \in X | d(x, a) < \varrho\right\} \subset sA$.*

Then $G(IntA) = X$, and $f : Y \to X$ is a covering .

Proof. Since X is connected and $G(\operatorname{Int} A)$ open in X, we will have $\overline{G(\operatorname{Int} A)} = X$ if we show that $G(\operatorname{Int} A)$ is closed in X. Now let $x \in \overline{G(\operatorname{Int} A)}$. Then there exist $t \in G$ and a $a \in \operatorname{Int} A$ such that $d(x, ta) < \varrho$, i.e., $d(t^{-1}x, a) < \varrho$. Hence there is an $s \in S$ such that $t^{-1}x \in s(\operatorname{Int} A)$; this implies that $x \in G(\operatorname{Int} A)$. $\square$

It follows from $G(\operatorname{Int} A) = X$ that f is onto. We now prove that $f : Y \to X$ is a covering.

1. *f is locally injective*. It is sufficient to prove that f is injective in a neighbourhood of any $q(e, a)$, with $a \in \operatorname{Int} A$. Now let U be a neighbourhood of e in G such that $U^{-1}U \subset S$. Since q^{-1}
53 $(q(U\chi \operatorname{Int} A)) = \bigcup_{s\in S}(Us\chi(A \cap s^{-1} \operatorname{Int} A)), q(U\chi \operatorname{Int} A)$ is a neighbourhood of $q(e\chi \operatorname{Int} A)$. We assert that f is injective on $q(U\chi \operatorname{Int} A)$. In fact let $(t, a), (t', a') \in U\chi \operatorname{Int} A$, and let $f(q(t, a)) = f(q(t', a'))$, i.e. $ta = t'a'$. Then, since $t^{-1}t' \in U^{-1}U \subset S$, we have $q(t, a) = q(t', a')$.

2. *f has a local section.* For any $x_o \in X$, let $B = \left\{x \in X | d(x_o, x) < \varrho/2\right\}$, and let $N = \left\{t \in G \middle| tA \supset B\right\}$. For each $t \in N$, we have a section $\sigma_t : B \to Y$ of f defined by $\sigma_t(z) = q(t, t^{-1}z), z \in B$. Note that, for t, t' in N, we have either $\sigma_t = \sigma_{t'}$ or $\sigma_t(B) \cap \sigma_{t'}(B) = \phi$.

3. *f is a covering.* Let $x_o \in X$. In view of 1) and 2), it is sufficient to show, with the notation of 2), that $f^{-1}(B) = \bigcup\limits_{t \in N} \sigma_t(B)$.

Let $q(r, a) \in f^{-1}(B)$, i.e. $ra \in B$. Let $s \in S$ be such that $sA \supset \left\{x \in X \middle| d(x, a) < \varrho\right\}$. Then $rsA \supset \left\{x \in X \middle| d(x, ra) < \varrho\right\} \supset B$, which means $rs \in N$. Then $\sigma_{rs}(ra) = q(rs, s^{-1}r^{-1}ra) = q(rs, s^{-1}a) = q(r, a)$.

This proves Lemma 1.

Theorem 1. *Let G be a topological group acting isometrically on a connected metric space X. Let A be a connected subset of X, and S a neighbourhood of e in G generating G such that the following conditions are satisfied:*

1. *$S \subset G(A/A)$,*

2. *$s, s' \in S, A \cap sA \cap s'A \neq \phi$ imply $s^{-1}s' \in S$;*

3. *there exists a $\varrho > 0$ such that for any $a \in A$, we have $s \in S$ with $sA \supset \left\{x \in X \middle| d(x, a) < \varrho\right\}$;*

4. *any connected covering of X admitting a section over A is trivial.*

Then $S = G(A|A)$. If moreover S is relatively compact in G, then **54**
the action of G on X is proper.

Proof. By Lemmas 1 and 2, $f : Y \to X$ is a connected covering; this covering admits a section over A, given by $a \rightsquigarrow q(e, a)$. Hence f is bijective. ◻

We now prove that $G(A|A) \subset S$. Let $t \in G(A|A)$. Then there exists $a, a' \in A$ such that $ta = a'$, i.e., $f(q(t,a)) = f(q(e,a'))$. Since f is bijective, we have $q(t,a) = q(e,a')$, i.e., $t \in S$.

The second assertion of the theorem now follows from the remark after Lemma 1, Chapter 3.

2 The rigidity of proper actions with compact quotients

Let G be a locally compact group, and X a locally compact metrisable space, We denote by $\mathscr{C} = \mathscr{C}(G\chi X, X)$ the space of all continuous mapping of $G\chi X$ into X, provided with the compact open topology, and we denote by $\underline{M}$ the subset of $\mathscr{C}$ consisting of continuous actions of G on X(with the induced topology). Also, we denote by $\underline{M}_P$ the set of *proper* actions of G on X, and by $\underline{M}_I$ the set of isometric actions (i.e., an action of G on X belongs to $\underline{M}_I$ if there exists a metric on X invariant under this action). By Theorem 2, chapter 1, we have $\underline{M}_P \subset \underline{M}_I$ (at least when X is connected).

Theorem 2. *Let G be a locally compact group, and X a connected, locally connected, locally compact metrisable space. Suppose that X has a compact subset K such that any connected covering of X admitting a section over K is trivial. Let $m_o \varepsilon \underline{M}_p$ be such that $m_o\, X$ is compact.*

55 Then there exists a neighbourhood W of m_o in $\underline{M}_I$ such that

a) $W \subset \underline{M}_p$

b) for every $m \varepsilon W$, $m\, X$ is compact

c) the action of G on $W \times X$ defined by $(s,(m,x)) \rightsquigarrow (m, m(s,x))$ is proper,

d) if G is a Lie group, then $\ker m \subset \ker m_o$ for any $m \varepsilon W$ (here, for any m, $\ker m = \left\{ g \varepsilon G | m(g,x) = x \text{for every} x \varepsilon X \right\}$.

Proof of a) and b). With the assumptions of the theorem, we shall prove that there exists a compact connected subset A of X containing K, a relatively compact open neighbourhood S of e in G, and a neighbourhood W of m_o in $\underline{M}_I$ such that, for every $m\varepsilon W, A$ and $S_m = S \cap G_m(A|A)$ satisfy the conditions of Theorem 1. Then W will satisfy *a*) and *b*).

Let C be a compact subset of X such that $m_o(G, C) = X$. Since X is locally connected, locally compact and connected, there exists a connected compact neighbourhood A of C containing K. Let B be an open relatively compact set in X, containing A. We set $S = G_{m_o}(B|B)$. Clearly S is a symmetric open relatively compact neighbourhood of e in G. For $m\varepsilon M$, we set $S_m = S \cap G_m(A|A)$. Clearly S_m is also a neighbourhood of e.

(i) *There exists a neighbourhood* W_1 *of* m_o in M *such, that, for any* $m\varepsilon W_1$, *and any* $s, s'\varepsilon S_m, A \cap m(s, A) \cap m(s', A) \neq \phi$ *implies* $s^{-1}s'\varepsilon S_m$.

In fact, $L = \bar{S}^2 - S$ is compact, and $m_o(L, A) \cap A = \phi$. Hence there exists a neighbourhood W_1 of m_o in $\underline{M}$ such that, for any **56**
$m \in W, m(L, A) \cap A = \phi$. It is easily verified that W_1 has the required property.

(ii) *There exists a neighbourhood* W_2 *of* m_o *in M such that, for any* $m \in W_2, S_m$ *generates G*.

Let C' be a compact neighbourhood of C contained in Int A Then $T = G_{m_o}(C'|C)$ generates G; in fact, since TC is a neighbourhood of C, and $T \supset G_{m_o}(C|C)$, the proof of Lemma 1, Chapter 3 is valid. We shall now show that $T \subset S_m$ is m sufficiently close to m_o.

For each $t \in T$, we have *a* $c(t) \in C$ such that $m_o(t, c(t)) \in C' \subset \text{Int}\, A$. Thus there exists a compact neighbourhood $V(t)$ of t such that $m_\circ(V(t), c(t)) \subset \text{Int}\, A$. Let $W(t)$ be a neighbourhood of $m_\circ$ in $\underline{\text{M}}$ such that $m(V(t), c(t)) \subset \text{Int}\, A$ for any $m \in W(t)$.

Since T is compact, there exists a finite subset T' of T such that $T \subset \bigcup_{t\in T'} V(t)$. If we take $W_2 = \bigcap_{t\in T'} W(t)$, we clearly have $T \subset G_m(A|A)$ for any $m \in W_2$. Since $T \subset S$, we have $T \subset S_m$; hence S_m generates G, for every $m \in W_2$.

(iii) *There exists a neighbourhood W_3 of m_o in M such that, for any $m \in W_3, (S_m, \operatorname{Int} A) \supset A$.*

We know that $m_o(S, \operatorname{Int} A) \supset A$. Thus for any $a \in A$, there exists an $s_a \in S$ such that $m_o(s_a, \operatorname{Int} A) \ni a$. Let U_a be a compact neighbourhood of a *in* A such that $U_a \subset m_o(s_a, \operatorname{Int} A)$,i.e. $m_o(s_a^{-1}, U_a) \subset \operatorname{Int} A$. Since A is compact, we have a finite subset F of A such that $\bigcup_{a \in F} U_a = A$. For each $a \in F$, let W_a be neighbourhood of m_o in $\underline{M}$ such that $m(s_a^{-1}, U_a) \subset \operatorname{Int} A$
57 for every $m \in W_a$. Clearly $W_3 = \bigcap_{a \in F} W_a$ has the required property.

We now set $W = \underline{M}_1 \cap W_1 \cap W_2 \cap W_3$, and assert that, for any $m \in W, A$ and S_m satisfy the conditions of Theorem 1. In view of the above considerations, our assertion will follow if we verify condition 3) of Theorem 1. Take any $m \in W$, and choose an invariant metric d on X with respect to m. By (*iii*) above, we have $A \subset \bigcup_{s \in S_m} m(s, \operatorname{Int} A) = U$ say. Let $\lambda = d(A, X - U)$, and let $A' = \left\{x \in X | d(x, A) \leq \lambda/2\right\}$. Then for the ρ of condition 3) we can take the minimum of $\lambda/2$ and the Lebesgue number of the covering $\{m(s, \operatorname{Int} A)\}_{s \in S_m}$ of A'.

Thus a) and b) are proved.

Proof of c). We shall prove that the action of G on $W \chi X$ is proper, where W is as above. Since $W \chi X$ is Hausdorff, it is enough to verify the condition (P) of Chapter 1 for the points $(m_o, x_1), (m_o, x_2), x_1, x_2 \in X$. Now, given $x_1, x_2 \in X$, we may assume by enlarging the A of the above considerations if necessary, that $x_1, x_2 \in \operatorname{Int} A$. For this A we obtain a neighbourhood $W' \subset W$ of m_o such that $G_m(A|A) \subset S$ for every $m \in W'$. $W' \chi A$ is a neighbourhood of (m_o, x_1) and (m_o, x_2) such that $G(W' \chi A | W' \chi A) \subset S$. Since S Since S is relatively compact, this proves c).

Proof of d). Let $K = \ker m_o$. Since the action m_o is proper, K is a *compact* normal subgroup of G. Let $q : G \to G/K$ be the canonical homomorphism. Let V be an open neighbourhood of $q(K)$ which contains no nontrivial subgroup of G/K. Now $F = \overline{S} - q^{-1}(V)$ is a compact set in G such that $\ker m_o \cap F = \phi$. hence there exists a neighbourhood $W' \subset W$
58 of m_o such that $\ker m \cap F = \phi$ for all $m \in W'$. Since, for any $m \in W$, we

have ker $m \subset G_m(A|A) \subset S$, we have, for $m \in W'$, $q(\ker m) \subset V$, i.e. ker $m \subset K$. This proves d).

Remark 1. It is not in general true that every $m_o \in \underline{M}_P$has a neighbourhood in $\underline{M}$ which is contained in $\underline{M}_P$, even if we suppose that m_o X is compact. For instance, let $G = \mathbb{Z}, X = \mathbb{R}$, and let $m_o \in \underline{M}_P$ be defined by $m_o(n, t) = t + n$. For any $a \in \mathbb{R}$, let $\varphi_a : \mathbb{R}\mathbb{R}$ be a differentiable function such that

$$\varphi_a(t) = \begin{cases} 1, & t \leq a \\ 0, & t > a + 2 \end{cases}$$
$$\varphi'_a(t) \geq -1.$$

Let m_a be defined by $m_a(x, t) = t + n\varphi_a(t)$. It is easy to check that $m_a \in \underline{M}$. It is also clear that if a is large enough, m_a is arbitrarily close to m_o. However, $m_a \notin \underline{M}_P$, since under this action $\mathbb{Z}$ leaves every point $\geq a + 2$ fixed.

Remark 2. In Theorem 2, the condition that $m_o x$ is compact is essential For instance, let $G = \mathbb{Z}, X = GL(2, \mathbb{C}).\mathbb{Z}$ operates on X by left multiplication, through the homomorphism h defined by

$$h(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

This action is proper. The action of $\mathbb{Z}$ on X defined by the homomorphism $h_n : \mathbb{Z} \to GL(2, \mathbb{C})$ which maps 1 on $\begin{pmatrix} e^{2\pi i/n} & 1 \\ 0 & 1 \end{pmatrix}$ is arbitrarily near this action if n is large, but is not proper. Note that all the above actions are in $\underline{M}_I$, since $GL(2, \mathbb{C})$ has a left-invariant metric.

3 Discrete subgroup of Lie group. Witt's Theorem

Theorem 3 (A. Weil [1]). *Let* Γ *be a discrete group,* G *a connected Lie group,and* $h_o : \Gamma \to G$ *a homomorphism such that* 59

(i) $\ker h_o$ *is finite;*

(ii) $h_o(\Gamma)$ *is discrete;*

(iii) $h_o(\Gamma)^G$ *is compact.*

Then there exists a neighbourhood W of h_o in Hom (Γ, G) (with the finite open topology), such that for any $h \in W$, (i), (ii) and (iii) hold with h_o replaced by h.

Proof. We may identify Hom(Γ, G) with a subspace of $\underline{M}$. Then, since there exists a left invariant metric on G, Hom $(\Gamma, G) \subset \underline{M}_I$. Also, (i) and (ii) imply that $h_o \in \underline{M}_p$; further $\prod_1(G)$ is finitely generated. Hence we may apply Theorem 2 to obtain Theorem 3.

Let G be a Lie group, and X a defferential manifold. By a *differentiable*(one-parameter) *family of actions*of G on X we mean a differentiable mapping $m : \mathbb{R}\chi G\chi X \to X$ such that for each $t \in \mathbb{R}, m_t : (s, x) \to m(t, s, x)$ is an action of G on X. □

Theorem 4. *Let G be a Lie group, and X a connected differentiable manifold such that $\prod_1(X)$ is finitely generated. Suppose given a differentiable family $m : \mathbb{R} \times G \times X \to X$ of actions of G on X such that $m_t \in \underline{M}_I$ for every $t \in \mathbb{R}$, and suppose that m_o is proper and m_o^X compact. Then there exists a neighbourhood W of* 0 *in $\mathbb{R}$, and for each $t \in W$ a differentiable automorphism a_t of X such that*

$$m_t(s, x) = a_t(m_o(s, a_t^{-1}(x)))$$

60 *for every $x \in X, s \in G, t \in W$.*

Proof. In view of Theorem 2, we can find a neighbourhood W_1 of 0 in $\mathbb{R}$, and a compact set A in X, such that the action of G on $W_1 \times X$ defined by $s(t, x) = (t, m_t(s, x))$ is proper, and such that $m_t(G, A) = X$ for every $t \in W_1$. Then there exists a G-invariant Riemannian metric on $W_1 \times X$ (Theorem 2, Chapter 1). Let $p : W_1 \times X \to W_1$ be the natural projection, and let H be the vector-field on $W_1 \times X$ orthogonal to the fibres of p such that $p^{T_H} = \dfrac{d}{dt}$. It is easily seen that H is G-invariant. □

Let the differentiable mapping

$$\varphi : \left\{\tau \in \mathbb{R} |\tau| <\in \right\} x W_2 \times U \to W_1 \times X$$

be the local one-parameter group generated by the vector-field H in a neighbourhood $W_2 \times U$ of $\{0\} \times A$ in $W \times X$. Since H is G-invariant and $G(W_1 \times A) = W_1 \times X, \varphi$ can be extended to a differentiable mapping

$$\varphi : \{\tau \in \mathbb{R} |\tau| <\in\} \times W_2 \times X \to W_1 \times X$$

by means of the equation

$$s\varphi_\tau(t, x) = \varphi_\tau(t, m_t(s, x)), t \in W_2. \qquad (*)$$

Since H projects on the vector- field $\dfrac{d}{dt}$, we have

$$\varphi_\tau(0, x) = (\tau, a_\tau(x))$$

where $a_\tau : X \to X$ is a diffeomorphism. Using the fact that φ_τ $(0, m_0$ $(s, x)) = s\varphi_\tau(o, x)$ (which is $(*)$ with $t = 0$), we see that the $a_t, |t| <\in$, 61
satisfy the conditions of the theorem.

For other applications, we need the following modification of Theorem 1.

Theorem 5. *Let G be a discrete group acting isometrically on a connected locally connected, simply connected metric space X. Let C be a connected compact subset of X, and S a finite subset of $G(C|C)$ such that*

(i) $e \in S$,

(ii) *for any $s, s' \in S, C \cap sC \cap s'C \neq \phi$ implies $s^{-1}s' \in S$,*

(iii) *SC is a neighbourhood of C,*

(iv) *S generates G.*

This $S = G(C|C)$, the action of G on X is proper, and $GC = X$.

Proof. Since C is compact, and S is finite, there exists a neighbour V of C such that $s, s' \in S, C \cap sC \cap s'C = \phi$, imply $V \cap sV \cap s'V = \phi$. Let A be the connected component of $V \cap SC$ which contains C. Since X is locally connected, A is a neighbourhood of C. A and S satisfy the conditions of Theorem 1. In fact, it is clear we need only check the condition 3) of Theorem 1, and for the ϱ of that condition we can take $d(C, X - A)$. Since $C \subset A \subset SC$, the assertions of Theorem 5 follows from Theorem 1 (and Lemma 1). □

Theorem 6 (E. Witt [1]). *Let G be the group generated by the set*
62 $\{r_1, \ldots, r_n\}$ *with the relations* $(r_i r_j)^{p_{ij}} = e, 1 \leq i, j \leq n$, *where the* p_{ij} *are integers satisfying*

$$P_{ii} = 1, p_{ij} = p_{ji} > 1 \text{ if } j \neq i, 1 \leq i, j \leq n.$$

Then G is finite if and only if the matrix $\left(-\cos \dfrac{\Pi}{p_{ij}}\right)$ is positive definite.

Proof. Let $(e_i)_{1 \leq i \leq n}$ denote the canonical basis of $\mathbb{R}^n$, and B the symmetric bilinear form on $\mathbb{R}^n$ defined by $B(e_i, e_j) = -\cos \dfrac{\Pi}{p_{ij}}$. We define the *standard representation* of G in $\mathbb{R}^n$ by setting

$$r_i e_j = e_j - 2B(e_i, e_j)e_i.$$

Clearly, B is invariant under G. □

a) *G is finite $\Rightarrow$ B is positive definite.*

We first prove that B is non-degenerate. Let $N = \left\{x \in \mathbb{R}^{n^n} | B(x, y) = 0 \text{ for every } y \in \mathbb{R}^n\right\}$. Since N is G-stable and G is finite, there exists a G-stable supplement N' to N. Now, for every $i, r_i|N =$ identity, and r_i is not identity on $\mathbb{R}^n$, hence there exists $ay_i \in N'$ such that $r_i y_i \neq y_i$, i.e., $B(e_i, y) \neq 0$. Since $r_i y_i - y_i = -2B(e_i, y)e_i$, we have $e_i \in N'$. Hence $N' = \mathbb{R}^n$, i.e. $N = 0$.

Then prove the positive-definiteness, we consider any non-trivial irreducible G-subspace L of $\mathbb{R}^n$. We see as above that there exists an $e_i \in L$.On the other hand, there exists on L a G-invariant positive definite bilinear form, say B_o, and (Since L is irreducible) $a\lambda \in \mathbb{R}$ such that $B|L = \lambda B_o$. Since $B(e_i, e_i) = 1$, we must have $\lambda > 0$. Hence $B|L$ is positive definite. Since B is non-degenerate, it follows that B **63**
is positive definite.

b) *B positive definite $\Rightarrow$ G is finite*. (The following proof is based on Buisson [1]). Let $C \subset \mathbb{R}^n$ be defined by

$$C = \left\{x \in \mathbb{R}^n \middle| B(x, e_i) \geq 0 \text{ for every } i\right\}.$$

We shall prove the following statements by induction on n.

1) G is finite

2) $GC = \mathbb{R}^n$

3) If $s \in G$ and $c \in C$ are such that $sc \in C$, then $sc = c$, and in fact s belongs to the subgroup of G generated by the r_i belonging to $G(c)$.

If $n = 2, B$ is automatically positive definite, and the above statements are easily verified. Thus let $n \geq 3$, and let us assume that 1), 2) and 3) are true for $n - 1$.

Let $\sum = \left\{x \in \mathbb{R}^n \middle| B(x, x) = 1\right\}$, and let $A = \sum \cap C$. For each i, let G_i be the subgroup of G generated by $r_1, \ldots, r_{i-1}, r_{i+1}, \ldots, r_n$ Note that $N_i = \sum\limits_{j \neq i} \mathbb{R} e_j$ is G_i-stable, and that the representation of G_i in N_i thus obtained is the standard representation of G_i in $\mathbb{R}^{n-1}$. By induction, each G_i is finite, hence the set $S = \bigcup\limits_{1 \leq i \leq n} G_i$ is finite. Clearly $e \in S$, and $S = S^{-1}$. Since G_i operates trivially on the orthogonal complement N'_i of N_i with respect to B, we have $G_i \subset G(A|A)$, hence $S \subset G(A|A)$. Also, S generates G, since $r_i \in S, 1 \leq i \leq n$.

Lemma. *If $a \in A$ and $s \in S$ are such that $sa \in A$, then $sa = a$; in fact s* **64**
belongs to the subgroup G_a of $G(a)$ generated by the $r_i \in G(a)$.

Proof of the lemma. *Let $s \in G_i$,a nd let $a = b + b'$, where $b \in N_i, b' \in N_i'$. Then both b and sb belong to $N_i \cap (C + N_i') \subset C_i$; hence $C_i = \left\{y \in N_i \middle| B(y, e_j) \geq 0 \text{ for every } j \neq i\right\}$. Hence, by induction, s belongs to the subgroup of G_i generated by the r_j which leave b(and hence a) fixed .*

The lemma implies in particular that if $s, s' \in S$ are such that $A \cap sA \cap s'A \neq \phi$, then $s^{-1}s' \in G_a$ for some $a \in A$. But clearly $G_a \subset G_i$ for some i, and then $s^{-1}s' \in G_i \subset S$.

We prove finally that SA is a neighbourhood of A in $\sum$. Since, for any $a \in A, G_a \subset S$, it is sufficient to check that G_aA is a neighbourhood of a in $\sim$, or equivalently that G_aC is a neighbourhood of a in $\mathbb{R}^n$.

Let $L = \sum\limits_{r_j \in G(a)} \mathbb{R}e_j$, and L' its orthogonal complement with respect to B. Clearly $a \in L'$, and there exists a neighbourhood V of a in $\mathbb{R}^n$ such that

$$V \cap C = V \cap \left\{x \in \mathbb{R}^n \middle| B(x, e_i) \geq 0 \text{ for all } e_i \in L\right\}.$$

Then if

$$C_a = \left\{y \in L | B(y, e_i) \geq 0 \text{ for all } e_i \in L\right\},$$

we have $V \cap C = V \cap (C_a + L')$. Assuming as we may that V is G_a-stable,
65 we have therefore

$$G_a(V \cap C) = V \cap (G_aC_a + L').$$

By induction, $G_aC_a = L$, hence $G_a(V \cap C) = V$, and G_aC is a neighbourhood of a.

Now, for the action of G on $\sum$, all the conditions of Theorem 5 are satisfied for A and S; note that A is connected and $\sum$ simply connected. Thus the action of G on $\sum$ is proper. Since $\sum$ is compact, this means that G finite. Moreover, since $GA = \sum$, we have $GC = \mathbb{R}^n$. This proves the statements 1) and 2); 3) follows from the lemma since $S = G(A|A)$. Hence the proof of the theorem is complete.

Remark. The proof of Theorem 6 shows that show when $\left(-\cos\frac{\Pi}{p_ij}\right)$ is positive definite, the standard representation of G in $\mathbb{R}^n$ is faithful.

Chapter 5

For proper action a discrete group Γ on a space with compact orbit space Γ/X, there are rather strong connections between the topological properties of X and the properties of Γ. The theory of ends, due to Freundenthal [1] and Hopf [1] is the most conspicuous example of such a connection. **66**

1

Let X be a connected topological space. We denote by $\mathfrak{L}$ the set of all sequences (a_i) of connected sets in X such that

(i) for every $i, a_i \neq \phi$,

(ii) $a_i \supset a_{i+1}$ for all i,

(iii) each a_i has compact boundary,

(iv) for every compact set K in X , there exists an i such that $a_i \cap K = \phi$.

For (a_i), $(b_i) \in \mathfrak{L}$, we write $(a_i) \sim (b_i)$ if for every i there exists a j such that $a_i \supset b_j$. The relation $\sim$ is an equivalence relation $\mathfrak{L}$. Indeed we need only check that it is symmetric. Let $(a_i) \sim (b_i)$. For any i, there exists a j such that $a_j \cap \partial b_i = \phi$. Since a_j is connected and $a_j \not\subset X - b_i$, it follows that $a_j \subset b_i$.

An equivalence class of $\mathfrak{L}$ with respect to the relation $\sim$ is called an *end* of X. The set of all ends of X is denoted by $\mathscr{E}(X)$.

Remark. For $(a_i), (b_i) \in \mathfrak{L}$ with $(a_i)\chi(b_i)$, there exists a k such that $a_k \cap b_k = \phi$. In fact, there exists an i such that, for every $j, a_i \not\supset b_j$. On
67 the other hand, for a sufficiently large j, we have $a_i \cap \partial b_j = \phi$. Then for $k > i, j$ we have $a_k \cap b_k = \phi$.

Let $a \in \mathscr{E}(X)$, and let $(a_i) \in \mathfrak{L}$ represent a. By a *neighbourhood* of a we mean any subset of X which contains a_i for some i. If V is a neighbourhood of a, it it clear that, for *any* (a_i) representing $a, V \supset a_i$ for some i.

We also need the notion of *ends* of graphs. Let X be a connected graph (see Chapter 3, §2). For any $A \subset X$, we define the *boundary* of A, denoted by ∂A, as the set

$$\left\{x \in X \middle| \Sigma(x) \cap A \neq \phi, \Sigma(x) \cap (X - A) \neq \phi\right\}.$$

It is easily seen that if $C \subset X$ is connected and $C \cap \partial A = \phi$, then either $C \subset A$ or $C \subset X - A$. Now let $\mathfrak{L}$ be the set all sequences (a_i) of connected subsets of X such that

(i) $a_i \neq \phi$ for every i,

(ii) $a_i \supset a_{i+1}$ for every i,

(iii) ∂a_i is finite for every i,

(iv) $\bigcap_i a_i = \phi$.

We define the equivalence relation $\sim$ in $\mathfrak{L}$ as in the topological case, and the quotient set is the set of *ends* of X, denoted by $\mathscr{E}(X)$. The *neighbourhoods* of points of $\mathscr{E}(X)$ are defined as in the topological case.

We note that a group which acts as a group of automorphisms on a connected space (or graph) X also acts on $\mathscr{E}(X)$ in a natural way.

The following theorem will enable us to speak of the "set of ends" of any finitely generated group.

68 **Theorem 1.** *Let X and Y be connected countable graphs finite at each point, and let $f : X \to Y$ be a homomorphism. Suppose that*

(1) *for every $y \in Y$, $f^{-1}(y)$ is finite.*

Then there exists a unique map $f^{\varepsilon}; \mathscr{E}(X) \to \mathscr{E}(Y)$ such that, for any $a \in \mathscr{E}(X)$ and any neighbourhood V of $f^{\varepsilon}(a)$, $f^{-1}(V)$ is a neighbourhood of a. If we further suppose that

(2) *f is surjective, and for each connected $C' \subset Y$ with C' finite, there exists a finite $H \subset Y$ such that $H \supset \partial(f(C))$ for every connected component C of $f^{-1}(C')$, then f^{ε} is surjective. Finally, if we suppose in addition that*

(3) *for every $y \in Y$, $f^{-1}(y)$ is connected, then f^{ε} is bijective.*

Proof. Let $K_1 \subset K_2 \subset \cdots$ be finite subsets of Y such that $\bigcup K_i = Y$. Let $a \in \mathscr{E}(X)$, and let $(a_i) \in \mathfrak{L}(X)$ represent a. Since, by (1), each $f^{-1}(K_i)$ is finite, there exists a $j(i)$ such that $f^{-1}(K_i) \cap a_{j(i)} = \phi$; we assume that $j(i)$ is the least integer with this property. Let b_i be the connected component of $f(a_{j(i)})$ in $Y - K_i$. We assert that $(b_i) \in \mathfrak{L}(Y)$. It is clear $b_i \neq \phi$ and $b_{i+1} \subset b_i$ for every i. And since $b_i \subset Y - K_i, \cap b_i = \phi$. Also, b_i being a connected component of $Y - K_i, \partial b_i \subset \partial(Y - K_i) = \partial K_i$ which is finite since Y is finite at each point. Hence $(b_i) \in \mathscr{L}(Y)$. Let b be the end of Y defined by (b_i). We set $f^{\varepsilon}(a) = b$. It is easily checked that f^{ε} is a well-defined map from $\mathscr{E}(X)$ to $\mathscr{E}(Y)$. □

Now let V be any neighbourhood of $b = f^{\varepsilon}(a)$. Then $V \supset b_i \supset f(a_{j(i)})$ for some i. Thus $f^{-1}(V) \supset a_{j(i)}$, and hence is a neighbourhood **69**
of a. Suppose $f^{\varepsilon_1} : \mathscr{E}(X) \to \mathscr{E}(Y)$ is any map having this property. We assert that $f_1^{\varepsilon} = f^{\varepsilon}$. Suppose in fact that $f_1^{\varepsilon}(a) \neq f^{\varepsilon}(a)$ for some $a \in \mathscr{E}(X)$. Let V, V_1 be neighbourhoods of $f^{\varepsilon}(a)$, $f_1^{\varepsilon}(a)$ respectively such that $V \cap V_1 = \phi$. Then $f^{-1}(V) \cap f^{-1}(V_1) = \phi$, contradicting the assumption that $f^{-1}(V)$, $f^{-1}(V_1)$ are neighbourhoods of a.

We now assume (2), and prove that f^{ε} is surjective. Let $b \in \mathscr{E}(Y)$ and let (b_i) represent b. For every i, we choose a finite subset H_i of Y such that $H_i \supset \partial(f(C_i))$ for every connected component C_i of $f^{-1}(b_i)$. Also let $j(i)$ be the least integer such that $H_i \cap b_{j(i)} = \phi$.

Let a_1 be any connected component of $f^{-1}(b_1)$ which meets f^{-1} $(b_{j(1)})$. Since $\partial(f(a_1)) \subset H_1$, we have $\partial(f(a_1)) \cap b_{j(1)} = \phi$.

Also, $f(a_1) \cap b_{j(1)} \neq \phi$, since $a_1 \cap f^{-1}(b_{j(1)}) \neq \phi$. Hence $f(a_1) \supset b_{j(1)'}$ i.e., $f(a_1)$ is a neighbourhood of b.

Assume inductively that we have a sequence $a_1 \supset a_2 \supset \cdots \supset a_n$ of subsets of X such that each a_i is a connected component of $f^{-1}(b_i)$ and $f(a_i)$ is a neighbourhood of b. Then we take for a_{n+1} any connected component of $f^{-1}(b_{n+1})$ which meets $a_n \cap f^{-1}(b_{j(n+1)})$; such a connected component exists since $f(a_n), b_{n+1}$ and $b_{j(n+1)}$ are all neighbourhoods of b so that $f(a_n) \cap b_{n+1} \cap b_{j(n+1)} \neq \phi$. It can be verified as in the case of a_1 that $f(a_{n+1}) \supset b_{j(n+1)}$ and hence is a neighbourhood of b. It is also clear that $a_{n+1} \subset a_n$. Since $\partial a_i \subset (f^{-1}(b_i)) \subset f^{-1}(\partial b_i), \partial a_i$ is finite for every
70 i. Also $\bigcap_{a_i} = \phi$. Thus the sequence (a_i) defines an end a in X. We have $f^{\varepsilon}(a) = b$, since every neighbourhood of $f^{\in}(a)$ is also a neighbourhood of b. Hence f^{ε} is surjective.

With the same assumptions, we assert that for any $a \in \mathscr{E}(X)$ and any neighbourhood U of a, $f(U)$ is a neighbourhood of $f^{\varepsilon}(a)$. Let $b = f^{\varepsilon}(a)$, and let (a_i),(b_i) represent a and b respectively. Since, for every i, $f^{-1}(b_i)$ is a neighbourhood of a, there exists a $j(i)$ such that $a_{j(i)} \subset f^{-1}(b_i)$. Let a'_ibe the connected component of $f^{-1}(b_i)$ which contains $a_{j(i)}$. Clearly, $(a'_i) \varepsilon \mathscr{L}(X)$. Since $a'_i \supset a'_{j(i)}$, it follows that $(a'_i) \sim (a_i)$, i.e. (a'_i) represents a. We now assert that $(f(a'_i)) \in \mathscr{L}(Y)$ and represents b. In fact, $(f(a'_i)) \in \mathscr{L}(Y)$ since, by (2),$\partial f(a'_i)$ is finite, and the other conditions are clearly satisfied. Since $f(a'_i) \subset b_i$, we have $(f(a'_i)) \sim (b_i)$. Thus every $f(a'_i)$ is a neighbourhood of b; it follows that $f(U)$is a neighbourhood of b.

Finally, we assume in addition that (3) holds and prove that f is also injective. Let $a, a' \in \mathscr{E}(X), a \neq a'$. Let V, V' be neighbourhoods of a, a' such that $V \cap V' = V \cap \partial V' = \phi$. Then $f(V) \cap f(V') = \phi$, Since $f(V)$, $f(V')$ are neighbourhoods of $f^{\varepsilon}(a)$, $f^{\varepsilon}(a')$ respectively, we must have $f^{\varepsilon}(a) \neq f^{\varepsilon}(a')$, and Theorem 1 is proved.

Let G be a (discrete) group. Let S be a set of generators for G such that $e \varepsilon S$, and $S = S^{-1}$. Then we know that S defines a left invariant connected graph structure $\sum_S$ on G, given by $\mathscr{E}_S(x) = xS, x \in G$, We denote by $\mathscr{E}_S(G)$ the set of ends of $(G, \sum_S)$.

71 **Theorem 2.** *Let G be a finitely generated group, and let S, S' be two finite symmetric sets of generators of G which contain e. Then there is*

a unique natural bijection $\varphi_{S,S'} : \mathscr{E}_S(G) \to \mathscr{E}'_S(G)$ *such that, for any* $a \in \mathscr{E}_S(G)$*, any neighbourhood of a is also a neighbourhood* $\varphi_{S,S'}(a)$*.*

Proof. The uniqueness of $\varphi_{S,S'}$, is obvious. To find $\varphi_{S,S'}$, we first assume that $S \subset S'$. Then the identity mapping of G is a graph homomorphism $\varphi : (G, \sum_s) \to (G, \sum'_S)$. We assert that the conditions of Theorem are satisfied for φ. In fact, we need only verify condition (2). Thus let C' be an S'-connected set with $\partial'_S C'$ finite. Let n an integer such that $S' \subset S^n$. We take $H = \partial'_S C'.S^r$, and claim that for any S-connected component C of $C', \partial_{S'} C \subset H$. In fact let $x \in \partial_{S'} C$. Then $\partial_{S'} xS' \cap C \neq \phi \neq xS' \cap (G - C).xS^r$ is S-connected, hence S'- connected. Since $xS^r \cap C \neq \phi$, we must have $xS^r \not\subset C'$, for otherwise $xS^r \subset C$, contradicting $xS' \cap (G - C) \neq \phi$. Hence $xS^r \cap \partial_{S'}, C' \neq \phi$, i.e., $x \in H$. □

Hence $\varphi_{S,S'}$ is the φ^ε of Theorem 1.

If $S \not\subset S''$, let $S'' = S \cup S'$, then we can take $\varphi_{S,S'} = \varphi^{-1}_{S'',S'} \circ \varphi_{S,S''}$.

In view of the above theorem, ends and their neighbourhoods are intrinsically defined for finitely generated groups.

Theorem 3. *Let G be a discrete group, operating properly on a connnected, locally connected locally compact space X such that* ${}_G\backslash^X$ *is compact (consequently G is finitely generated). Then there exists a unique* 72
map $f : \mathscr{E}(G) \to \mathscr{E}(X)$ *such that for* $a \in \mathscr{E}(G)$ *and any neighbourhood* V *of* $f(a), G(V|\{x\})$ *is a neighbourhood of a for any* $x \in X$*. Moreover,* f *is bijective, and commutes with the operation of G.*

Proof. We first prove the uniqueness. Let f_1, f_2 be two maps $\mathscr{E}(G) \to \mathscr{E}(X)$ having the properties stated in the theorem. Let $a \in \mathscr{E}(G)$, and $f_i(a) = b_i$; let V_i be any neighbourhood of $b_i (i = 1, 2)$. Then, for any $x \in X, G(V_1|\{x\}) \cap G(V_2|\{x\})$ is a neighbourhood of a, and hence nonempty. Hence $V_1 \cap V_2 \neq \phi$. It follows that $b_1 = b_2$. Hence $f_1 = f_2$. □

We now prove the existence of f. There exists a compact connected subset K of X such that $GK = X$. Let $S = G(K|K)$. Then $S = S^{-1}$is finite, contains e, and generates G; and SK is a neighbourhood of K. We put on G the graph structure defined by S.

Let $a \in \mathscr{E}(G)$, and let (a_i) represent a. Let $b_i = a_i K$. We want to prove that $(b_i) \varepsilon \mathscr{L}(X)$. Clearly, $b_i \supset b_{i+1}$, and each b_i is connected. Also, for any compact set H in $X, G(H|K)$ is finite, hence $a_i \cap G(K'|K) = \phi$ for all large i, i.e., $b_i \cap K' = \phi$ for all large i. Now, for any $t \in a_i - \partial a_i$, we have $tS \subset a_i$, hence $tK \subset tSK \subset b_i$; since SK is a neighbourhood of K, we have $tK \subset \operatorname{Int} b_i$. Since $(tK)_{t \in a_i}$ is locally finite . $b_i = a_i K$ is closed , hence it follows that $\partial b_i \subset \partial a_i K$. Since ∂a_i is finite, we have finally that ∂b_i is compact. Hence $(b_i) \in \mathscr{L}(X)$.

73 Let b denote the end defined by (b_i). We set $f(a) = b$. Clearly $f : \mathscr{E}(G) \to \mathscr{E}(X)$ is then well defined. Now let V be any neighbourhood of $b = f(a)$, and let $x \varepsilon X$. Since S generates G,and $GK = X$, there exists an integer n such that $x \in S^n K$. It is easily seen that $(a_j S^n K)$ represents b. Thus $V \supset a_j S^n K$ for some j. Hence $a_j \subset G(V|\{x\}|)$, i.e., $G(V|\{x\})$ is a neighbourhood of a.

We now prove that f is bijective. Let $b \in \mathscr{E}(X)$, and let (b_i) represent b. We set $a_i = G(b_i|K)$. Clearly $a_i \neq \phi, a_i \supset a_{i+1}$ and $\cap a_i = \phi$. Further, since K and b_i are connected, and the family $(gK)_{g \in G}$ is locally finite, we see easily that the a_i are connected. Now, if $tS \cap a_i = \phi$, we have $t \in G(b_i|K)S = G(b_i|SK)$. Similarly $tS \cap (G - a_i) \neq \phi$implies $t \in G((X - b_i)|SK)$. Since SK is connected, it follows that $\partial a_i \subset G(\partial b_i|SK)$, and hence is finite. Thus, (a_i) defines an end $f'(b)$ of G. Clearly $b \rightsquigarrow f'(b)$ is a well - defined map of $\mathscr{E}(X)$ into $\mathscr{E}(G)$, and f' is easily seen to be the inverse of f.

Finally, for any $t \in G, t^{-1} o f o t : \mathscr{E}(G) \to \mathscr{E}(X)$, also has the properties mentioned in the theorem, hence we have, by the uniqueness, $t^{-1} o f o t = f$, i.e. $f o t = t o f$. This completes the proof of the theorem.

2

Lemma 1. *Let X be a connected graph, and let A, B, H be connected subsets such that $\partial A \subset H$ and $\partial B \subset A - H$. Then either $B \subset A$ or $A \cup B = X$.*

Proof. Since $\partial B \cap H = \phi$ and H is connected we have either $H \subset B$ or $H \subset X - B$. If $H \subset B$, we have $\partial(A \cup B) \subset \partial A \cup \partial B \subset H \cup A \subset B \cap A$.

74 Since X is connected, we have $A \cap B = X$ or ϕ. □

If $H \subset X-B$, we have $\partial A \cap B = \phi$. Hence either $B \subset A$ or $B \subset X-A$. But since $\partial B \subset A$, we must have $B \subset A$ or $B \subset X-A$. But since $\partial B \subset A$, we must have $B \subset A$.

Theorem 4. *Let G be a finitely generated group, and let $a^{(1)}, a^{(2)}, a^{(3)}$, their distinct ends of G. Then for every neighbourhood $V of a^{(3)}$, there exists a $t \in G$ such that V is a neighbourhood of at least of $ta^{(1)}$, $ta^{(2)}$, $ta^{(3)}$.*

Proof. Let S be a finite set of generators for G defining a graph structure. Let $a_i^{(j)}$ represent $a^{(j)}$, $j = 1, 2, 3$. We may assume that, for every i, the $a^{(j)}$, $j = 1, 2, 3$, are mutually disjoint. Now let V be a neighbourhood of $a^{(3)}$, say $V \supset a_i^{(3)}$. Let n be an integer such that $S^n \supset \bigcup\limits_j a_i^{(j)}$. Take any $t \in a_i^{(3)} - S^{2n}$. Since tS^n is connected, and since $tS^n \cap \partial a_i^{(3)} \subset tS^n \cap S^n = \phi$, it follows that $tS^n \cap a_i^{(3)} - S^n$. Hence Lemma 1 can be applied, with $A = a_i^{(3)}, H = S^n$, and $B = ta_i^{(j)}$. Since the $ta_i^{(j)}$, $j = 1, 2, 3$ are mutually disjoint, we must have $ta_i^{(j)} \subset a_i^{(3)}$ for at least two the $j's$. This proves the theorem. □

Corollary 1. *Let G be a finitely generated group. If G has three distinct ends, then every neighbourhood of an end of G is the neighbourhood of two distinct ends; in particular, the set of ends is finite.*

Corollary 2. *If the finitely generated group G has two invariant ends, it has no other ends.*

Proof. Let a, b be two invariant ends of G. If possible let c be another end of G. By Theorem 3, there exists, for every neighbourhood V of 75
$c, at \in G$ such that V is a neighbourhood of at least one of $ta = a, tb = b$,. Hence $c = a$ or b, a contradiction. □

Remark. It is known whether a group with one invariant end can have infinitely may ends (Freundenthal [1]).

Examples. 1) The group $\mathbb{Z}$ has two invariant ends.

2) The group $\mathbb{Z}X\mathbb{Z}$ has just one end.

3) The free product of the cyclic group of order 2 with the cyclic group of order 3 (which is isomorphic to the classical modular group) has infinitely many ends, none of which is invariant. This example shows incidentally that in Theorem 3, the assumption that ${}_G\backslash^X$ is compact cannot be dropped.

Chapter 6

Discrete linear groups acting properly on convex open cones in real vector spaces are of special interest for the applications. In that case, the existence of a stable lattice or, more generally, of certain stable discrete subsets gives rise to special methods of constructing fundamental domains. The material here is due to Koecher [1] and Siegel [1]. **76**

1

Let E be a real vector space, of dimension $n \geq 2$. A subset Ω of E is called a *cone* if $t\Omega \subset \Omega$ for every real $t > 0$. The cone Ω^* in the dual E^* of E, defined by

$$\Omega^* = \left\{ X^* \in E^* \middle| < x^*, x >> 0 \text{ for all } x \in \Omega - \{0\} \right\}$$

is called the *dual cone* of Ω.Ω^* is always open in E^*; in fact, if $\sum$ denotes the unit sphere in E (with respect to some norm on E), we have $\Omega^* = \left\{ X^* \in E^* \middle| < x^*, x >> 0 \text{ for all } x \in \bar{\Omega} \cap \sum - \right\}$.

Assume E to be a Euclidean vector space with scalar product $\langle , \rangle$. If, under the canonical identification of E^* with E, we have $\Omega^* = \Omega$, we say that Ω is a *self- dual* cone (or a *positivity domain*). Clearly, Ω is self - dual if and only if $\Omega = \left\{ x \in E \middle| \langle x, y \rangle > 0 \text{ for all } y \in \bar{\Omega} - \{0\} \right\}$.

Examples. (i) In $\mathbb{R}^n$(with the usual Euclidean structure),

$$\Omega = \left\{(t_1, \ldots, t_n)\Big|t_i > 0 \text{ for all } i\right\}$$

and
$$\Omega\left\{(t_1, \ldots, t_n)|t_1^2 + \cdots + t_{n-1}^2 < t_n^2, t_n > 0\right\}$$

are selfdual cones.

77 (ii) Let E be the vector of real $n \times n$ symmetric matrices, with the scalar product $\langle A, B\rangle = T_r(AB)$. Then the set Ω of positive definite matrices of E is a selfdual cone. To see this, we note first that $\Omega^* \subset \Omega$. In fact, let $A \in \Omega^*$, and let $e_1, \ldots, e_n$ be an orthonormal basis of $\mathbb{R}^n$ such that $Ae_i = \lambda_i e_i, \lambda_i \in \mathbb{R}, i = 1. \ldots, n$. Let $P_i \in E$ be defined by $P_i e_j = \delta_{ij} e_i$. Then $P_i \in \overline{\Omega} - \{0\}$, and $\langle A, P_i\rangle = \lambda_i$. Hence $\lambda_i > 0$ for all i, thus $A \in \Omega$. Conversely, let $A \in \Omega$, and $B \in \overline{\Omega} - \{0\}$. Let $\sqrt{A} \in \Omega$ and $\sqrt{B} \in \Omega - \{0\}$ be the positive square roots of A and B respectively. Then

$$\begin{aligned}\langle A, B\rangle = T_r(AB) &= T_r(\sqrt{A}\sqrt{A}\sqrt{B}\sqrt{B})\\ &= T_r(\sqrt{B}\sqrt{A}\sqrt{A}\sqrt{B})\\ &= T_r((\sqrt{A}\sqrt{B})'(\sqrt{A}\sqrt{B})) > 0.\end{aligned}$$

2

We now state elementary properties of cones and their duals.

(i) For any cone Ω, Ω^* is convex.

(ii) If the cone Ω in E contains a basis of E the $\overline{\Omega^*}$ is a non-degenerate convex cone (A convex set *non- degenerate* if it does not contain any straight line). In fact, let $x^*, y^* \in E^*$, and suppose $x^* + ty^* \in \Omega^*$ for every $t \in \mathbb{R}$. Then, for every $z \in \overline{\Omega} - \{0\}$, we have $0 \leq \langle x^* + ty^*, z\rangle = \langle x^*, z\rangle + t\langle y^*, z\rangle$, for every $t \in \mathbb{R}$.

78 Hence $\langle y^*, z\rangle = 0$ for every $z \in \overline{\Omega} - \{0\}$, hence $y^* = 0$.

Using (i) and (ii) we have (iii) If Ω contains a basis of E, then

$$x^* \in \overline{\Omega}^*, -x^* \in \overline{\Omega}^* \text{ imply } x^* = 0.$$

Lemma 1. *Given any compact subset K of Ω^*, we have $\rho(K) > 0$ such that $\langle x^*, y\rangle \geq \rho(K)|y|$ for every $x^* \in K$ and $y \in \overline{\Omega}$. Here $||$ denotes some norm on E.*

Proof. Let $\sum$ be the unit sphere in E. Then the function $(x^*, y) \rightsquigarrow \langle x^*, y\rangle$ on $Kx(\overline{\Omega} \cap \sum)$ is continuous and > 0. We can take $\rho(K)$ to be the infimum of this function. ◻

Remark. If Ω is open, the statement analogous to that of Lemma 1, with the roles of Ω and Ω^* interchanged, is also true; the proof is the same.

3

Let $\Omega \subset E$ be a non-degenerate cone; we then have $\Omega^* \neq \phi$. Let D be a discrete subset of E contained in $\overline{\Omega} - \{0\}$. For any $x^* \in \Omega^*$, we define

$$\mu(x^*) = \inf_{d \in D} \langle x^*, d\rangle.$$

We see by lemma 1 that $\mu(x^*) < 0$, and that the set

$$M(x^*) = \left\{d \in D \middle| \langle x^*, d\rangle = \mu(x^*)\right\}$$

is non-empty and finite.

Lemma 2. *For any $x^* \in \Omega^*$ and $\in > 0$, there exists a neighbourhood $U \subset \Omega^*$ of x^* such that, for any $y^* \in U$, $\left|\mu(y^*) - \mu(x^*)\right| < \in$ and $M(Y^*) \subset M(x^*)$.* 79

Proof. Let $K \subset \Omega^*$ be a compact neighbourhood of x^*. Let $\rho = \rho(K)$ be as in Lemma 1. Let $D' = \left\{ d \in D \middle| |d| \leq (\mu(x^*) + \in)/\rho \right\}$. Clearly D' is finite, and for any $y^* \in K$ and $d \in D - D'$, we have

$$< y^*, d > \mu(x^*) + \in .$$

In particular, we have $M(x^*) \subset D'$. Clearly there exists $a > 0$ such that

$$\langle x^*, d\rangle > \mu(x^*) + \frac{a}{2}$$

for $d \in D' - M(x^*)$. (We may suppose that $\frac{a}{2} < \in$.) Thus, there exists a neighbourhood $V_i \subset \Omega^*$ of x^* such that $y^* \in V_i$ implies

$$\langle y^*, d\rangle > \mu(x^*) + \frac{a}{2}, d \in D' - M(x^*).$$

Finally there exists neighbourhood $V_2 \subset \Omega^*$ of x^* such that $y^* \in V_2$ implies

$$|\langle y^*, d > -\mu(x^*)\big| < \frac{a}{2}; d \in M(x^*).$$

Clearly, $U = K \cap V_1 \cap V_2$ satisfies conditions of the lemma. □

A point x^* of Ω^* is called *perfect* it $M(x^*)$ contains a basis of E. Since, for any $\lambda > 0, M(\lambda x^*) = M(x^*)$, we shall assume that , for a perfect point $x^*, \mu(x^*) = 1$.

80 **Lemma 3.** *Let $y^* \in \Omega^*$ be not perfect, and let $M \subset M(y^*), M \neq \phi$. Then, for every $x^* \in E^*$ with $\langle x^*, M(y^*)\rangle \geq 0$ and $\langle x^*, M\rangle = 0$, we have either*

(i) $\mu(y^* + tx^*) = \mu(y^*)$ *for every $t \geq 0$ such that $y^* + tx^* \in \Omega^*$, or*

(ii) *there exists $t_o > 0$ such that*

(a) $y^* + t_o x^* \in \Omega^*$

(b) $\mu(y^* + t_o x^*) = \mu(y^*)$

(c) $M \subset M(y^* + t_o x^*)$

(d) $\dim M(y^* + t_o x^*) > \dim M$,

(where, for any subset S of E, $\dim\ S$ denotes the dimensions of the subspace generated by S).

Proof. Suppose that (i) does not hold. Since, for any $d \in M$ and any $t \in \mathbb{R}$, we have $\langle y^* + tx^*, d\rangle = \mu(y^*)$, it follows that $\mu(y^* + tx^*) \leq \mu(y^*)$ if $y^* + tx^* \in \Omega^*$. Hence there exists $\theta > 0$ such that $y^* + \theta x^* \in \Omega^*$, and

$\mu(y^* + \theta x^*) < \mu(y^*)$. Let $\mathfrak{B} = \left\{ d \in D \left\langle x^*, d >< 0 \right. \right\}$. $\mathfrak{B}$ is non-empty, since $\mathfrak{B} \supset M(y^* + \theta x^*)$. For $d \in \mathfrak{B}$, we set

$$\varphi(d) = (\mu(y^*) - \langle y^*, d\rangle)/\langle x^*, d\rangle\cdot$$

Clearly, $\varphi(d) > 0$, and for $d \in M(y^* + \theta x^*)$, we have $\varphi(d) < \theta$. On other hand, if $\varphi(d) < \theta$, we have $\mu(y^*) - \langle y^* + \theta x^*, d\rangle > 0$. Hence if $\rho = \rho(K)$ of Lemma 1 with $K = \left\{ y^* + x^* \right\}$, we have $|d| \leq \mu(y^*)/\rho$. Hence $\varphi < \theta$ only on a (non-empty) finite subset of $\mathfrak{B}$. Hence φ attaints its infimum in $\mathfrak{B}$, let $t_o = \inf\limits_{d \in \mathfrak{B}} \varphi(d)$, and let $\varphi(d_o) = t_0$. We assert t_o has the **81**
properties stated in (ii) of the lemma. □

Since Ω^* is convex, and $0 < t_0 < \theta$, we have $y^* + t_o x^* \in \Omega^*$. We observe that for $d \in M, \langle y^* + t_o x^*, d\rangle = \mu(y^*)$. Hence ($b$) and ($c$) of ($ii$) will be proved if we show that

$$\langle y^* + t_o x^* \rangle \geq \mu(y^*) \tag{I}$$

for every $d \in D$. This is obvious for $d \in D - \mathfrak{B}$. For $d \in \mathfrak{B}$, we have $\varphi(d) \geq t_o$, i.e., $\mu(y^*) - \langle y^*, d\rangle \leq t_o < x^*, d\rangle$. Hence (I) follows, and (b), (c) are proved. Finally, it is clear that $d_o \in M(y^* + t_o x^*)$; since $\langle x^*, M\rangle = 0$, while $< x^*, d_\circ >< 0$, (d) follows.

From now on, we shall suppose that Ω is an open non-degenerate convex cone; we then have $(\Omega^*)^* = \Omega$. For any finite subset S of Ω, the set $PS = \left\{ \sum t_i s_i | s_i \in S, t_i \geq 0 \right\}$ is called the *pyramid on S*. If $x^* \in \Omega^*$ is a perfect point, then $PM(x^*)$ is called a *perfect pyramid*.

Lemma 4. *For any $x^*, y^* \in \Omega^*$, we have*

$$PM(x^*) \cap PM(y^*) = P(M(x^*) \cap M(y^*))$$

and

$$\langle \mu(x^*)y^* - \mu(y^*)x^*, PM(x^*) \cap PM(y^*)\rangle = 0.$$

Proof. Obviously, $P(M(x^*) \cap M(y^*)) \subset PM(x^*) \cap PM(y^*)$. Conversely, let $z \in PM(x^*) \cap PM(y^*)$. Let $z = \sum a_i x_i; x_i \in M(x^*), a_i > 0$. Similarly, let $z = \sum b_j y_j; y_j \in M(y^*), b_j > 0$. We have **82**

$$\begin{aligned}\langle x^*, z\rangle &= \sum_i a_i\langle x^*, x_i\rangle = \mu(x^*)\sum_i a_i\\ &= \sum_j b_j\langle x^*, y_j\rangle \geq (x^*)\sum_j b_j\end{aligned}$$

Since $\mu(x^*) \neq 0$, we have $\sum a_i \geq \sum b_j$, hence, by symmetry $\sum a_i = \sum b_j$. It follows that $\langle x^*, y_j\rangle = \mu(x^*)$, i.e. $y_j \in M(x^*)$ for every j. Similarly, $x_i \in M(y^*)$ for every i, i.e. $z \in PM(x^*) \cap PM(y^*)$. The first assertion of the lemma is therefore proved. The second is then clear. $\square$

4

Definition. *The discrete set D in E (contained in $\overline{\Omega} - \{0\}$) is said to satisfy the density condition if, for each $z^* \in \overline{\Omega^*} - \Omega^*, \mu(x^*) \to 0$ as $x^*(\in \Omega^*) \to z^*$.*

Examples. (i) Let $\Omega C \mathbb{R}^2$ be the (self - dual) cone defined by $\Omega = \left\{(t_1, t_2)\middle| t_1, t_2 > 0\right\}$. Then $D = \{(1,0) \cup (0,1)\}$ also satisfies the density condition. The set $D = \left\{(\exp n, \exp(-n)) \middle| n \in \mathbb{Z}\right\}$ also satisfies the density condition.

(ii) Let Ω be the (self - dual) cone of positive definite matrices in the space E of real $n \times n$ symmetric matrices. Let D be the set $\{UU' | U \in \mathbb{Z}^n, U \neq 0\}$. Clearly D is a discrete set in E, and $D \subset \Omega - \{0\}$. D satisfies the density condition. We shall prove in fact that for any $A \in \Omega$,

$$\mu(A)^n \leq (2^{2n}.\det A)/_{\rho_n^2},$$

83 where ρ_n is the volume of the unit ball in $\mathbb{R}^n$. Let $A \in \Omega$, and $U \in \mathbb{Z}^n - \{0\}$. We have

$$\mu(A) \leq Tr(AUU') = TrU'AU = U'BBU = |BU|^2,$$

where $B \in \Omega$ is the square root of A. Thus, the convex symmetric set $C = \left\{x \in \mathbb{R}^n \middle| |Bx|^2 < \rho(A)\right\}$ does not contain any non-zero integral point. Hence, by a theorem of Minkowski, vol $C \leq 2^n$. However, the volume of C is easily seen to be $\rho_n \mu(A)^{n/2}/(\det A)^{\frac{1}{2}}$, and we get the required inequality.

Remark. If D satisfies the density condition, then

$$\overline{\Omega^*} = \left\{x^* \in E^* \middle| \langle x^*, d\rangle \geq 0 \text{ for all } d \in D\right\}. \tag{I}$$

In fact, it is clear that Ω is contained in the right hand side of (I). Now let $x^* \in E^* - \Omega^*$. Then, for any $y^* \in \Omega^*$, there exists $t_o, 0 < t_o < 1$, such that $t_o x^* + (1 - t_o) y^* \in \overline{\Omega^*} - \Omega^*$. For any $d \in D$, we have

$$\begin{aligned}\langle t_o x^* + (1 - t_0) y^*, d\rangle &= t_o \langle x^*, d\rangle + (1 - t_o)\langle y^*, d\rangle \\ &\geq t_o\langle x^*, d\rangle + (1 - t_o)\mu(y^*).\end{aligned}$$

Since $\mu(tx^* + (1 - t)y^*) \to 0$ as t increases to t_0, it follows that $\langle x^*, d\rangle < 0$ for some $d \in D$, and (I) is proved. However, the density condition is not necessary for (I) to hold.

Lemma 5. *If D satisfies the density condition, then the set $\mathscr{P}$ of perfect points is discrete in E^*.*

Proof. Let (x_i^*) be a sequence in $\mathbb{P}$ converging to $x^* \in E^*$. Clearly **84**
$x^* \in \overline{\Omega^*}$, and in view of the density condition, we must have $x^* \in \Omega^*$. Then $\mu(x^*) = 1$, and $M(x_i^*) \subset M(x^*)$ for large i (Lemma 2). Since x_i^* is perfect, it follows by Lemma 4 that $x^* = x_i^*$. □

Lemma 6. *If D satisfies the density condition, every point of Ω belongs to a perfect pyramid.*

Proof. We first note that, since D satisfies the density condition, the first alternative of Lemma 3 can never hold if $x^* \notin \overline{\Omega^*}$. Hence we see by Lemma 3 that $\mathbb{P} \neq \phi$. Now let $z \in \Omega$, and let y^* be any perfect point. If $z \in PM(y^*)$, there is nothing to prove. Let $z \notin PM(y^*)$. □

Then there exists $x^* \in E^*$ such that $(a)\langle x^*, PM(y^*)\rangle \geq O, (b)\langle x^*, z\rangle < O, (c) x^*$ vanishes on a subset M of $M(y^*)$containing $n - 1$ linearly independent points. On account of $(b), x^* \notin \overline{\Omega^*}$. Hence the second alternative of Lemma 3 holds, and there exists $t_o > O$ such $y^* + t_o x^* \in \Omega^*, \mu(y^* + t_o x^*) = 1, M \subset M(y^* + t_o x^*)$. and $\dim M(y^* + t_o x^*) > \dim M$. Clearly $y_1^* = y^* + t_o x^*$ is perfect. Moreover, $\langle y_1^*, z\rangle < \langle y^* z, \rangle$.

If $z \in PM(y_1^*)$ we are through. Otherwise, we repeat the above procedure with y_1^*, and obtain $y_2^* \in \mathbb{P}$ such that $\langle y_2^*, z\rangle < \langle y_1^*, z\rangle$. This process must terminate after a finite number of steps, since $\mathbb{P}$ is discrete, and since (Remark following Lemma 1) there is a constant $\varrho = \varrho(z)$ such that

$$|y_i^*| \leq \varrho\langle y_i^*, z\rangle < \varrho\langle y^*, z\rangle$$

for any i. We thus obtain a perfect pyramid containing z.

85 **Lemma 7.** *Any compact set K in Ω is met by only finitely many perfect pyramids.*

Proof. Let $x^* \in \mathbb{P}$, and let $y \in K \cap PM(x^*)$. Then if $\varrho(K)$ is as in Lemma 1, we have $\langle x^*, y\rangle \geq \varrho(K)|x^*|$, *i.e.* $|x^*| \leq \dfrac{\langle x^*, y\rangle}{\varrho(K)}$. On the other hand, the convex closure of D does not contain O, and hence there exists $\varrho'(K) > O$ such that for every $x^* \in \mathbb{P}, \langle x^*, y\rangle < \varrho'(K)$ on $K \cap PM(x^*)$. Since $\mathbb{P}$ is discrete, the lemma follows. □

Remark. It follows from the above lemma that, for any $x^* \in \mathbb{P}$, the set $\{y^* \in \mathbb{P} | PM(x^*) \cap PM(y^*) \cap \Omega \neq \phi\}$ is finite: in view of Lemma 4, this set is the set of $y^* \in \mathbb{P}$ such that $PM(y^*) \cap K \neq \phi$, where K is, for instance the (finite) set in Ω consisting of those of the barycentres of the subsets of $M(x^*)$ which lie in Ω.

5

Let Ω be an open non-degenerate convex cone in a real vector space E. Let $G(\Omega) = G$ be the subgroup of $GL(E)$ which maps Ω into itself. Then G is a closed subgroup of $GL(E)$. For any $x \in \Omega, G(x)$ is compact. In fact, the set $\Omega \cap \{x - z | z \in \Omega\}$ is stable under the action of $G(x)$. Since Ω is non-degenerate, this is a bounded open set. Hence $G(x)$ is compact.

$G = G(\Omega)$ also acts on Ω^*: for $s \in G$ and $x^* \in \Omega^*$, we define sx^* by $\langle sx^*, y\rangle = \langle x^*, s^{-1}y\rangle$; this identifies G with $G(\Omega^*)$. Let D be a discrete subset of $\underline{\Omega} - \{O\}$, and let Γ be a subgroup of G such that $\Gamma D = D$. Then clearly $\mu(sx^*) = \mu(x^*)$ and $M(sx^*) = sM(x^*)$ for any $x^* \in \Omega^*$ and
86 $s \in \Gamma$. Thus Γ also acts on the set of perfect points and the set of perfect

pyramids. Note that if $x^* \in \mathbb{P}, \Gamma(x^*) = \{s \in \Gamma | PM(x^*) = sPM(x^*)\}$.

Assume now that D satisfies the density condition. Then for any compact set K in Ω, we can find a finite subset $\mathbb{R}$ of $\mathbb{P}$ such that $K \subset \bigcup_{x^* \in \mathbb{R}} PM(x^*)$ (Lemma 6 and 7), thus

$\Gamma(K|K) \subset \bigcup_{x^*, y^* \in \mathbb{R}} \Gamma(K \cap PM(x^*) | PM(y^*))$. Since, for any $x^* \in \mathbb{P}$, $\left\{ y^* \in \mathbb{P} | K \cap PM(x^*) \cap PM(y^*) \neq \phi \right\}$ is finite, it follows that $\Gamma(K \cap PM(x^*) | PM(y^*))$ is finite for $x^*, y^* \in \mathbb{P}$); hence $\Gamma(K|K)$ is finite. Thus Γ is a discrete subgroup of $G(\Omega)$, and acts properly on Ω.

Remark. It can be proved that $G(\Omega)$ itself acts properly on Ω.

Also there is a natural $G(\Omega)$-invariant Riemannian metric on Ω. For any $x \in \Omega$, we define

$$(N(x))^{-1} = \int_{\Omega^*} \exp(-\langle y^*, x \rangle) dy^*.$$

Then integral is finite, in view of Lemma 1. It is easy to verify that for $s \in G(\Omega), (N(sx))^{-1} = |\det s| (N(x))^{-1}$. The 2-form $-\dfrac{\partial^2 \log N}{\partial x_i \partial x_j} dx_i dx_j$ gives a $G(\Omega)$-invariant Riemannian metric on Ω.

Theorem 1. *Let Ω be an open convex non-degenerate cone in a real vector space $E, D \subset \bar{\Omega} - \{\circ\}$ a discrete subset of E satisfying the density condition, and Γ a discrete subgroup of $G(\Omega)$ such that $\Gamma D = D$. Let $\mathbb{P}(\subset \Omega^*)$ be the set perfect points. Assume that there exists a finite subset* **87**
L of $\mathbb{P}$ such that $\Gamma L = \mathbb{P}$. Then, if

$$A = \Omega \cap \bigcup_{x^* \in L} PM(x^*),$$

we have

(a) $\Gamma A = \Omega$,

(b) $\Gamma(A|A)$ *is finite,*

(c) $\Gamma(A|A)A$ *is a neighbourhood of* A *in* Ω,

(d) Γ *is finitely presentable.*

Proof. Using Lemma 6 and the fact that $\Gamma L = \mathbb{P}$ (and since $sM(x^*) = M(sx^*), x^* \in \mathbb{P}, s \in \Gamma$), it is easy to see that $\Gamma A = \Omega$. The proof of (b) is similar to that of the fact that the action of Γ on Ω is proper; we have only to use the remark following Lemma 7. Since $\Gamma A = \Omega, (c)$ follows. (Note that $\left\{sA|s \in \Gamma\right\}$ is a locally finite family of closed sets in Ω.) Since Ω is convex, it is connected, locally connected and simply connected. Hence the conditions of Theorem 1, Chapter 3 are satisfied, and the assertion (d) follows. □

Remark. The condition in the above theorem that $\Gamma\backslash\mathbb{P}$ be finite is satisfied if we assume the following: there exists a finite subset B of D such that, for every $y^* \in \mathbb{P}$, there exists $s \in \Gamma$ such that convex envelope of $M(sy^*) \cap B$ meets Ω. In fact, let $y^* \in \mathbb{P}$, and let s be as above. Then $b = \dfrac{1}{r} \sum\limits_{a \in M(sy^*) \cap B} a \in \Omega$, where r =number of elements of $M(sy^*) \cap B$. Now,

$$\langle sy^*, b\rangle = \frac{1}{r} \sum \langle sy^*, a\rangle = 1.$$

88 Hence $|sy^*| \leq \dfrac{1}{\varrho_b}$, where $\varrho_b = \varrho(K)$ of Lemma 1 with $K = \{b\}$.

The number of points b is finite. Since $\mathbb{P}$ is discrete, our assertion follows.

6

We now apply the preceding results to the case of $GL(n, \mathbb{Z})$ acting on the space of symmetric positive definite matrices. Thus let E be the vector space of all real $n \times n$ symmetric matrices with the scalar product $\langle A, B\rangle = Tr\,(AB)$, and let Ω be the (self-dual) cone of positive definite matrices in E. $\Gamma = GL(n, \mathbb{Z})$ acts on $\Omega : (S, A) \rightsquigarrow SAS' = S[A]$.

Let $D = \left\{UU'|U \in \mathbb{Z}^n, U \neq \circ\right\}$. We have seen that D satisfies the density condition. It is clear that $\Gamma[D] = D$.

For any $A \in \Omega$, let $\tilde{M}(A) = \left\{U \in \mathbb{Z}^n | UU' \in M(A)\right\}$. For $S \in \Gamma$, we clearly have $\tilde{M}(S[A]) = S^* \tilde{M}(A)$, where $S^* = (S')^{-1}$.

Lemma 8. *If A is perfect, then $\tilde{M}(A)$ contains n linearly independent elements.*

Proof. Let B be any matrix such that $BU = O$ for every $U \in \tilde{M}(A)$. Then $\langle B, UU' \rangle = Tr(BUU') = U'BU = O$ for every $U \in \tilde{M}(A)$, *i.e.*$\langle B, M(A) \rangle = O$. Since A is perfect, this implies $B = O$. □

Lemma 9. *Let A be perfect, and let $U_1, \ldots, U_n \in \tilde{M}(A)$ be linearly independent. Let $C = (U_1, \ldots, U_n)$ be the matrix whose i-th column is U_i. Then $|\det C| \leq 2^n/\varrho_n$, where ϱ_n is the volume of the unit ball in $\mathbb{R}^n$.*

Proof. We have $C'AC \in \Omega$, and the diagonal elements of $C'AC$ are equal to 1. Hence by a known lemma we have $\det C'A \leq 1$, *i.e.*$(\det C)^2 \leq (\det A)^{-1}$. However, we have seen (Example (*ii*), p.82) that $(\det A)^{-1} \leq 2^{2n}/\varrho_n^2$. □ **89**

Lemma 10. *There exists a finite subset L of $\mathbb{Z}^n$ such that, for any $A \in \mathbb{P}$, there exists $S \in \Gamma$ such that $\dim(L \cap S\tilde{M}(A)) = n$.*

Proof. Let $L = \left\{(v_1, \ldots, v_n) \in \mathbb{Z}^n - \{O\} | \circ \leq v_i \leq 2^n/_{\varrho_n}\right\}$. Let $A \in \mathbb{P}$. By Lemma 8, $\tilde{M}(A)$ contains n independent elements $U_1 \ldots, U_n$. □

Now there exists $S \in \Gamma$ such that, for $1 \leq i \leq n$,

$$SU_i = \begin{pmatrix} a_i^1 \\ \vdots \\ a_i^i \\ O \\ \vdots \\ O \end{pmatrix}, O \leq a_i^j < a_i^i \text{ for } 1 \leq j < i$$

The proof of this fact is analogous to the Elementary Divisor Theorem (see van der Waerden [1]). Clearly the SU_i are independent. Since $SU_i \in S\tilde{M}(A) = \tilde{M}(S^*[A])$, we have bu Lemma 9,

$$\det(SU_1, \ldots, SU_n) = a_{11}a_{12} \cdots a_{1n} \leq 2^n/\rho_n.$$

Thus $a_i^j \leq 2^n/_{\varrho_n}, 1 \leq i, j \leq n$. Thus $SU_i \in L$, and the lemma is proved.

Lemma 11. *There exists a finite subset $\mathbb{B}$ of D such that for every $A \in \mathbb{P}$. there exists $S \in \Gamma$ such that the convex envelope of $\mathbb{B} \cap M(S^*[A])$ meets Ω.*

90 *Proof.* Let $\mathbb{B} = \left\{UU' | U \in L\right\}$, where L is as in Lemma 10. Let $A \in \mathbb{P}$. By Lemma 10, there exists $S \in \Gamma$ such that $L \cap S\tilde{M}(A)$ contains n independent elements $V_1, \ldots, V_n$. Then $V_i V_i' \in \mathbb{B} \cap M(S^*[A]), 1 \leq i \leq n$. Clearly $\dfrac{1}{n} \sum V_i V_i'$ is positive definite, and belongs to the convex closure of $\mathbb{B} \cap M(S^*[A])$. □

Remark. The above lemma and the remark following Theorem 1 show that Theorem 1 is applicable to the case of $GL(n, \mathbb{Z})$ acting on the positive definite matrices. It follows in particular that $GL(n, \mathbb{Z})$ is finitely presentable.

7

Let Ω be an open non-degenerate convex cone in a real vector space E. Let G be a subgroup of $G(\Omega)$, and let $\chi : G \to \mathbb{R}^+$ be a homomorphism ($\mathbb{R}^+$denotes the group of real numbers $> O$).

Definition. *A* norm *on Ω (with respect to $\chi : G \to \mathbb{R}^+$) is a continuous map $\gamma : \Omega \to \mathbb{R}^+$ such that*

(i) $\nu(sx) = \chi(s)\nu(x)$ *for* $s \in G, x \in \Omega$,

(ii) $\nu(x) \to \circ$ *as* $x \to \bar{\Omega} - \Omega$

(iii) *for every* $x \in \Omega$*and* $r \geq O$,

$(x + \bar{\Omega}) \cap \left\{x \in \Omega | \nu(x) \leq r\right\}$ *is compact.*

Examples. i) If Ω is the cone of positive definite real $n \times n$ matrices and $G = GL(n, \mathbb{R})$, then $\nu(A) = \det A, A \in \Omega$, is a norm on Ω for $\chi : G \to \mathbb{R}^+$ defined by $\chi(S) = (\det S)^2$.

ii) For any Ω, and $G = G(\Omega)$

$$\nu(x) = \left(\int\limits_{\Omega^*} e^{-\langle x, y^* \rangle} dy^* \right)^{-1}$$

is a norm for $\chi(s) = |\det s|$. **91**

Theorem 2. *Let Ω be an open non-degenerate convex cone in a real vector space E of dimension n. Let L be a lattice in E, and let D be a subset of $L \cap (\overline{\Omega} - \{O\})$ satisfying the density condition. Let Γ be a discrete subgroup of $G(\Omega)$ such that $\Gamma D = D$, and assume that ${}_{\Gamma}\backslash\mathbb{P}$ is finite. Then the subset A of Ω constructed in Theorem 1 has the property: for any norm ν on Ω and any $r > \circ, A \cap L$ contains only finitely many points x with $\nu(x) \leq r$.*

We first prove the following

Lemma 12. *For any $y^* \in \mathbb{P}$, there exists $a \in \Omega$ such that $PM(y^*) \cap \Omega \cap L \subset a + \overline{\Omega}$.*

Proof. We first remark that for any compact set $K \subset \Omega$, there exists $\vartheta \in \Omega$ such that $K \subset \vartheta + \Omega$. Now, for any $y^* \in \mathbb{P}, PM(y^*)$ is a finite union of pyramids PM_i, where the $M_i \subset M(y^*)$ consists of precisely n independent elements. It is sufficient to find each i precisely n independent elements. It is sufficient to find for each i an $a_i \in \Omega$ such that $PM_i \cap \Omega \cap L \subset a_i + \overline{\Omega}$; for, by the remark above, there exists $a \in \Omega$ such that $a_i \in a + \Omega$ for each i, and clearly a will satisfy the condition of the lemma. ◻

Let $M = \left\{a_1, \ldots, a_n\right\}$ be any one of the M_i. We have $M \subset L$; let L_o be the sublattice of L generated by M. Let $p > O$ be an integer such that $pL \subset L_o$. For any $x \in PM \cap \Omega \cap L$, let $px = \sum \lambda_i a_i, \lambda_i \in \mathbb{Z}$. Since $x \in PM$, and the a_i are independent, we have $\lambda_i \geq O$ for every **92**
i. Let $a_x = \dfrac{1}{p} \sum\limits_{\lambda_i \neq O} a_i$. Since $x \in \Omega$, we see easily that $a_x \in \Omega$. Also, $x - a_x = \dfrac{1}{p}(\sum \lambda_i a_i - \sum\limits_{\lambda_i \neq O} a_i) \in PM \subset \overline{\Omega}$. The set $\left\{a_x | x \in PM \cap \Omega \cap L\right\}$

is clearly finite. Hence we have $a \in \Omega$ such that $a_x \in a + \Omega$ for every x. Clearly $PM \cap \Omega \cap L \subset a + \Omega$, and this proves the lemma.

Proof of Theorem 2. Let $A = \Omega \cap \bigcup_{i \in I} PM(y_i^*)$ be as in Theorem 1. Let $a_i \in \Omega$ be such that $\Omega \cap PM(y_i^*) \cap L \subset a_i + \bar{\Omega}$. For any r, let $V_r = \left\{ z \in \Omega | \nu(z) \leq r \right\}$. Then $V_r \cap (a_i + \bar{\Omega})$ is compact. Hence $V_r \cap (a_i + \bar{\Omega}) \cap L$ is finite. It follows that $A \cap L \cap V_r$is finite.

Examples. (i) Let Ω be the cone or real positive definite $n \times n$ matrices, $\Gamma = GL(n, \mathbb{Z}), D = \left\{ UU' | U \in \mathbb{Z}^n - \{\circ\} \right\}, L$ = the lattice of all integral $n \times n$ matrices, $\nu(A) = (\det A)^2$ for $A \in \Omega$. Since $\nu$is constant on the orbits of Γ, Theorem 2 implies in particular that the number of orbits of Γ in $A \cap L$ with determinant less than a given r is finite.

(ii) Let $\Omega = \left\{ (x, y, z) \in \mathbb{R}^3 | z > O \text{ and } x^2 + y^2 - 3z^2 < O \right\}$. Let $L = \mathbb{Z}^3$, and let Γ be the subgroup of $GL(\mathbb{R}^3)$ generated by the matrices

$$\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{and} \begin{pmatrix} 2 & 0 & 3 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

93 It is easy to verify that $\Gamma\Omega = \Omega$. Let $D = \Gamma\{(0, 0, 1\}.D$ satisfies the density condition. The fact that ${}_{\Gamma}\backslash\mathbb{P}$ is finite is a consequence of the following remark: if $D \subset \Omega$ satisfies the density condition and $\Gamma\backslash D$ is finite, then $\Gamma\backslash\mathbb{P}$ is finite. In fact let B be a finite subset of D such that $\Gamma B = D$. Then for any $y^* \in \mathbb{P}$, there exists an $s \in \Gamma$ such that $M(sy^*) \cap A \neq \phi$. The remark follows, since $PM(z^*) \cap A \neq \phi$ for only finitely many $z^* \in \mathbb{P}$-note that by assumption, $A \subset \Omega$.

(iii) Let K be a totally real extension of $\mathbb{Q}$ of degree n.

Let Γ be the group of totally positive units of K. Let $\sigma_1, \ldots, \sigma_n$ be n distinct isomorphisms of K into $\mathbb{R}$. We make Γ act on the self dual cone

$\Omega = \Big\{(t_1, \ldots, t_n) \in \mathbb{R}^n | t_i > O \text{ for all } i\Big\}$ by setting

$$\varepsilon(t_1, \ldots, t_n) = (\sigma_1(\varepsilon)t_1, \ldots, \sigma_n(\varepsilon)t_n),$$

Let $D = \Gamma\Big\{(1, \ldots, 1\Big\}$. It is a classical result that for any $i, 1 \leq i \leq n$, there exists $\in\in \Gamma$ such that $\sigma_i(\varepsilon) > 1$, and $\sigma_j(\varepsilon) < 1$ for $j \neq i$.

Using this, we verify that D satisfies the density condition. Let $(t_1, \ldots, t_n) \in \overline{\Omega}$; let $t_i = \circ$. Let ε be chosen as above. Then

$$\langle \varepsilon^p(1, \ldots, 1), (t_1, \ldots, t_n)\rangle = \sum_{j \neq i} t_j(\sigma_j(\varepsilon))^p$$

which tends to zero as $p \to \infty$. It follows as in Example (*ii*) that $_\Gamma\backslash\mathbb{P}$ is finite.

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