Presheaves of chain complexes

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Introduction

This paper was written to express a personal attitude about derived categories of presheaves and sheaves of chain complexes, much of which has existed for some time but has not previously appeared in the literature.

The attitude begins this way: the category of presheaves of positively graded chain complexes on a Grothendieck site $\mathcal{C}$ is equivalent to the category of presheaves of simplicial abelian groups under the Dold-Kan correspondence, and thus inherits a good closed model structure from the standard closed model structure for simplicial presheaves on $\mathcal{C}$. Unbounded, or $\mathbb{Z}$-graded, presheaves of chain complexes correspond to spectrum objects in the category of simplicial abelian presheaves, and the category of such things thus acquires the usual strict and stable closed model structures from the closed model structure for presheaves of spectra. In other words, a $\mathbb{Z}$-graded chain complex is best viewed as a stable homotopy type.

Precise versions of these statements are proved in the first three sections of this paper. The unstable model structure appears here as Lemma 1.5, and the stable structures appear in Theorem 2.5, Theorem 2.6 and Theorem 3.6. All results are stated in terms of presheaves of chain complexes and spectrum objects over a presheaf of rings R which is commutative and has a unit.

All model structures discussed in this paper are cofibrantly generated. The model structures for presheaves of chain complexes and simplicial modules given here specialize (see Remark 1.8) to the standard structures [15] for the usual categories of chain complexes of modules over an ordinary ring. It is important to note, however, that the classical description of these structures does not globalize, precisely because the naive approach to cofibrant generation breaks down for presheaves and sheaves.

The fibrant objects on the presheaf level are interesting, and behave like chain complexes of injectives in the sense that chain complexes of injective sheaves I which are bounded above satisfy descent: any weak equivalence $I \to J$ with J fibrant induces a homology isomorphism in each section. This is a fundamental point which has been understood in one form or another for some time [1], [8]; it is expressed in this paper as Theorem 2.7. The outcome is that bounded complexes of injective sheaves and their fibrant models have the same homological properties, and play essentially the same role in the theory.

Another basic aspect of the theory is that it makes no appeal to projective resolutions. Instead, one has cofibrant resolutions which appear universally, and are the basis of the construction of all left derived functors. In particular, the higher derived functors $\mathrm{Tor}_i(A, B)$ of the tensor product are defined as sheaves of homotopy groups by first taking cofibrant resolutions $X \to A$ and $Y \to B$ (ie. quasi-isomorphisms such that A and B are cofibrant in the ambient model structures); then one sets

$$\mathrm{Tor}_i(A, B) = \pi_i(X \otimes Y).$$

The description of the tensor product is of interest in its own right. The monoidal structures found in this theory do not arise from the classical tensor product of chain complexes; they instead are induced by the tensor product of simplicial modules. The two constructions are homologically equivalent, but the tensor product of simplicial modules is transparently symmetric monoidal in a homotopical sense, while the classical tensor product of presheaves of chain complexes is not.

The simplicial module tensor product can be bootstrapped to a symmetric monoidal tensor product on the category of $\mathbb{Z}$-graded chain complexes through the various equivalences given in the first three sections of this paper, but at the cost of introducing a suitable category of symmetric spectrum objects of presheaves of simplicial modules. These objects are defined by analogy with the symmetric spectra of Hovey, Shipley and Smith [7], and the presheaves of symmetric spectra of [13] and [14]. Analogues of the standard results hold: there is a stable closed model structure on symmetric spectrum objects in the category of simplicial modules (Theorem 4.9), and an equivalence of the associated homotopy category with the stable homotopy category of ordinary spectrum objects in the category of simplicial modules (Theorem 4.10). Lemma 4.16 shows that the stable model structure on symmetric spectrum objects and the tensor product arising from simplicial modules together satisfy the monoidal property. These results are the subject of Section 4 of this paper.

The interesting part of the proof of Theorem 4.9 begins with the definition of stable equivalence of symmetric spectrum objects. This definition is external in the sense that a map $A \to B$ of symmetric spectrum objects in simplicial modules is defined to be a stable equivalence if the underlying map of presheaves of symmetric spectra is a stable equivalence in the sense of [13]. This approach forces one to show that the the free module functor on presheaves of symmetric spectra preserves trivial stable cofibrations — this is the substance of Lemma 4.8, and its proof involves some of the more delicate arguments in the paper. The key point is Proposition 4.7, which in effect gives explicit weak equivalences relating abelian homology functors for a spectrum X to the effect of smashing X with Eilenberg-Mac Lane spectra.

Section 5 contains an elaboration of the basic properties of the higher Tor functors, for simplicial modules and for spectrum objects respectively. There is a spectral sequence of the form

$$E_2^{p,q} = \mathrm{Tor}_p(A, \pi_q B) \Rightarrow \mathrm{Tor}_{p+q}(A, B) \tag{0.1}$$

in both cases (Lemma 5.7, Lemma 5.16); this spectral sequence converges in the simplicial module case, and converges if A is bounded below in the stable case. In both cases the E_2-term can be calculated with a subsidiary spectral sequence

$$E_2^{r,s} = \mathrm{Tor}_r(\pi_s A, \pi_q B) \Rightarrow \mathrm{Tor}_{r+s}(A, \pi_q B)$$

which arises as a special case of (0.1).

The derivation of the spectral sequence (0.1) in the stable case uses a reformulation of the definition of the Tor functors for spectrum objects in terms of a naive tensor product which appears in Lemma 5.12. This result is the abelian analogue of a result explicitly relating naive smash products of spectra with smash products of symmetric spectra that appears in Lemma 5.9.

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1 Simplicial modules

Suppose that $\mathcal{C}$ is a small Grothendieck site, and let R be a presheaf of commutative rings with unit on $\mathcal{C}$. We shall use the notation Mod_R to denote the category of presheaves of left R-modules on $\mathcal{C}$; an R-module M is a presheaf of abelian groups which carries a R-module structure $R \times M \to M$. The category of $\mathbb{Z}$-graded R-chain complexes in Mod_R will be denoted by $\mathrm{Ch}(\mathrm{Mod}_R)$, and the category of ordinary, or positively graded R-chain complexes will be denoted by $\mathrm{Ch}_+(\mathrm{Mod}_R)$. We shall write $s\,\mathrm{Mod}_R$ for the category of simplicial R-modules.

In all that follows, the tensor product $M \otimes N = M \otimes_R N$ of two presheaves of R-modules will be defined (in each section) over the presheaf of rings R.

Say that a map $f : A \to B$ of simplicial R-modules is a *weak equivalence* if and only if (equivalently) f is a local weak equivalence of the underlying simplicial presheaves [10], or f induces an quasi-isomorphism $f_* : MA \to MB$ of Moore complexes; the latter means that f induces an isomorphism $f_* : H_*MA \cong H_*MB$ in all homology sheaves. Roughly speaking, a local weak equivalence of simplicial presheaves is a map which induces an isomorphism on all possible sheaves of homotopy groups; in the presence of stalks, the local weak equivalences are those maps which induce weak equivalences of simplicial sets in all stalks.

The *Moore complex* MA of a simplicial R-module A is a chain complex with n-chains defined by

$$MA_n = A_n$$

and boundary map $\partial : MA_n \to MA_{n-1}$ specified as an alternating sum of face maps by

$$\partial = \sum_{i=0}^{n} (-1)^i d_i.$$

A map $p : X \to Y$ is a *fibration* if and only if the underlying simplicial presheaf map is a global fibration [10]. A *global fibration* of simplicial presheaves is a map which has the right lifting property with respect to all maps which are cofibrations (ie. inclusions) and local weak equivalences. Say that $i : C \to D$ is a *cofibration* if it has the left lifting property with respect to all maps of simplicial R-modules which are fibrations and weak equivalences.

Lemma 1.1. *Suppose that $f : X \to Y$ is a local weak equivalence of simplicial presheaves. Then the induced map $f_* : RX \to RY$ is a weak equivalence of simplicial R-modules.*

If X is a pointed simplicial presheaf, write $\tilde{R}X$ for the kernel of the map $RX \to R*$, equivalently for the cokernel of the map $R* \to RX$ induced by the inclusion of the base point.

Proof. The free R-module functor $X \mapsto RX$ preserves pointwise equivalences, so we can assume that $f : X \to Y$ is a cofibration. It therefore suffices to show that the cofibre Y/X has trivial reduced homology sheaves in the sense that

$\tilde{R}(Y/X)$ is acyclic. We can replace Y/X up to pointwise weak equivalence by a pointwise fibrant simplicial presheaf Z, and then the idea is to show that $\tilde{R}Z$ is acyclic.

Every homology class $x \in \pi_n \tilde{R}Z(U)$ is carried on a finite pointed complex K in the sense that there is a map $\alpha : K \to Z(U)$ and a class $y \in \pi_n \tilde{R}(U)K$ such that $\alpha_*(y) = x$. The map $Z \to *$ is a trivial local fibration, so there is a covering sieve $\{\phi : V \to U\}$ for U such that α factors through the cone CK for K over R in the sense that there are commutative diagrams

$$\begin{array}{ccc} K & \xrightarrow{\alpha} & Z(U) \\ \downarrow & & \downarrow{\scriptstyle \phi^*} \\ CK & \longrightarrow & Z(V) \end{array}$$

and hence induced diagrams

$$\begin{array}{ccc} \tilde{R}(U)K & \xrightarrow{\alpha_*} & \tilde{R}Z(U) \\ \downarrow & & \downarrow{\scriptstyle \phi^*} \\ \tilde{R}(V)CK & \longrightarrow & \tilde{R}Z(V) \end{array}$$

But then $\phi^*(x) = \phi^*\alpha_*(y) = 0$ for all members ϕ of the covering sieve, so that x maps to 0 in the associated homology sheaf. □

Lemma 1.2. *With these definitions, the category $s\,\mathrm{Mod}_R$ of simplicial R-modules satisfies the axioms for a proper closed simplicial model category. Every cofibration is a monomorphism.*

Proof. The category $s\,\mathrm{Mod}_R$ of simplicial R-modules is complete and cocomplete, giving **CM1**. The weak equivalence axiom **CM2** and the retract axiom **CM3** are both trivial to verify.

The maps $RY \to RL_U\Delta^n$ freely associated to the simplicial presheaf inclusions $Y \subset L_U\Delta^n$ generate the cofibrations. There is a set of generating trivial cofibrations $A \subset B$ for the closed model structure on the category $s\,\mathrm{Pre}(\mathcal{C})$ of simplicial presheaves, and the induced maps $RA \to RB$ generate the trivial cofibrations of the simplicial R-module category; all such maps $RA \to RB$ are weak equivalences by Lemma 1.1. The factorization axiom **CM5** follows, by a standard transfinite induction, and then **CM4** is a formal consequence in the usual way. It also follows that every cofibration is a retract of a monomorphism and is hence a monomorphism.

The *function complex* $\mathbf{hom}(A, B)$ is the simplicial R-module having n-simplices consisting of all simplicial R-module maps $A \otimes R\Delta^n \to B$. The simplicial presheaf B^K is defined in sections by

$$B^K(U) = \mathbf{hom}(K, Y(U))$$

for a simplicial set K and a simplicial R-module B. The object B^K inherits the structure of a simplicial R-module from B. Also, if $p : A \to B$ is a fibration of simplicial R-modules and $i : K \to L$ is a cofibration of simplicial sets, then the induced map of simplicial R-modules

$$(i^*, p_*) : A^L \to A^K \times_{B^K} B^L$$

is a fibration of simplicial R-modules, which is trivial if either i is a weak equivalence of simplicial sets or p is a weak equivalence of simplicial R-modules. This follows from the corresponding statement for simplicial presheaves. In particular, the category $s\,\mathrm{Mod}_R$ has a closed simplicial model structure.

Weak equivalences are preserved by pullback along fibrations because this is true in the simplicial presheaf category, and one shows that weak equivalences are preserved by pushout along cofibrations by looking at induced long exact sequences arising from pushouts of Moore complexes, where we note that the free R-module functor preserves monomorphisms. □

Lemma 1.3. *With the definitions given above, the simplicial R-module tensor product $\otimes$ gives the category $s\,\mathrm{Mod}_R$ of simplicial abelian groups the structure of a monoidal proper closed simplicial model category.*

Proof. We will only verify that the closed model structure is monoidal. For this, it is enough to take cofibrations $i : K \subset L$ and $j : X \subset Y$ of simplicial presheaves, and show that the induced map

$$(RL \otimes RX) \cup_{(RK \otimes RX)} (RK \otimes RY) \to RL \otimes RY \tag{1.1}$$

is a cofibration of simplicial R-modules which is trivial if either i or j is a weak equivalence of simplicial presheaves.

But there is a natural isomorphism

$$RX \otimes RK \cong R(X \times K)$$

and so the map (1.1) is isomorphic to the map

$$R((L \times X) \cup_{(K \times X)} (K \times Y)) \to R(L \times Y)$$

which is induced by applying the free R-module functor to the cofibration

$$(L \times X) \cup_{(K \times X)} (K \times Y) \subset L \times Y \tag{1.2}$$

which is induced by i and j. It follows that the map (1.1) is a cofibration. The map (1.2) is a trivial cofibration of simplicial presheaves if either i or j is trivial, so the same is true of the map (1.1). □

Remark 1.4. One can use a spectral sequence argument to show that the functor $A \mapsto X \otimes A$ preserves weak equivalences of simplicial R-modules.

The Dold-Kan correspondence (N, Γ) [3] is used to give the category of positively graded R-chain complexes $\mathrm{Ch}_+(\mathrm{Mod}_R)$ the structure of a monoidal proper closed simplicial model category. In effect, say that a map $f : C \to D$ of R-chain complexes is a *weak equivalence* (respectively *fibration*, *cofibration*) if the induced map $f_* : \Gamma C \to \Gamma D$ is a weak equivalence (respectively fibration, cofibration) of simplicial R-modules. Then we have

Lemma 1.5. *With these definitions, the category* $\mathrm{Ch}_+(\mathrm{Mod}_R)$ *of chain complexes of R-modules, together with the tensor product functor* $\otimes$ *has the structure of a monoidal proper closed simplicial model category. The functors* Γ *and* N *induce an equivalence of homotopy categories*

$$\mathrm{Ho}(\mathrm{Ch}_+(\mathrm{Mod}_R)) \simeq \mathrm{Ho}(s\,\mathrm{Mod}_R).$$

Remark 1.6. The simplicial R-module tensor product gives the R-chain complex category $\mathrm{Ch}_+(\mathrm{Mod}_R)$ the structure of a monoidal model category. In effect, one defines

$$C \otimes D = N(\Gamma C \otimes \Gamma D).$$

This is **not** the standard R-chain complex tensor product.

Recall [10] that a map $p : F \to G$ of presheaves on the site $\mathcal{C}$ is said to be a *local epimorphism* if for all $x \in G(U)$ there is a covering sieve $R \subset \hom(\ ,U)$ such that $\phi^*(x) = p(y_\phi)$ for all $\phi \in R$. A *pointwise epimorphism* is a map $f : C \to D$ of presheaves such that all maps of sections $f : C(U) \to D(U)$ are surjective.

We shall need to know that every global fibration $p : X \to Y$ of simplicial presheaves induces Kan fibrations $p : X(U) \to Y(U)$ of simplicial sets in all sections [10].

Lemma 1.7. *1) Suppose that* $p : C \to D$ *is a fibration of* $\mathrm{Ch}_+(\mathrm{Mod}_R)$*. Then* $p : C_n \to D_n$ *is a pointwise epimorphism for all* $n \geq 1$*.*

2) Suppose $p : C \to D$ *is a trivial fibration of* $\mathrm{Ch}_+(\mathrm{Mod}_R)$*. Then* $p : C_n \to D_n$ *is a pointwise epimorphism for* $n \geq 0$*, and the kernel of* p *is acyclic in each section.*

Proof. If p is a fibration, then all induced maps $\Gamma C(U) \to \Gamma D(U)$ in sections are fibrations, so all lifting problems

$$\begin{array}{ccc} \Lambda^n_n & \xrightarrow{0} & \Gamma C(U) \\ \downarrow & \nearrow & \downarrow{\scriptstyle p_*} \\ \Delta^n & \xrightarrow[x]{} & \Gamma D(U) \end{array}$$

can be solved for $x \in D_n(U) = N\Gamma D_n(U)$, giving the first statement.

If $p : C \to D$ is an acyclic fibration with kernel K, then each map $\Gamma K(U) \to 0$ is a trivial fibration of simplicial sets, so that $K(U)$ must be acyclic. Also, all lifting problems above can be solved, and the lifting problem

$$\begin{array}{ccc} \emptyset & \longrightarrow & \Gamma C(U) \\ \downarrow & \nearrow & \downarrow p_* \\ \Delta^0 & \xrightarrow[x]{} & \Gamma D(U) \end{array}$$

for $x \in D_0(U) = ND_0(U) = \Gamma ND_0(U)$ has a solution, so that p is surjective in degree 0. □

Remark 1.8. The converse of both statements of Lemma 1.7 hold for ordinary chain complexes of modules over a ring, with the result that the model structure of Lemma 1.5 specializes to the standard model structure for these objects [15]. This is not true in general for presheaves of chain complexes: the presheaves of Kan complexes $K(A, n)$, for example, are almost never globally fibrant since they represent sheaf cohomology in the homotopy category of simplicial presheaves.

If X and Y are pointed simplicial presheaves, then the pushout square

$$\begin{array}{ccc} R(X \vee Y) & \longrightarrow & R(X \times Y) \\ \downarrow & & \downarrow \\ R* & \longrightarrow & R(X \wedge Y) \end{array}$$

can be used to show that there is a natural exact sequence

$$R(X \vee Y) \to R(X \times Y) \to \tilde{R}(X \wedge Y) \to 0$$

At the same time, it is easily seen that there is an exact sequence

$$0 \to (RX \otimes R*) + (R * \otimes RY) \to RX \otimes RY \to \tilde{R}X \otimes \tilde{R}Y \to 0$$

It follows that there is an isomorphism of simplicial R-modules

$$\tilde{R}(X \wedge Y) \cong \tilde{R}X \otimes \tilde{R}Y$$

and that this isomorphism is natural in pointed simplicial presheaves X and Y.

If C is a R-chain complex and $n \in \mathbb{Z}$, identify C with a $\mathbb{Z}$-graded R-chain complex by inserting 0 in negative degrees, and define $C[n]$ to be the R-chain complex with

$$C[n]_p = \begin{cases} C_{p+n} & \text{for } p > 0 \\ \ker \partial : C_n \to C_{n-1} & \text{if } p = 0. \end{cases}$$

In other words $C[-n]$ shifts n times while $C[n]$ is the good truncation if $n \geq 0$. If $n > 0$ there are natural isomorphisms

$$C[-n] \cong C \otimes R[-n] \cong C \otimes R[-1]^{\otimes n}$$

where the R-chain complex $R[-n]$ consists of a copy of the ring R concentrated in degree n. In this case, the displayed tensor product is the ordinary R-chain complex tensor product.

Recall that there is a natural isomorphism

$$N(\overline{W}A) \cong (NA)[-1]$$

for simplicial R-modules A, where $A \mapsto \overline{W}A$ is Eilenberg-Mac Lane $\overline{W}$ construction. We shall also use the fact [12, IV.4] that there is an isomorphism of R-chain complexes

$$N(\overline{W}A) \cong (NA)[-1]$$

which is natural in simplicial R-modules A.

Lemma 1.9. *The shift functor $A \mapsto A[-1]$ preserves cofibrations of R-chain complexes.*

Proof. It is enough to show that any cofibration of simplicial presheaves $j : K \to L$ induces a cofibration $j_* : (NRK)[-1] \to (NRL)[-1]$ of R-chain complexes. In order to do this, we show that the map $j_* : \overline{W}RK \to \overline{W}RL$ is a cofibration of simplicial R-modules.

We have a natural isomorphism

$$RX \cong \tilde{R}(X_+)$$

where X_+ denotes X with a disjoint base point attached. There is also a natural isomorphism

$$\overline{W}\tilde{R}Y \cong \tilde{R}(\Sigma Y)$$

for pointed simplicial presheaves Y, where ΣY denotes the Kan suspension of Y [12]. If $j : X \to Y$ is a pointed cofibration (respectively trivial cofibration), then the induced map $j_* : \Sigma X \to \Sigma Y$ is a cofibration (respectively trivial cofibration) of pointed simplicial sets. Finally, the functor $X \mapsto \tilde{R}X$ is left adjoint to the inclusion of simplicial R-modules (pointed by 0) in pointed simplicial presheaves, and hence takes pointed cofibrations to cofibrations of simplicial R-modules. □

2 $\mathbb{Z}$-graded chain complexes

A R-chain complex spectrum A consists of R-chain complexes A^n, $n \geq 0$, and R-chain maps $\sigma : A^n[-1] \to A^{n+1}$. A morphism $f : A \to B$ of R-chain complex spectra consists of R-chain maps $f : A^n \to B^n$ which respect structure in the sense that the diagram

$$\begin{array}{ccc} A^n[-1] & \xrightarrow{\sigma} & A^{n+1} \\ {\scriptstyle f[-1]}\downarrow & & \downarrow{\scriptstyle f} \\ B^n[-1] & \xrightarrow[\sigma]{} & B^{n+1} \end{array}$$

Write $\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R))$ for the category of R-chain complex spectra. This category is complete and cocomplete.

Say that a map $f : C \to D$ of R-chain complex spectra is a *strict weak equivalence* (respectively *strict fibration*) if all constituent maps $f : C^n \to D^n$ are weak equivalences (respectively fibrations) of R-chain complexes.

A morphism $g : A \to B$ of R-chain complex spectra is a *cofibration* if the following hold:

1) the R-chain map $g : A^0 \to B^0$ is a cofibration of R-chain complexes, and

2) the morphisms
$$B^n[-1] \cup_{A^n[-1]} A^{n+1} \to B^{n+1}$$
are cofibrations of R-chain complexes.

It follows from Lemma 1.9 that all maps $g : A^n \to B^n$ are cofibrations of R-chain complexes if g is a cofibration of R-chain complex spectra.

Lemma 2.1. *The category* $\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R))$*, together with the definition of strict weak equivalence, strict fibration and cofibration given above, satisfies the axioms for a proper closed model category.*

Proof. The axioms **CM1**, **CM2** and **CM3** are trivial to verify. The factorization axiom **CM5** and the lifting axiom **CM4** are verified directly, as for ordinary spectra. The properness assertion follows from properness for the R-chain complex category $\mathrm{Ch}_+(\mathrm{Mod}_R)$. □

The identification of an ordinary R-chain complex C with an integer graded R-chain complex by inserting 0 in all negative degrees defines a fully faithful functor $i_0 : \mathrm{Ch}_+(\mathrm{Mod}_R) \to \mathrm{Ch}(\mathrm{Mod}_R)$ with right adjoint $T_0 : \mathrm{Ch}(\mathrm{Mod}_R) \to \mathrm{Ch}_+(\mathrm{Mod}_R)$ defined by the good truncation in degree 0. This means that

$$T_0E_p = \begin{cases} E_p & \text{if } p > 0, \\ \ker \partial : E_0 \to E_{-1} & \text{if } p = 0, \\ 0 & \text{if } p < 0. \end{cases}$$

Observe that T_0 preserves quasi-isomorphisms. There are analogously defined good truncation functors T_n for all $n \in \mathbb{Z}$. Note that the adjoint (and inverse) of the shift functor $E \mapsto E[n]$ on $\mathrm{Ch}(\mathrm{Mod}_R)$ is the shift $F \mapsto F[-n]$.

Suppose that A is a R-chain complex spectrum. Then the bonding morphisms $\sigma : A^n[-1] \to A^{n+1}$ induce a string of morphisms

$$i_0A^0 \to (i_0A^1)[1] \to (i_0A^2)[2] \to \cdots$$

Write SA for the filtered colimit of this diagram of complexes in $\mathrm{Ch}(\mathrm{Mod}_R)$. Say that a map $f : A \to B$ of R-chain complex spectra is a *stable equivalence* if and only if it induces a homology isomorphism $f_* : SA \to SB$ of $\mathbb{Z}$-graded R-chain complexes. The following is a trivial observation:

Lemma 2.2. *Every strict weak equivalence of* $\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R))$ *is a stable weak equivalence.*

Suppose that E is an object of $\mathrm{Ch}(\mathrm{Mod}_R)$. Then the good truncations T_nE, $n \leq 0$ line up in a diagram

$$T_0E \to T_{-1}E \to T_{-2}E \to \cdots$$

with colimit E. These morphisms therefore determine a sequence of R-chain complexes

$$(T_{-n}E)[-n] = T_0(E[-n])$$

and maps of R-chain complexes

$$(T_{-n+1}E)[-n+1][-1] \to (T_{-n}E)[-n].$$

These constructions are functorial in Z-graded R-chain complexes E, so that we have a functor $T : \mathrm{Ch}(\mathrm{Mod}_R) \to \mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R))$. Observe that there is a natural isomorphism

$$\epsilon : STE \xrightarrow{\cong} E \tag{2.1}$$

induced by taking colimits, for all $\mathbb{Z}$-graded R-chain complexes E. There is also a natural morphism of R-chain complex spectra

$$\eta : A \to TSA \tag{2.2}$$

which is induced by the canonical map $i_0A^n[n] \to SA$.

There is a natural commutative triangle

$$\begin{array}{ccc} SA & \xrightarrow{S\eta} & STSA \\ & {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \epsilon}\,\cong \\ & & SA \end{array} \tag{2.3}$$

It follows that $S\eta$ is an isomorphism for all A, so that $\eta : A \to TSA$ is a stable equivalence for all R-chain complex spectra A. In particular, $f : A \to B$ is a stable equivalence if and only if the induced map $f_* : TSA \to TSB$ is a strict equivalence. Also, applying T to the diagram shows that $TS\eta$ is an isomorphism for all A. We have in particular proved the following

Lemma 2.3. *The maps* $\eta_{TSA}, TS\eta_A : TSA \to (TS)^2A$ *are strict weak equivalences for all R-chain complex spectra A.*

Say that a map $p : C \to D$ of $\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R))$ is a *stable fibration* if it has the right lifting property with respect to all maps which are cofibrations and stable equivalences.

Suppose that A is a R-chain complex. There is a R-chain complex spectrum F_nA such that

$$(F_nA)^p = \begin{cases} 0 & \text{if } p < n, \\ A[n-p] & \text{if } p \geq n. \end{cases}$$

There is a natural bijection

$$\hom(F_nA, B) \cong \hom_{\mathrm{Ch}_+(\mathrm{Mod}_R)}(A, B^n)$$

If $i : A \to B$ is a cofibration of R-chain complexes, then $i_* : F_nA \to F_nB$ is a cofibration of R-chain complex spectra which is strictly trivial if i is a quasi-isomorphism. A stable fibration $p : X \to Y$ has the right lifting property with respect to all maps which are cofibrations and strict weak equivalences. All of its constituent maps $p : X^n \to Y^n$ must therefore be fibrations of R-chain complexes. In other words, every stable fibration is a strict fibration.

Suppose that $p : C \to D$ is a strict fibration with kernel K. Then $p : C^n \to D^n$ is pointwise surjective above degree 0 for all n. Factorize p as

$$\begin{array}{ccc} C_n & \xrightarrow{\pi} & E_n \\ & {\scriptstyle p}\searrow & \downarrow{\scriptstyle j} \\ & & D_n \end{array}$$

for all n, such that π is epi and j is monic. Then the maps π and j define morphisms of R-chain complex spectra, and $j : E \to D$ is a stable equivalence. It follows that every strict fibre sequence

$$0 \to K \xrightarrow{i} C \xrightarrow{p} D$$

determines a long exact sequence

$$\cdots \to H_{n+1}SD \xrightarrow{\partial} H_nSK \xrightarrow{i_*} H_nSC \xrightarrow{p_*} H_nSD \xrightarrow{\partial} \cdots \tag{2.4}$$

in homology sheaves (aka. sheaves of stable homology groups).

If $i : A \to B$ is a cofibration then $i : A^n \to B^n$ is a cofibration of R-chain complexes and hence a monomorphism for all n. Then if C is the cokernel of i there is an induced long exact sequence

$$\cdots \to H_{n+1}SC \xrightarrow{\partial} H_nSA \xrightarrow{i_*} H_nSB \xrightarrow{p_*} H_nSC \xrightarrow{\partial} \cdots \tag{2.5}$$

in homology sheaves.

Lemma 2.4. *1) Suppose given a pullback square*

$$\begin{array}{ccc} A \times_D C & \xrightarrow{g_*} & C \\ \downarrow & & \downarrow p \\ A & \xrightarrow[g]{} & D \end{array}$$

in which p is a stable fibration and g is a stable equivalence. Then g_* is a stable equivalence.

2) Suppose given a pushout square

$$\begin{array}{ccc} A & \xrightarrow{f} & C \\ i \downarrow & & \downarrow \\ B & \xrightarrow[f_*]{} & B \cup_A C \end{array}$$

in which i is a cofibration and f is a stable equivalence. Then f_* is a stable equivalence.

Proof. This follows from the long exact sequences (2.4), (2.5) for strict fibrations and cofibrations displayed above. Recall that every stable fibration is a strict fibration. □

We have now assembled a proof of the following

Theorem 2.5. *The category* $\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R))$ *together with the classes of cofibrations, stable weak equivalences and stable fibrations as defined above, satisfies the axioms for a proper closed model category.*

Proof. This is a consequence of Lemma 2.2, Lemma 2.3 and Lemma 2.4, together with Theorem X.4.1 of [3]. □

One can show that the triangle of functors

$$\begin{array}{ccc} TC & \xrightarrow{\eta} & TSTC \\ & {\scriptstyle 1}\searrow & \downarrow T\epsilon \\ & & TC \end{array} \tag{2.6}$$

commutes — this follows from the commutativity of the triangle (2.3), plus the fact that ϵ is a natural isomorphism: one applies the functor S to the diagram (2.6) to show that it commutes. It follows that the stabilization functor S is left adjoint to the truncation functor T.

The game is now to find a model structure on the full R-chain complex category $\mathrm{Ch}(\mathrm{Mod}_R)$ such that the functors S and T form a Quillen equivalence.

The *weak equivalences* are obvious: these should be the quasi-isomorphisms, meaning the maps which induce isomorphisms in all homology sheaves.

We want the functor T to take fibrations to stable fibrations. We know that a map $p : A \to B$ of $\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R))$ is a stable fibration if and only if it is a strict fibration and the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TSA \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p_*} \\ B & \xrightarrow[\eta_B]{} & TSB \end{array}$$

is a strict homotopy cartesian diagram [3, IV.4.8]. But the map $\eta_{TC} : TC \to TSTC$ is an isomorphism, so it is enough to define a map $f : C \to D$ in $\mathrm{Ch}(\mathrm{Mod}_R)$ to be a *fibration* if and only if all induced maps $(T_{-n}C)[-n] \to (T_{-n}D)[-n]$ are fibrations of R-chain complexes. A *cofibration* of $\mathrm{Ch}(\mathrm{Mod}_R)$ is a map which has the left lifting property with respect to all maps which are fibrations and quasi-isomorphisms.

Theorem 2.6. *With these definitions, the category* $\mathrm{Ch}(\mathrm{Mod}_R)$ *of* $\mathbb{Z}$*-graded* R*-chain complexes has the structure of a proper closed model category. The functors* S *and* T *form a Quillen equivalence*

$$S : \mathrm{Ho}(\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R))) \overset{\simeq}{\leftrightarrows} \mathrm{Ho}(\mathrm{Ch}(\mathrm{Mod}_R)) : T$$

Proof. The axioms **CM1** – **CM3** are trivial. A map f is a weak equivalence of $\mathrm{Ch}(\mathrm{Mod}_R)$ if and only if the induced map Tf is a strict weak equivalence; it follows that the functor S preserves cofibrations.

There are generating sets I of cofibrations and J of trivial cofibrations for the strict model structure on $\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R))$. It follows that a R-chain complex map $f : C \to D$ is a fibration (respectively trivial fibration) if and only if it has the right lifting property with respect to all maps $Si : SA \to SB$ associated to maps $i : A \to B$ in the generating set I (respectively in the set J).

By a small object argument, every R-chain complex map $f : C \to D$ has factorizations

$$\begin{array}{ccc} D & \xrightarrow{i} & E \\ {\scriptstyle j}\downarrow & \searrow^{f} & \downarrow{\scriptstyle p} \\ F & \xrightarrow[q]{} & D \end{array}$$

where p is a fibration and i is a trivial cofibration which has the left lifting property with respect to all fibrations, and q is a trivial fibration and j is a cofibration. This gives **CM5**, and **CM4** follows by a standard argument.

The functors S and T both preserve weak equivalences in the respective model structures, the adjunction map $\eta : A \to TSA$ is a stable equivalence, and the adjunction map $\epsilon : STC \to C$ is an isomorphism. These observations guarantee that the functors T and S form a Quillen equivalence — see [4, p.19]. ∎

This section closes with the statement and proof of a descent result for chain complexes of injective sheaves which are bounded above. The following result implies, for example (see Remark 2.8 below), that sheaf cohomology can be recovered from the model structures discussed in Lemma 1.5 and Theorem 2.6. The basic point behind the proof has been understood for some time [1], [8], but has not been expressed in the present form before. The proof given here is a sketch — the technology of Brown's categories of fibrant objects [1] lurks behind — but an attempt has been made for completeness, for the benefit of the reader.

Theorem 2.7. *Suppose that I is a $\mathbb{Z}$-graded chain complex of R-modules such that all objects I_p are injective sheaves of R-modules and such that $I_p = 0$ for $p > 0$. Suppose that $f : I \to J$ is a weak equivalence such that J is fibrant. Then the induced maps $I(U) \to J(U)$ are homology isomorphisms for all objects U of the site $\mathcal{C}$. The weak equivalence f induces homology isomorphisms $\Gamma_* I \to \Gamma_* J$ in global sections.*

Proof. Suppose that A is a chain complex which is concentrated in non-negative degrees in the sense that $A_q = 0$ for $q < 0$. Then the bicomplex

$$\hom(A, I)^{p,q} = \hom(A_p, I_{-q})$$

has a total complex $\operatorname{Tot}\hom(A, I)$ with homology

$$H^n \operatorname{Tot}(A, I) = \pi(A, T_0(I[-n]))$$

where $\pi(\ ,\ )$ denotes chain homotopy classes of maps. The functors $\hom(\ , I_q)$ are exact, so there is a spectral sequence with

$$E_2^{p,q} = H^q \hom(H_p A, I_{-*}) \Rightarrow \pi(A, T_0(I[-p-q])),$$

where $H_p A$ denotes the p^{th} homology sheaf of A (this is where we require that the modules I_p are sheaves). It follows that the functors $A \mapsto \pi(A, T_0(I[-n]))$ take weak equivalences in A to isomorphisms.

Say that a chain map $\pi : C' \to C$ which is a weak equivalence and induces a sheaf epimorphism in all degrees is a *local trivial fibration*. Any local trivial fibration π induces a map

$$\pi(C', D) \to [C, D]$$

which is defined on chain homotopy classes of maps from C' to D and takes values in morphisms $[C, D]$ in the derived category: this map is defined by taking a chain map represented by $f : C' \to D$ to the map $f_* \pi_*^{-1}$ in the homotopy category. The collection of all such maps defines a morphism

$$\phi_{C,D} : \varinjlim_{C' \xrightarrow{[\pi]} D} \pi(C', D) \to [C, D].$$

where the colimit is defined on the comma category of chain homotopy classes of maps $C' \to D$ which are represented by local trivial fibrations. In the presence of the model structure of Lemma 1.5, it is an exercise to show that this colimit is filtered (although it is an avatar of a much more general calculus of fractions argument [1]).

One also knows that every chain map $g : D \to D'$ has a factorization

$$\begin{array}{ccc} D & \xrightarrow{g} & D' \\ & {\scriptstyle\sigma}\searrow \quad \nearrow{\scriptstyle\pi} & \\ & E & \end{array}$$

where π is a local fibration in the sense that it induces a sheaf epimorphism in all degrees greater than 0 and σ is a section of a trivial local fibration. In particular, if g is a weak equivalence, then π is a trivial local fibration. The existence of this factorization is a basic property of Brown's categories of fibrant objects (see [1] again), but it's really just an instance of the standard replacement of a map by a fibration up to weak equivalence which holds formally in the chain complex setting. The factorization can be used to show that every weak equivalence $g : D \to D'$ induces an isomorphism

$$\varinjlim_{C' \xrightarrow{[\pi]} D} \pi(C', D) \cong \varinjlim_{C' \xrightarrow{[\pi']} D} \pi(C', D'),$$

because g can be replaced by a trivial local fibration.

One then verifies that the map $\pi_{C,D}$ is an isomorphism in general. In effect, there is a weak equivalence $D \to D'$ where D' is fibrant for the model structure of Lemma 1.5, and the map $\phi_{C,D'}$ is an isomorphism since Lemma 1.7 implies that the chain homotopy classes represented by local trivial fibrations $\pi : C' \to C$ with C' cofibrant are cofinal in the filtered system.

Suppose that $f : I \to J$ is a weak equivalence such that J is fibrant. Suppose that the chain complex A is cofibrant. Then all chain complexes $T_0(J[-n])$ are fibrant and there are commutative diagrams of isomorphisms

$$\begin{array}{ccc} \pi(A, T_0(I[-n])) & \longrightarrow & \pi(A, T_0(J[-n])) \\ {\scriptstyle\cong}\downarrow & & \downarrow{\scriptstyle\cong} \\ [A, T_0(I[-n])] & \xrightarrow[\cong]{} & [A, T_0(J[-n])] \end{array} \tag{2.7}$$

For each object U of the site $\mathcal{C}$, the presheaf of chain complexes $R(U)[q]$ consisting of the free R-module $R(U)$ concentrated in degree q is cofibrant. It follows that all ordinary chain complex maps in sections

$$T_0(I[-n])(U) \to T_0(J[-n])(U)$$

are homology isomorphisms for all objects U of $\mathcal{C}$, and hence that all maps $I(U) \to J(U)$ of ordinary $\mathbb{Z}$-graded complexes are homology isomorphisms.

The claim about the homology isomorphism in global sections arises from the diagram (2.7) in the same way, by using the cofibrant object $A = R[q]$ consisting of a copy of R concentrated in degree q. □

Remark 2.8. Theorem 2.7 has many consequences, including the fact that fibrant resolutions can be used to define sheaf cohomology. In particular, if B is a sheaf of R-modules, then an injective resolution

$$B \to I_0 \to I_{-1} \to I_{-2} \to \dots$$

in the category of sheaves of R-modules can be viewed as a chain complex concentrated in negative degrees, as displayed. Then the q^{th} sheaf cohomology group $H^q(\mathcal{C}, B)$ of the site $\mathcal{C}$ with coefficients in B can be defined by

$$H^q(\mathcal{C}, B) = H_{-q}\Gamma_* I.$$

Theorem 2.7 implies that if $B[0] \to J$ is a weak equivalence with J fibrant, then there is an isomorphism

$$H^q(\mathcal{C}, B) = H_{-q}\Gamma_* J.$$

In other words, in the language of spectra, sheaf cohomology with coefficients in B appears as stable homotopy groups of global sections of a stably fibrant model J of the Eilenberg-Mac Lane object $B[0]$.

3 Simplicial module spectra

Suppose that K is a pointed simplicial presheaf and that A is a simplicial R-module. Write

$$K \otimes A = \tilde{R}K \otimes A.$$

In this notation, if L is a second pointed simplicial set, then there is an isomorphism of simplicial R-modules

$$K \otimes \tilde{R}L \cong \tilde{R}(K \wedge L).$$

A simplicial R-module spectrum A consists of simplicial R-modules A^n, $n \geq 0$, together with simplicial R-module homomorphisms $\sigma : S^1 \otimes A^n \to A^{n+1}$, which are often called bonding maps. A morphism of $f : A \to B$ of simplicial R-module spectra consists of simplicial R-module homomorphisms $f : A^n \to B^n$ which respect structure in the sense that the following diagram commutes:

$$\begin{array}{ccc} S^1 \otimes A^n & \xrightarrow{\sigma} & A^{n+1} \\ {\scriptstyle S^1 \otimes f}\downarrow & & \downarrow{\scriptstyle f} \\ S^1 \otimes B^n & \xrightarrow[\sigma]{} & B^{n+1} \end{array}$$

The simplicial R-module spectra and their morphisms form a category, which will be denoted by $\mathbf{Spt}(s\,\mathrm{Mod}_R)$. This category is complete and cocomplete.

A morphism $f : A \to B$ of simplicial R-module spectra is said to be a *strict weak equivalence* (respectively *strict fibration*) if all maps $f : A^n \to B^n$ are weak equivalences (respectively fibrations) of simplicial R-modules. A map $i : C \to D$ is said to be a *cofibration* if

1) the map $i : C^0 \to D^0$ is a cofibration of simplicial R-modules, and

2) all maps
$$S^1 \otimes D^n \cup_{S^1 \otimes C^n} C^{n+1} \to D^{n+1}$$
are cofibrations of simplicial R-modules.

Lemma 3.1. *With these definitions of strict weak equivalence, strict fibration and cofibration, the category* $\mathbf{Spt}(s\,\mathrm{Mod}_R)$ *satisfies the axioms for a proper closed simplicial model category.*

Proof. The assertion that $\mathbf{Spt}(s\,\mathrm{Mod}_R)$ is a proper closed model category follows from the standard argument (see the proof of Lemma 2.1). For this we need to know that if $i : A \to B$ is a cofibration (respectively trivial cofibration) of simplicial R-modules, then the induced map $i_* : S^1 \otimes A \to S^1 \otimes B$ is a cofibration (respectively trivial cofibration). More generally, the functor $A \mapsto X \otimes A$ preserves cofibrations and trivial cofibrations since
$$X \otimes RK \cong X \otimes \tilde{R}(K_+) \cong \tilde{R}(X \wedge K_+)$$
and the functor $K \mapsto X \wedge K_+$ preserves cofibrations and trivial cofibrations of simplicial presheaves. Finally, the free R-module functor $X \mapsto RX$ preserves weak equivalences of simplicial presheaves by Lemma 1.1.

The simplicial structure is given by the functors $A \mapsto A \otimes K$ and the functors $B \mapsto B^K = \mathbf{hom}(K, A)$; the latter is defined for simplicial presheaves K in the simplicial presheaf category. The n-simplices of the function complex $\mathbf{hom}(A, B)$ are the morphisms $A \otimes \Delta^n \to B$. As in the simplicial R-module case, Quillen's axiom **SM7** is most easily proved by observing that if $p : A \to B$ is a strict fibration and $K \subset L$ is an inclusion of simplicial presheaves, then the induced map
$$A^L \to B^L \times_{B^K} A^K$$
is a strict fibration, on account of the validity of **SM7** in the simplicial presheaf category. □

It follows from Lemma 4.48 of [12, p.120] that there is a natural homotopy equivalence of simplicial R-modules
$$S^1 \otimes A \simeq \overline{W}A. \tag{3.1}$$
In effect, there is an isomorphism $S^1 \otimes A \cong \tilde{\mathbb{Z}}S^1 \otimes_{\mathbb{Z}} A$ with R-module structure coming from A, while $\overline{W}A$ is formed in the simplicial abelian group category. The quoted result implies a natural homotopy equivalence of simplicial abelian groups, and one checks that the data for the homotopy equivalence is R-linear; the most efficient way to do this is to compare coend constructions for the functors $A \mapsto S^1 \otimes A$ and $A \mapsto \overline{W}A$.

The existence of the natural homotopy equivalence (3.1) means that there are natural transformations
$$S^1 \otimes A \xrightarrow{\mu} \overline{W}A \xrightarrow{\nu} S^1 \otimes A$$

and natural homotopies $h : S^1 \otimes A \otimes \Delta^1 \to S^1 \otimes A$ from $\nu \cdot \mu$ to the identity, and $H : \overline{W}A \otimes \Delta^1 \to \overline{W}A$ from $\mu \cdot \nu$ to the identity on $\overline{W}A$.

Every simplicial R-module spectrum A determines a R-chain complex spectrum NA. The R-chain complex NA^n is the normalized complex of the simplicial R-module A^n as the notation suggests, and the bonding map $\sigma : S^1 \otimes A^n \to A^{n+1}$ induces a composite

$$NA^n[-1] \cong N\overline{W}A^n \xrightarrow{\nu_*} N(S^1 \otimes A^n) \xrightarrow{N\sigma} NA^{n+1}$$

This assignment determines a functor

$$N : \mathbf{Spt}(s\,\mathrm{Mod}_R) \to \mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R))$$

which preserves strict fibrations and strict weak equivalences. Simlarly, if C is a R-chain complex spectrum, then there is a composite

$$S^1 \otimes \Gamma C^n \xrightarrow{\mu} \overline{W}\Gamma C^n \cong \Gamma(C^n[-1]) \xrightarrow{\Gamma\sigma} \Gamma C^{n+1}$$

and so there is an induced functor

$$\Gamma : \mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R)) \to \mathbf{Spt}(s\,\mathrm{Mod}_R).$$

The functor Γ plainly preserves strict weak equivalences and strict fibrations.

Lemma 3.2. *The functors N and Γ induce an equivalence*

$$\mathrm{Ho}_{strict}(\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R))) \simeq \mathrm{Ho}_{strict}(\mathbf{Spt}(s\,\mathrm{Mod}_R))$$

of strict homotopy categories.

Proof. See the proof of Theorem 4.52 of [12]. We use a telescope construction in the categories of R-chain complex and simplicial R-module spectra to show that there are natural strict weak equivalences

$$B \xleftarrow{\simeq} TB \xrightarrow{\simeq} N\Gamma B$$

for cofibrant objects B in $\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R))$ and natural strict weak equivalences

$$A \xleftarrow{\simeq} TA \xrightarrow{\simeq} \Gamma NA$$

for cofibrant simplicial R-module spectra A. ∎

The relation between the functors N and Γ can be expressed in a different way. First of all, there is the following

Lemma 3.3. *1) The canonical map $\eta : A \to \Omega(S^1 \otimes A)$ of simplicial abelian groups is a weak equivalence.*

2) The canonical map $\epsilon : S^1 \otimes \Omega A \to A$ induces an isomorphism in π_n for $n \geq 1$.

Proof. For the first statement, the functor $B \mapsto S^1 \otimes B$ shifts homotopy groups so that a triangle identity implies that η is a split monomorphism in all homotopy groups. By truncating suitably, it's enough to show that η is an isomorphism in all homotopy groups for simplicial abelian groups of the form $K(C,n)$. If C is a copy of $\mathbb{Z}$ or is a finite abelian group, then η is an isomorphism in homotopy groups for $K(C,n)$. Both functors involved in η preserve fibrations and filtered colimits of simplicial R-modules, so that η is an isomorphism in π_* for all $K(C,n)$ with C finitely generated, and hence for all C.

The second statement is an easy consequence of the first. $\square$

Corollary 3.4. *1) The canonical map $\eta : A \to \Omega(S^1 \otimes A)$ of simplicial R-modules is a strict weak equivalence.*

2) The canonical map $\epsilon : S^1 \otimes \Omega A \to A$ induces an isomorphism in homotopy group sheaves π_n for $n \geq 1$.

Suppose given a simplicial R-module homomorphism $\sigma : A \to \Omega B$. The natural composition

$$\overline{W}\Omega B \xrightarrow{\nu} S^1 \otimes \Omega B \xrightarrow{\epsilon} B$$

induces a natural R-chain map $\nu_* : N\Omega B \to (NB)[1]$ by adjointness, and there is more generally a commutative diagram

$$\begin{array}{ccccccc} NA & \xrightarrow{\cong} & NA[-1][1] & \xrightarrow{\cong} & N\overline{W}A[1] & \xrightarrow{N\nu[1]} & N(S^1 \otimes A)[1] \\ {\scriptstyle N\sigma}\downarrow & & & & & & \downarrow{\scriptstyle N\sigma_*[1]} \\ N\Omega B & & & \xrightarrow{\nu_*} & & & NB[1] \end{array} \tag{3.2}$$

for any morphism $\sigma : A \to \Omega B$. The composite $\nu_* \cdot N\sigma : NA \to NB[1]$ therefore coincides with the adjoint of the map $NA[-1] \to NB$ which is induced by the map $\sigma_* : S^1 \otimes A \to B$. The picture (3.2) and Corollary 3.4 together imply that the map $\nu_* : N\Omega B \to NB[1]$ is a quasi-isomorphism.

Suppose that K and L are pointed simplicial presheaves and that A is a simplicial R-module. The functor $K \mapsto \tilde{R}K$ is left adjoint to the forgetful functor $A \mapsto uA$ from simplicial R-modules to pointed simplicial presheaves, where uA is pointed by 0. There is a natural map

$$\gamma : K \wedge uA \to u(\tilde{R}K \otimes A) = u(K \otimes A)$$

which is given in sections by the assignment $k \wedge a \mapsto k \otimes a$. This map is initial among all maps $f : K \wedge uA \to uB$ which determine R-module homomorphisms $a \mapsto f(x \wedge a)$ in all degrees. In other words, given such a map f, there is a unique simplicial R-module homomorphism $f_* : \tilde{R}K \otimes A \to B$ which makes the diagram

$$\begin{array}{ccc} K \wedge uA & \xrightarrow{\gamma} & u(K \otimes A) \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle uf_*} \\ & & uB \end{array}$$

commute.

The adjunction map $\epsilon : S^1 \otimes \mathbf{hom}(S^1, B) \to B$ can be defined as the unique simplicial R-module homomorphism which makes the diagram

$$\begin{array}{ccc} S^1 \wedge u\mathbf{hom}(S^1, B) & \xrightarrow{=} & S^1 \wedge \mathbf{hom}(S^1, uB) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle ev} \\ u(S^1 \otimes \mathbf{hom}(S^1, B)) & \xrightarrow[u\epsilon]{} & uB \end{array}$$

commute, where ev is the standard evaluation map. It follows that there is a commutative diagram

$$\begin{array}{ccc} uB & \xrightarrow{\eta} & \mathbf{hom}(S^1, S^1 \wedge uB) \\ {\scriptstyle u\eta}\downarrow & & \downarrow{\scriptstyle \gamma_*} \\ u\mathbf{hom}(S^1, S^1 \otimes B) & \xrightarrow[=]{} & \mathbf{hom}(S^1, u(S^1 \otimes B)) \end{array} \tag{3.3}$$

relating the simplicial presheaf and simplicial R-module adjunction maps η. The simplicial presheaf adjunction $uA \to \mathbf{hom}(S^1, uB)$ of the composite

$$S^1 \wedge uA \xrightarrow{\gamma} u(S^1 \otimes A) \xrightarrow{u\sigma} uB$$

therefore coincides with the simplicial presheaf map underlying the simplicial R-module adjoint $\sigma_* : A \to \mathbf{hom}(S^1, B)$ of the simplicial R-module homomorphism $\sigma : S^1 \otimes A \to B$.

There is a natural commutative diagram

$$\begin{array}{ccc} K \wedge L & \xrightarrow{1 \wedge \eta} & K \wedge u\tilde{R}L \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ u\tilde{R}(K \wedge L) & \xrightarrow[uc]{\cong} & u(\tilde{R}K \otimes \tilde{R}L) \end{array} \tag{3.4}$$

where $c : \tilde{R}(K \wedge L) \to \tilde{R}K \otimes \tilde{R}L$ is the canonical isomorphism. It follows by adjointness that there is a commutative diagram

$$\begin{array}{ccc} \tilde{R}(K \wedge uA) & \xrightarrow[\cong]{c} & \tilde{R}K \otimes \tilde{R}uA \\ {\scriptstyle\tilde{R}\gamma}\downarrow & & \downarrow{\scriptstyle 1 \otimes \epsilon} \\ \tilde{R}u(\tilde{R}K \otimes A) & \xrightarrow[\epsilon]{} & \tilde{R}K \otimes A \end{array} \tag{3.5}$$

There are also natural commutative diagrams

$$\begin{array}{ccc} K \wedge L \wedge uA & \xrightarrow{1 \wedge \gamma} & K \wedge u(L \otimes A) \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\gamma} \\ u((K \wedge L) \otimes A) & \xrightarrow[u(c \otimes 1)]{\cong} & u(K \otimes L \otimes A) \end{array} \tag{3.6}$$

The forgetful functor u and the reduced free R-module functor $\tilde{R}$ determine functors

$$u : \mathbf{Spt}(s\,\mathrm{Mod}_R) \to \mathbf{Spt}$$

and

$$\tilde{R} : \mathbf{Spt} \to \mathbf{Spt}(s\,\mathrm{Mod}_R).$$

Here, **Spt** denotes the category of presheaves of spectra on the site $\mathcal{C}$. Explicitly, if A is a simplicial R-module spectrum, then uA is the presheaf of spectra with $(uA)^n = u(A^n)$, and with bonding maps

$$S^1 \wedge uA^n \xrightarrow{\gamma} u(S^1 \otimes A^n) \xrightarrow{u\sigma} uA^{n+1}.$$

Also, if X is a presheaf of spectra then $\tilde{R}X$ is the simplicial R-module spectrum with $(\tilde{R}X)^n = \tilde{R}(X^n)$, and having bonding maps

$$S^1 \otimes \tilde{R}X^n \xrightarrow{\cong} \tilde{R}(S^1 \wedge X^n) \xrightarrow{\tilde{R}\sigma} \tilde{R}X^{n+1}$$

It follows from the commutativity of the diagram (3.5) that the adjunction maps $\epsilon : \tilde{R}uA^n \to A^n$ for a simplicial R-module spectrum A assemble to give a natural map

$$\epsilon : \tilde{R}uA \to A$$

of simplicial R-module spectra. Similarly, it follows from the commutativity of the diagram (3.4) that the maps $\eta : X^n \to u\tilde{R}X^n$ for a presheaf of spectra X determine a natural map of spectra

$$\eta : X \to u\tilde{R}X.$$

In particular, the spectrum level functor $\tilde{R}$ is left adjoint to the spectrum level version of the forgetful functor u.

The forgetful functor u preserves strict weak equivalences and strict fibrations, while the functor $\tilde{R}$ preserves strict weak equivalences and cofibrations.

Say that a map $f : A \to B$ is a *stable weak equivalence* (respectively *stable fibration*) of simplicial R-module spectra if the underlying map $uf : uA \to uB$ is a stable equivalence (respectively stable fibration) of presheaves of spectra.

Lemma 3.5. *The functor $X \mapsto \tilde{R}X$ preserves stable weak equivalences of presheaves of spectra X.*

Proof. It is enough to show that $\tilde{R}$ takes maps which are cofibrations and stable equivalences to stable equivalences of simplicial R-module spectra. Suppose that $i : X \to Y$ is a cofibration of spectra with cofibre Y/X. Then the sequence

$$0 \to \tilde{R}X \to \tilde{R}Y \to \tilde{R}(X/Y) \to 0$$

is exact in all levels, and hence induces a long exact sequence in sheaves of stable homotopy groups

$$\cdots \to \pi_{n+1}u\tilde{R}(Y/X) \xrightarrow{\partial} \pi_n u\tilde{R}X \to \pi_n u\tilde{R}Y \to \pi_n u\tilde{R}(Y/X) \xrightarrow{\partial} \cdots$$

It suffices, therefore, to show that $\pi_* u\tilde{R}Z = 0$ if $\pi_* Z = 0$, for all spectra Z.

We can assume that Z is level fibrant. Since $\pi_* Z = 0$, the presheaf of spectra QZ consists of simplicial presheaves QZ^n such that the maps $QZ^n \to *$ are trivial local fibrations. It follows that if K is a finite pointed simplicial set, and $\alpha : K \to Z^n(U)$ is a pointed simplicial set map, then there is a covering sieve $R \subset \hom(\ ,U)$ such that α determines a commutative diagram of maps of ordinary spectra

$$\begin{array}{ccc} \Sigma^\infty K[-n] & \xrightarrow{\alpha} & Z(U) \\ \simeq \uparrow & & \downarrow \phi^* \\ \Sigma^\infty(S^m \wedge K)[-n-m] & \xrightarrow[\beta]{} & Z(V) \end{array}$$

where $\beta = 0$ in the ordinary stable category: in effect, α factors locally through the cone CK of K, stably. Thus, any class $\alpha : \Sigma^\infty K[-n] \to Z(U)$ dies after refinement along a covering sieve. All elements x of the presheaf of stable homotopy groups for $\tilde{R}Z$ are carried by such maps α, so each presheaf of stable homotopy groups for $\tilde{R}Z$ maps to 0 in its associated sheaf. This means that $\pi_* u\tilde{R}Z = 0$, as required. □

Theorem 3.6. *1) With the definitions of stable weak equivalence, stable fibration and cofibration given above the category* $\mathbf{Spt}(s\,\mathrm{Mod}_R)$ *of simplicial R-module spectra satisfies the axioms for a proper closed simplicial model category.*

2) The functors N and Γ induce an equivalence of associated stable homotopy categories

$$\mathrm{Ho}_{stable}(\mathbf{Spt}(\mathrm{Ch}_+(\mathrm{Mod}_R)) \simeq \mathrm{Ho}_{stable}(\mathbf{Spt}(s\,\mathrm{Mod}_R)).$$

Proof. The simplest way to demonstrate the existence of the stable model structure on $\mathbf{Spt}(s\,\mathrm{Mod}_R)$ begins by recalling [2] that the category of presheaves of spectra $\mathbf{Spt}$ has sets of generating cofibrations and generating trivial cofibrations. Thus, there is a set of trivial cofibrations $X \to Y$ of presheaves of spectra such that a map $p : A \to B$ of objects of $\mathbf{Spt}(s\,\mathrm{Mod}_R)$ is a stable fibration if and only if it has the right lifting property with respect to all induced map $\tilde{R}X \to \tilde{R}Y$. Similarly, there is a set of cofibrations $U \to V$ of presheaves of spectra such that $p : A \to B$ is a stable fibration and a stable equivalence if and only if it has the right lifting property with respect to all $\tilde{R}U \to \tilde{R}V$. The functor $\tilde{R}$ takes cofibrations of pointed simplicial presheaves to cofibrations of simplicial R-modules, and hence takes cofibrations of presheaves of spectra to cofibrations of simplicial R-module spectra. We also know from Lemma 3.5 that $\tilde{R}$ takes stable equivalences to stable equivalences. It follows that every map

$f : A \to B$ has a factorization

$$\begin{array}{ccc} A & \xrightarrow{i} & C \\ {\scriptstyle j}\downarrow & \searrow^{f} & \downarrow{\scriptstyle p} \\ D & \xrightarrow[q]{} & B \end{array}$$

where p is a stable fibration and i is a cofibration and a stable equivalence which also has the left lifting property with respect to all fibrations, and q is a stable fibration and a stable equivalence and j is a cofibration. This gives **CM5**, and then **CM4** by the standard argument. The axioms **CM1** – **CM3** are trivial to verify. The axiom **SM7** is a consequence of the corresponding axiom for spectra.

Stable equivalences are preserved by pullback along fibrations, as a result of the corresponding statement for the category **Spt** of presheaves of spectra. Every cofibration $i : A \to B$ is a levelwise monomorphism, and such a map induces a long exact sequence

$$\dots \to \pi_n uA \xrightarrow{i_*} \pi_n uB \to \pi_n u(B/A) \xrightarrow{\partial} \pi_{n-1} uA \to \dots$$

in sheaves of stable homotopy groups (aka. homology groups). A comparison of long exact sequences is used to show that stable equivalences are preserved by pushout along cofibrations.

Suppose that A is a simplicial R-module spectrum. Write QA^n for the filtered colimit in the simplicial R-module category of the string of homomorphisms

$$A^n \xrightarrow{\sigma_*} \Omega A^{n+1} \xrightarrow{\Omega\sigma_*} \Omega^2 A^{n+2} \to \cdots$$

Then there is an isomorphism of pointed simplicial presheaves

$$uQA^n \cong Q(uA)^n,$$

since the simplicial R-module adjoint of a map $f : S^1 \otimes B \to C$ coincides as a map of simplicial presheaves with the adjoint of the composite

$$S^1 \wedge uB \xrightarrow{\gamma} u(S^1 \otimes B) \xrightarrow{uf} uC$$

on account of the commutativity of diagram (3.3). It follows that a map $g : A \to B$ of simplicial R-module spectra is a stable equivalence if and only if all induced maps $g_* : QA^n \to QB^n$ of simplicial R-modules are weak equivalences.

The diagram (3.2) implies that there are commutative diagrams of R-chain maps

$$\begin{array}{ccc} NA^n & \xrightarrow{N\sigma} & N\Omega A^{n+1} \\ & {\scriptstyle \sigma_*}\searrow & \downarrow{\scriptstyle \nu_*} \\ & & NA^{n+1}[1] \end{array}$$

where ν_* is the natural canonical quasi-isomorphism. It follows that $g : A \to B$ is a stable equivalence of simplicial R-module spectra if and only if $g_* : NA \to NB$ is a stable equivalence of R-chain complex spectra. Statement 2) now follows from the proof of Lemma 3.2. $\square$

Lemma 3.7. *Any short exact sequence*

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0$$

in $\mathbf{Spt}(s\,\mathrm{Mod}_R)$ *is a stable homotopy fibre sequence in* $\mathbf{Spt}$*.*

Proof. The map p has a factorization

$$\begin{array}{ccc} B & \xrightarrow{p} & C \\ {\scriptstyle j}\downarrow & \nearrow_{\pi} & \\ D & & \end{array}$$

in $\mathbf{Spt}(s\,\mathrm{Mod}_R)$, where the map j is a stable equivalence and a cofibration and π is a stable fibration. The map π is an epimorphism (in all sections and levels) since p is an epimorphism. Let $F = \ker(\pi)$. Then there is a comparison diagram of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & {\scriptstyle j_*}\downarrow & & {\scriptstyle j}\downarrow & & \downarrow{\scriptstyle 1} & & \\ 0 & \longrightarrow & F & \longrightarrow & D & \xrightarrow[\pi]{} & C & \longrightarrow & 0 \end{array}$$

The resulting comparison in long exact sequences of stable homotopy groups implies that the induced map $j_* : A \to F$ is a stable equivalence. $\square$

Corollary 3.8. *Fibre and cofibre sequences of simplicial R-module spectra coincide up to stable equivalence.*

Proof. Every cofibre sequence

$$A \xrightarrow{i} X \to X/A$$

is short exact, and is therefore a fibre sequence by Lemma 3.7.

Given a fibre sequence

$$0 \to A \xrightarrow{j} X \xrightarrow{p} Y,$$

form the diagram

$$\begin{array}{ccccc} A & \xrightarrow{i} & \tilde{X} & \longrightarrow & \tilde{X}/A \\ {\scriptstyle 1_A}\downarrow & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi_*} \\ A & \xrightarrow[j]{} & X & \xrightarrow[p]{} & Y \end{array}$$

where i is a cofibration and π is a trivial cofibration. This map is a comparison of fibre sequences by Lemma 3.7, so the induced map π_* is a stable equivalence. $\square$

Lemma 3.9. *Suppose that A_i, $i \in I$ is a set of simplicial R-module spectra. Then the canonical map*

$$\omega : \bigvee_{i \in I} uA_i \to u(\bigoplus_{i \in I} A_i)$$

is a stable equivalence of spectra.

Proof. The functors on both sides of the transformation ω preserve filtered colimits in I. The set I is a filtered colimit of its finite subsets, so it suffices to assume that I is finite. In this case, ω is the canonical equivalence relating a finite wedge of spectra to the corresponding finite product. □

4 Symmetric simplicial module spectra

A symmetric simplicial R-module spectrum A consists of simplicial R-modules A^n, $n \geq 0$, such that each A^n has a symmetric group action $\Sigma_n \times A^n \to A^n$. The object A also comes equipped with bonding maps $S^1 \otimes A^n \to A^{n+1}$ such that all composites

$$S^r \otimes A^n \cong S^1 \otimes \cdots \otimes S^1 \otimes A^n \to A^{r+n}$$

are Σ_{r+n}-equivariant, where Σ_r acts by permuting smash factors in the simplicial r-sphere

$$S^r = S^1 \wedge \cdots \wedge S^1.$$

A morphism $f : A \to B$ of symmetric simplicial R-module spectra consists of Σ_n-equivariant simplicial R-module morphisms $f : A^n \to B^n$ which respect the bonding maps. The resulting category of symmetric simplicial R-module spectra is denoted by $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$. This category is complete and cocomplete.

Write $\mathbf{Spt}^{\Sigma}$ for the category of presheaves of symmetric spectra. The forgetful functor u and the pointed free abelian group functor $\tilde{R}$ determine functors

$$u : \mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R) \to \mathbf{Spt}^{\Sigma}$$

and

$$\tilde{R} : \mathbf{Spt}^{\Sigma} \to \mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$$

as in the ordinary spectrum case. If A is a symmetric simplicial R-module spectrum, then uA is the presheaf of symmetric spectra with $(uA)^n = u(A^n)$, and with bonding maps

$$S^1 \wedge uA^n \xrightarrow{\gamma} u(S^1 \otimes A^n) \xrightarrow{u\sigma} uA^{n+1}.$$

The symmetry condition, namely that the composite map

$$S^r \wedge uA^n \to uA^{r+n}$$

should be $(\Sigma_r \times \Sigma_n)$-equivariant, is a consequence of the commutativity of (3.6). If X is a presheaf of symmetric spectra then $\tilde{R}X$ is the symmetric simplicial R-module spectrum with $(\tilde{R}X)^n = \tilde{R}(X^n)$, with bonding maps

$$S^r \otimes \tilde{R}X^n \xrightarrow{\cong} \tilde{R}(S^r \wedge X^n) \xrightarrow{\tilde{R}\sigma} \tilde{R}X^{r+n}$$

It follows from the commutativity of the diagram (3.5) that the adjunction maps $\epsilon : \tilde{R}uA^n \to A^n$ assemble to give a natural map

$$\epsilon : \tilde{R}uA \to A$$

of symmetric simplicial R-module spectra. It follows from the commutativity of the diagram (3.4) that the maps $\eta : X^n \to u\tilde{R}X^n$ determine a natural map of presheaves of symmetric spectra

$$\eta : X \to u\tilde{R}X.$$

In particular, the spectrum level functor $\tilde{R}$ is left adjoint to the spectrum level version of the forgetful functor u.

It is shown in [11] that the category $\mathbf{Spt}^{\Sigma}$ has a proper closed simplicial model structure such that the following properties hold:

1) the cofibrations and weak equivalences are defined levelwise,

2) the fibrations, or *injective fibrations*, are defined to have the right lifting property with respect to all maps which are level weak equivalences and level cofibrations

3) there are generating sets I of trivial cofibrations and J of cofibrations.

The generating set I (respectively J) for the injective model structure on the category $\mathbf{Spt}^{\Sigma}$ of presheaves of symmetric spectra can be described as the collection of all maps $F_nK \to F_nL$ associated to a generating set of pointed trivial cofibrations (respectively cofibrations) $K \to L$ of simplicial presheaves, where F_n is the left adjoint to the level n functor $X \mapsto X^n$.

A map $f : A \to B$ of symmetric simplicial R-module spectra is a *levelwise weak equivalence* (respectively *cofibration*, *fibration*) if all maps $g : A^n \to B^n$ are weak equivalences (respectively cofibrations, fibrations) of simplicial R-modules.

Say that a map $f : A \to B$ of symmetric simplicial R-module spectra is an *injective fibration* if the underlying map $f_* : uA \to uB$ of presheaves of symmetric spectra is an injective fibration. An *injective cofibration* is a map which has the left lifting property with respect to all morphisms which are levelwise weak equivalences and injective fibrations. Observe that all level cofibrations $i : X \to Y$ of presheaves of symmetric spectra induce injective cofibrations $i_* : \tilde{R}X \to \tilde{R}Y$ of symmetric simplicial R-module spectra.

Lemma 4.1. *The category $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$ together with the levelwise weak equivalences, injective fibrations and injective cofibrations, satisfies the axioms for a proper closed simplicial model category.*

Proof. The existence of the model structure is the standard argument: the sets $\tilde{R}I$ and $\tilde{R}J$ give generating families of trivial cofibrations and fibrations for $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$.

The simplicial structure is given by the functors $A \mapsto A \otimes \tilde{R}(K_+)$ and $B \mapsto B^K$, and the fibration version of Quillen's axiom **SM7** is a consequence of the corresponding result for presheaves of symmetric spectra.

The generating cofibrations are levelwise cofibrations, so that all cofibrations are levelwise cofibrations. All injective fibrations are levelwise fibrations. Properness for this model structure on $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$ therefore follows from properness for simplicial R-modules. □

Remark 4.2. The model structure given in Lemma 4.1 is called the *injective model structure* for $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$. Unlike either symmetric spectra or presheaves of symmetric spectra, the injective model structure is not used to construct a stable model structure for the category $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$ of symmetric simplicial R-modules. A construction analogous to the injective fibrant model is, however, required for the proof of Lemma 4.14 below.

We shall need an explicit description of the left adjoint F_nA of the level n functor $B \mapsto B^n$ for a simplicial R-module A. First of all, write

$$A \otimes \Sigma_n = \bigoplus_{\sigma \in \Sigma_n} A$$

Then $A \otimes \Sigma_n$, concentrated in degree n is the free symmetric sequence G_nA on the simplicial R-module A, and then F_nA is specified by the symmetric sequence tensor product

$$F_nA = \tilde{R}S \otimes G_nA$$

The symmetric spectrum object F_nA can also be described at level p by the identification

$$(F_nA)_p = (\tilde{R}S^{p-n} \otimes (\bigoplus_{\sigma \in \Sigma_n} A)) \otimes_{\Sigma_{p-n} \times \Sigma_n} \Sigma_p$$

For a simplicial R-module B, there is a symmetric simplicial simplicial R-module spectrum $\Sigma^{\infty} B$, specified by the sequence of simplicial R-modules

$$B, S^1 \otimes B, S^2 \otimes B, \ldots$$

It is clear that $F_0B = \Sigma^{\infty} B$.

The functor $B \mapsto \Sigma^{\infty} B$, interpreted to take values in $\mathbf{Spt}(s\,\mathrm{Mod}_R)$ is left adjoint to the 0-level functor $A \mapsto A^0$ on the category of simplicial R-module spectra. In general, given a simplicial R-module spectrum A and an integer n, the spectrum object $A[n]$ is specified in levels by

$$A[n]^k = \begin{cases} A^{n+k} & \text{if } n+k \geq 0 \\ 0 & \text{if } n+k < 0 \end{cases}$$

If $n \geq 0$ then $B \mapsto \Sigma^\infty B[-n]$ is left adjoint to the level n functor $A \mapsto A^n$.

There is an obvious functor

$$U_R : \mathbf{Spt}^\Sigma(s\,\mathrm{Mod}_R) \to \mathbf{Spt}(s\,\mathrm{Mod}_R)$$

which forgets the symmetric structure. This functor has a left adjoint V_R which is determined by the assignment

$$V_R(\Sigma^\infty A[-n]) = F_n A$$

More generally, every simplicial R-module spectrum B has a layer filtration $B = \varinjlim L_n B$, where $L_n B$ is specified by the sequence

$$B^0, \dots, B^n, S^1 \otimes B^n, S^2 \otimes B^n, \dots$$

There are natural pushouts

$$\begin{array}{ccc} \Sigma^\infty(S^1 \otimes B^n)[-n-1] & \longrightarrow & L_n B \\ \downarrow & & \downarrow \\ \Sigma^\infty B^{n+1}[-n-1] & \longrightarrow & L_{n+1} B \end{array}$$

It follows that $V_R = \varinjlim V_R L_n B$, where the objects $V_R L_n B$ are specified inductively by the pushouts

$$\begin{array}{ccc} F_{n+1}(S^1 \otimes B^n) & \longrightarrow & V_R L_n B \\ \downarrow & & \downarrow \\ F_{n+1} B^{n+1} & \longrightarrow & V_R L_{n+1} B \end{array}$$

Observe that if X is a presheaf of spectra, then

$$V_R \tilde{R} X \cong \tilde{R} V X$$

where $V : \mathbf{Spt} \to \mathbf{Spt}^\Sigma$ is the left adjoint to the forgetful functor $U : \mathbf{Spt}^\Sigma \to \mathbf{Spt}$, as defined in [13].

Lemma 4.3. *Suppose that $i : A \to B$ is a cofibration of simplicial R-module spectra. Then the map $i_* : V_R A \to V_R B$ is a levelwise cofibration of symmetric simplicial R-module spectra.*

Proof. It is enough to prove the result for cofibrations

$$V_R \tilde{R} X \to V_R \tilde{R} Y$$

arising from cofibrations of presheaves of spectra $X \to Y$. There is a natural isomorphism

$$V_R \tilde{R} X \cong \tilde{R} V X$$

so the result follows from Lemma 3 of [13]. □

A map $f : A \to B$ of symmetric simplicial R-module spectra is said to be a *stable equivalence* (respectively *stable fibration*) if the underlying map $f_* : uA \to uB$ is a stable equivalence (respectively stable fibration) of presheaves of symmetric spectra.

Lemma 4.4. *Suppose given a pushout diagram*

$$\begin{array}{ccc} A & \longrightarrow & C \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i_*} \\ B & \longrightarrow & D \end{array}$$

in $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$*, where* i *is a level cofibration and a stable equivalence. Then the map* i_* *is a level cofibration and a stable equivalence.*

Proof. We only have to show that i_* is a stable equivalence. The maps i and i_* have a common cokernel E. The sequence

$$A \xrightarrow{i} B \to E$$

is a stable homotopy fibre sequence of the underlying presheaves of spectra, by Lemma 3.7, and is therefore a stable homotopy cofibre sequence of presheaves of symmetric spectra.

This analysis holds for any short exact sequenc, so one can finish the proof by showing that, given a cofibre sequence

$$X \xrightarrow{j} Y \to Y/X$$

of presheaves of symmetric spectra, the map j is a stable equivalence if and only if Y/X is stably equivalent to a point.

This last claim follows from a standard argument. Certainly trivial cofibrations are preserved by pushout in this category, so that one statement is obvious. For the other, recall that a map $i : A \to B$ is a weak equivalence if and only if the induced map $i^* : \mathbf{hom}(B, Z) \to \mathbf{hom}(A, Z)$ is a weak equivalence of simplicial sets for all stably fibrant injective objects Z. Any such Z has a delooping $Z \simeq \Omega Z[1]$ (where $Z \mapsto Z[1]$ is the symmetric spectrum shift), and so there is a fibre sequence

$$\begin{aligned} \mathbf{hom}(B, Z) \xrightarrow{i^*} \mathbf{hom}(B, Z) &\to \mathbf{hom}(B/A, Z[1]) \\ &\to \mathbf{hom}(B, Z[1]) \xrightarrow{i^*} \mathbf{hom}(A, Z[1]). \end{aligned}$$

If B/A is stably trivial, it follows that $i^* : \mathbf{hom}(B, Z) \to \mathbf{hom}(A, Z)$ is a map of H-spaces which induces an isomorphism in all homotopy groups, and is therefore a weak equivalence. $\square$

We shall need to know that the functor $Y \mapsto Y \wedge X$ takes stable trivial cofibrations of presheaves of symmetric spectra to stable equivalences, for any presheaf of symmetric spectra X. This is a consequence of the proof of Proposition 4.19 of [14], but see also Theorem 5.3.6 of [7].

If A is a symmetric simplicial R-module spectrum and K is a pointed simplicial presheaf, write $A \otimes K$ for the symmetric simplicial R-module spectrum $A \otimes \tilde{R}(K)$. There is a natural map of presheaves of symmetric spectra

$$\gamma : uA \wedge K \to u(A \otimes K) \tag{4.1}$$

which is defined at level n by the map $\gamma : uA^n \wedge K \to u(A^n \otimes K)$ discussed above. The commutativity of the diagram (3.6) implies that the maps γ respect the symmetric spectrum structures.

Lemma 4.5. *Suppose that A is a symmetric simplicial R-module spectrum and that K is a pointed simplicial presheaf. Then the map*

$$\gamma : uA \wedge K \to u(A \otimes K)$$

is a stable equivalence of the underlying presheaves of spectra.

Proof. The map γ is the diagonal of maps of simplicial presheaves of symmetric spectra

$$\gamma : uA \wedge K_n \to u(A \otimes K_n)$$

and all such maps are stable equivalences by Lemma 3.9. □

The *tensor product* $A \otimes_\Sigma B$ of the symmetric simplicial R-module spectra A and B is defined, by analogy with the smash product of symmetric spectra, as a coequalizer

$$\tilde{R}S \otimes A \otimes B \rightrightarrows A \otimes B \to A \otimes_\Sigma B$$

in the symmetric sequence category of simplicial R-modules. The defining maps $\tilde{R}S \otimes A \otimes B$ are the multiplication map $\sigma \otimes 1 : \tilde{R}S \otimes A \otimes B \to A \otimes B$ and the composite

$$\tilde{R}S \otimes A \otimes B \xrightarrow{\tau \otimes 1} A \otimes \tilde{R}S \otimes A \xrightarrow{1 \otimes \sigma} A \otimes B$$

A map $f : A \otimes_\Sigma B \to C$ in $\mathbf{Spt}^\Sigma(s\,\mathrm{Mod}_R)$ is therefore determined by simplicial R-module homomorphisms $f_{p,q} : A^p \otimes B^q \to C^{p+q}$ such that the following diagrams commute:

$$\begin{array}{ccc} S^r \otimes A^p \otimes B^q & \xrightarrow{\sigma \otimes 1} & A^{r+p} \otimes B^q \\ {\scriptstyle 1 \otimes f_{p,q}} \downarrow & & \downarrow {\scriptstyle f_{r+p,q}} \\ S^r \otimes C^{p+q} & \xrightarrow[\sigma]{} & C^{r+p+q} \end{array}$$

$$\begin{array}{ccccc} S^r \otimes A^p \otimes B^q & \xrightarrow{\tau \otimes 1} & A^p \otimes S^r \otimes B^q & \xrightarrow{1 \otimes \sigma} & A^p \otimes B^{r+q} \\ {\scriptstyle \sigma \otimes 1} \downarrow & & & & \downarrow {\scriptstyle f_{p,r+q}} \\ A^{r+p} \otimes B^q & \xrightarrow[f_{r+p,q}]{} & C^{r+p+q} & \xrightarrow[c_{r,p} \otimes 1]{} & C^{p+r+q} \end{array}$$

Here, τ flips tensor factors and $c_{r,p} \in \Sigma_{r+p}$ shuffles the first r elements of the set $\{1, \dots, r+p\}$ past the last p elements.

In particular, the bonding maps $\sigma : S^p \otimes A^q \to A^{p+q}$ assemble to define a natural isomorphism

$$\sigma : \tilde{R}S \otimes_\Sigma A \xrightarrow{\cong} A$$

for all symmetric simplicial R-module spectra A.

If Y is a presheaf of symmetric spectra, the composites

$$uA^n \wedge Y^m \xrightarrow{\gamma} u(A^n \otimes Y^m) \to u(A \otimes Y)^{n+m}$$

determine a map $\gamma : uA \wedge Y \to u(A \otimes_\Sigma Y)$ which is natural in A and Y, and specializes to the map $\gamma : uA \wedge K \to u(A \otimes K)$ of (4.1) if $Y = \Sigma^\infty K$.

Observe as well that the functor

$$K \mapsto A \otimes_\Sigma F_n(\tilde{R}K)$$

takes pointed cofibrations of simplicial sets $K \subset L$ to level cofibrations, since there are canonical isomorphisms

$$A \otimes_\Sigma F_n(\tilde{R}K) \cong (A \otimes_\Sigma F_n(\tilde{R}S^0)) \otimes K.$$

Lemma 4.6. *Suppose that A is a symmetric simplicial R-module spectrum. Then the map*

$$\gamma : uA \wedge V(S[-r]) \to u(A \otimes_\Sigma V(S[-r]))$$

is a stable equivalence of underlying presheaves of spectra.

Proof. The map γ can be identified with the canonical map $\gamma : (uA)[-r] \to u(A[-r])$ from the set theoretic r-shift functor to the R-module r-shift functor. Explicitly,

$$(uA)[-r]^p = \bigvee_{\Sigma_p/(\Sigma_{p-r} \times \Sigma_r)} (uA)^{p-r} \wedge (\Sigma_r)_+$$

If we identify $\Sigma_p/(\Sigma_{p-r} \times \Sigma_r)$ with the set of subsets of cardinality $p-r$ in the set of numbers $\underline{p} = \{1, \dots, p\}$, then the bonding map

$$S^1 \wedge (uA)[-r]^p \to (uA)[-r]^{p+1}$$

is defined on the summand corresponding to a subset $I \subset \underline{r} = \{1, \dots, r\}$ by the composite

$$\begin{array}{l} S^1 \wedge (uA)^{p-r} \wedge (\Sigma_r)_+ \\ \qquad \downarrow \sigma \\ (uA)^{p+1-r} \wedge (\Sigma_r)_+ \xrightarrow[in_{1 \oplus s(I)}]{} \bigvee_{\Sigma_{p+1}/(\Sigma_{p+1-r} \times \Sigma_r)} (uA)^{p+1-r} \wedge (\Sigma_r)_+, \end{array}$$

where the shift map $s : \{1, \dots, p\} \to \{1, \dots, p+1\}$ is defined by $s(i) = i+1$.

In general, write $T_nX = X[-n][n]$ for a presheaf of spectra X and $n \geq 0$, so that

$$T_nX^i = \begin{cases} X^i & \text{if } i \geq n \\ * & \text{if } i < n. \end{cases}$$

Then it follows from the analysis above that there is a natural isomorphism of presheaves of spectra

$$\begin{aligned} uA[-r] \cong & (uA \wedge (\Sigma_r)_+)[-r] \vee (\bigvee_{\binom{r}{1}} T_{r+1}((uA \wedge (\Sigma_r)_+)[-r]) \vee \\ & (\bigvee_{\binom{r+1}{2}} T_{r+2}((uA \wedge (\Sigma_r)_+)[-r]) \vee \dots \end{aligned}$$

A corresponding decomposition

$$\begin{aligned} A[-r] \cong & (A \otimes (\Sigma_r)_+)[-r] \oplus (\bigoplus_{\binom{r}{1}} T_{r+1}((A \otimes (\Sigma_r)_+)[-r]) \oplus \\ & (\bigoplus_{\binom{r+1}{2}} T_{r+2}((A \otimes (\Sigma_r)_+)[-r]) \oplus \dots \end{aligned}$$

holds for the symmetric simplicial R-module spectrum shift $A \mapsto A[-r]$. Note, for example, that $\binom{r+1}{2}$ is the number of 2-element subsets of $\underline{r+2}$ which do not contain the element 1. The map $\gamma : (uA)[-r] \to u(A[-r])$ is induced in all summands by the obvious map $uA \wedge (\Sigma_r)_+ \to u(A \otimes (\Sigma_r)_+)$, which is a stable equivalence of underlying presheaves of spectra by Lemma 3.9. It follows from the same result that γ is a stable equivalence of underlying presheaves of spectra. □

Proposition 4.7. *Suppose that X is a cofibrant presheaf of spectra and that A is a symmetric simplicial R-module spectrum. Then the canonical map*

$$\gamma : uA \wedge VX \to u(A \otimes_\Sigma VX)$$

is a stable equivalence of underlying presheaves of spectra.

Proof. The functors on both sides of the transformation γ preserve cofibre sequences of cofibrant presheaves of spectra. In effect, for any presheaf of symmetric spectra Y the functor $X \mapsto Y \wedge VX$ preserves cofibre sequences of cofibrant presheaves of spectra because Y can be replaced up to level equivalence by a cofibrant model and the functor $? \wedge VX$ preserves level equivalences since X is cofibrant. On the other side, any cofibre sequence $X \to Y \to Y/X$ induces a level cofibre sequence $VX \to VY \to V(Y/X)$ of presheaves of symmetric spectra, and hence induces an exact sequence

$$A \otimes_\Sigma VX \to A \otimes_\Sigma VY \to A \otimes_\Sigma V(Y/X) \to 0 \tag{4.2}$$

Finally, the functor $X \mapsto A \otimes_\Sigma VX$ takes cofibrations of the form $F_nK \to F_nL$ (and hence all cofibrations) to level cofibrations, so the sequence (4.2) is actually short exact, and is therefore a stable cofibre sequence of the underlying presheaves of spectra by Lemma 3.7.

The layer filtration L_nX for X is defined by pushout squares

$$\begin{array}{ccc} \Sigma^\infty(S^1 \wedge X^n)[-n-1] & \longrightarrow & L_nX \\ {\scriptstyle \sigma_*}\downarrow & & \downarrow \\ \Sigma^\infty X^{n+1}[-n-1] & \longrightarrow & L_{n+1}X \end{array}$$

in which all maps σ_* are cofibrations since X is cofibrant. It follows that all induced maps $L_nX \to L_{n+1}X$ are cofibrations and that all objects L_nX are cofibrant. It suffices therefore, by comparing cofibres, to show that all maps

$$\gamma : uA \wedge \Sigma^\infty K[-r] \to u(A \otimes_\Sigma V(\Sigma^\infty K[-r]))$$

are stable equivalences.

There is an isomorphism

$$uA \wedge \Sigma^\infty K[-r] \cong uA \wedge V(S[-r]) \wedge K$$

and the map γ for this object can be identified with the composite

$$uA \wedge V(S[-r]) \wedge K \xrightarrow{\gamma \wedge 1} u(A \otimes_\Sigma V(S[-r])) \wedge K \xrightarrow{\gamma} u(A \otimes_\Sigma V(S[-r]) \otimes K)$$

These maps are stable equivalences of underlying presheaves of spectra by Lemmas 4.6 and 4.5. □

Lemma 4.8. *Every map $f : A \to B$ of symmetric simplicial R-module spectra has a natural factorization*

$$\begin{array}{ccc} A & \xrightarrow{i} & C \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} \\ & & B \end{array}$$

where p is a stable fibration, i is a stable equivalence, and i is a level cofibration which has the left lifting property with respect to all stable fibrations.

Proof. This is a transfinite small object argument, which is based on the observation that a map p is a stable fibration if and only if it has the right lifting property with respect to all maps $\tilde{R}VX \to \tilde{R}VY$ arising from a set of trivial stable cofibrations $j : X \to Y$ of presheaves of spectra. The map i is a filtered colimit of a sequence of pushouts of maps of this form. The cofibre Y/X of the test map j is a cofibrant presheaf of spectra such that the map $Y/X \to *$ is a stable equivalence. It follows from Proposition 4.7 (with $A = \tilde{R}S$) that the induced map $\tilde{R}V(Y/X) \to 0$ is a stable equivalence, and so the map $j : \tilde{R}VX \to \tilde{R}VY$

is a stable equivalence as well as a level cofibration. Thus, for any pushout diagram

$$\begin{array}{ccc} \tilde{R}VX & \longrightarrow & A \\ \downarrow & & \downarrow{\scriptstyle i_*} \\ \tilde{R}VY & \longrightarrow & B \end{array}$$

Lemma 4.4 implies that the level cofibration i_* is stable equivalence.

Filtered colimits of stable equivalences are stable equivalences in category of presheaves of symmetric spectra, by a standard argument. □

Say that a map $i : A \to B$ of $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$ is a *stable cofibration* if it has the left lifting property with respect to all maps which are stable fibrations and stable weak equivalences. The map i in the factorization of Lemma 4.8 has the left lifting property with respect to all stable fibrations, and is therefore a stable cofibration.

Observe that a map $p : A \to B$ of $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$ is a stable fibration and a stable weak equivalence if and only if all maps $p : A^n \to B^n$ are weak equivalences and fibrations of simplicial R-modules. It follows that all level cofibrations

$$F_n\tilde{R}K \to F_n\tilde{R}L$$

arising from cofibrations $K \subset L$ of simplicial presheaves are stable cofibrations. There is a generating set for all such cofibrations, so that every map $f : A \to B$ has a factorization

$$\begin{array}{ccc} A & \xrightarrow{j} & D \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle q} \\ & & B \end{array}$$

such that j is a stable cofibration and q is stable fibration and a stable weak equivalence.

Theorem 4.9. *With the definition of stable fibration, stable equivalence and stable cofibration given above the category $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$ of symmetric simplicial R-module spectra satisfies the axioms for a proper closed simplicial model category.*

Proof. The factorization axiom **CM5** follows from Lemma 4.8 and the remarks preceding the statement of Theorem 4.9. The axiom **CM4** is a consequence of Lemma 4.8. The simplicial model axiom **SM7** is a consequence of the corresponding statement for the stable structure on $\mathbf{Spt}(s\,\mathrm{Mod}_R)$. The cofibration part of properness follows from a long exact sequence argument which is based on Lemma 3.7, while the fibration part is a consequence of properness for presheaves of symmetric spectra. □

The forgetful functor $U_R : \mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R) \to \mathbf{Spt}(s\,\mathrm{Mod}_R)$ preserves stable fibrations and stable equivalences between stably fibrant objects — for the latter, use the fact that a stable equivalence of stably fibrant objects is a level equivalence, since this is so for presheaves of symmetric spectra. It follows the adjoint pair (V_R, U_R) is a Quillen pair, but more is true:

Theorem 4.10. *The adjoint pair*

$$V_R : \mathbf{Spt}(s\,\mathrm{Mod}_R) \leftrightarrows \mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R) : U_R$$

is a Quillen equivalence, and therefore induces an equivalence of stable homotopy categories

$$\mathrm{Ho}_{stable}(\mathbf{Spt}(s\,\mathrm{Mod}_R)) \simeq \mathrm{Ho}_{stable}(\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)).$$

The forgetful functor U_R reflects stable equivalences, so it suffices to show [4] that the composite map

$$A \xrightarrow{\eta} U_R V_R A \xrightarrow{j_*} U_R(V_R A)_s$$

is a stable equivalence of simplicial R-module spectra if A is cofibrant, and where $j : V_R A \to (V_R A)_s$ is a choice of stably fibrant model in $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$. The demonstration of this statement requires some preliminary lemmas.

Lemma 4.11. *Suppose that A is a simplicial R-module spectrum. Then the composite*

$$A \xrightarrow{\eta} \Omega(A \otimes S^1) \xrightarrow{j_*} \Omega(A \otimes S^1)_s$$

is a stable equivalence, where $j : A \otimes S^1 \to (A \otimes S^1)_s$ is a choice of stably fibrant model in $\mathbf{Spt}(s\,\mathrm{Mod}_R)$.

Proof. The loop functor Ω on $\mathbf{Spt}(s\,\mathrm{Mod}_R)$ preserves stable equivalences, so it suffices to show that the map $\eta : A \to \Omega(A \otimes S^1)$ induces an isomorphism on stable homotopy groups. The map η is actually a strict weak equivalence, on account of Corollary 3.4. □

Lemma 4.12. *Suppose that $f : A \to B$ is a map of symmetric simplicial R-module spectra. Then f is a stable equivalence if and only if the induced map $f_* : A \otimes S^1 \to B \otimes S^1$ is a stable equivalence.*

Proof. There is a commutative diagram

$$\begin{array}{ccc} uA \wedge S^1 & \xrightarrow{f_* \wedge 1} & uB \wedge S^1 \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle \gamma} \\ u(A \otimes S^1) & \xrightarrow[f_*]{} & u(B \otimes S^1) \end{array}$$

in which the vertical maps γ are stable equivalences by Lemma 4.5. Thus, the map $f_* : A \otimes S^1 \to B \otimes S^1$ is a stable equivalence if and only if $f_* \wedge 1$ is a stable equivalence of presheaves of symmetric spectra. But the corresponding fact about smashing with S^1 for presheaves of symmetric spectra (see the proof of Lemma 4.25 in [14]) implies that $f_* \wedge 1$ is a stable equivalence of presheaves of symmetric spectra if and only if $f : A \to B$ is a stable equivalence of symmetric simplicial R-module spectra. □

Corollary 4.13. *Suppose that Y is a cofibrant symmetric simplicial R-module spectrum.*

1) The composite

$$Y \xrightarrow{\eta} \Omega(Y \otimes S^1) \xrightarrow{j_*} \Omega(Y \otimes S^1)_s$$

is a stable equivalence of symmetric simplicial R-module spectra. for any choice of stably fibrant model $j : Y \otimes S^1 \to (Y \otimes S^1)_s$ for $Y \otimes S^1$.

2) Suppose that Y is stably fibrant. Then any stably fibrant model $j : Y \otimes S^1 \to (Y \otimes S^1)_s$ is a stable equivalence of the underlying simplicial R-module spectra.

Proof. The first statement follows from Lemma 4.12 and the commutativity of the diagram

$$\begin{array}{ccc} Y \otimes S^1 & \xrightarrow{(j_*\eta)\otimes 1} & \Omega(Y \otimes S^1)_s \otimes S^1 \\ & {}_{j}\searrow & \downarrow{ev} \\ & & (Y \otimes S^1)_s \end{array}$$

The map ev in the diagram above is a stable equivalence of the underlying simplicial R-module spectra (consequence of Corollary 3.4). For the second statement, note that $j_*\eta : Y \to \Omega(Y \otimes S^1)_s$ is a stable equivalence of stably fibrant objects, and hence a level equivalence of the underlying simplicial R-module spectra, so the diagram above implies that j is a stable equivalence of the underlying simplicial R-module spectra. □

Lemma 4.14. *Suppose that B is a simplicial R-module, and write $S \otimes B = F_0 B$ for the corresponding suspension object in $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$. Then any stably fibrant model $j : S \otimes B \to (S \otimes B)_s$ induces a stable equivalence $j_* : U_R(S \otimes B) \to U_R(S \otimes B)_s$ of simplicial R-module spectra.*

Proof. The objects $(S^* \otimes B) \otimes S^m$ comprise a spectrum object $(S \otimes B) \otimes S$ in $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$. Equivalently,

$$S \otimes (B \otimes S) \cong (S \otimes B) \otimes S$$

is a symmetric spectrum object in $\mathbf{Spt}(s\,\mathrm{Mod}_R)$. There is a generating set of trivial stable cofibrations for $\mathbf{Spt}(s\,\mathrm{Mod}_R)$, so an argument similar to that for Lemma 4.1 shows that there is a map

$$j : S^n \otimes (B \otimes S^m) \to Y^{n,m}$$

of symmetric spectrum objects in $\mathbf{Spt}(s\,\mathrm{Mod}_R)$, such that each simplicial R-module spectrum map $S^n \otimes (B \otimes S) \to Y^{n,*}$ is a stable equivalence, with $Y^{n,*}$ stably fibrant for each n. The adjoint bonding maps $Y^{n,*} \to \Omega Y^{n+1,*}$ are stable equivalences of stably fibrant simplicial R-module spectra by Lemma 4.11, so that the symmetric simplicial R-module spectrum $Y^{*,0}$ at level 0 is stably fibrant. The map j induces an isomorphism of sheaves of stable homotopy groups of the underlying presheaves of bispectra, and the sheaves of stable homotopy groups of $Y^{*,0}$ and $S \otimes B$ are isomorphic to the sheaves of stable homotopy groups of the presheaves of bispectra Y and $(S \otimes B) \otimes S$ (the latter by Lemma 4.11 again). In summary, $Y^{*,0}$ is a stably fibrant object in $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$, and the map $S \otimes B \to Y^{*,0}$ is a stable equivalence of the underlying simplicial R-module spectra. □

Lemma 4.15. *Suppose that A is a cofibrant simplicial R-module spectrum. Then the composite morphism*

$$A \xrightarrow{\eta} U_R V_R A \xrightarrow{j_*} U_R(V_R A)_s$$

is a stable equivalence if and only if the composite

$$A \otimes S^1 \xrightarrow{\eta} U_R V_R(A \otimes S^1) \xrightarrow{j_*} U_R(V_R(A \otimes S^1))_s$$

is a stable equivalence of simplicial R-module spectra.

Proof. This proof boils down to knowing that the map $j \otimes 1 : (V_R A) \otimes S^1 \to (V_R A)_s \otimes S^1$ is a stable cofibration and a stable equivalence of $\mathbf{Spt}^{\Sigma}(s\,\mathrm{Mod}_R)$, so that there is a map $f : (V_R A)_s \otimes S^1 \to V_R(A \otimes S^1)_s$ which makes the diagram

$$\begin{array}{ccc} (V_R A) \otimes S^1 & \xrightarrow{j \otimes 1} & (V_R A)_s \otimes S^1 \\ \cong \downarrow & & \downarrow f \\ V_R(A \otimes S^1) & \xrightarrow[j]{} & V_R(A \otimes S^1)_s \end{array}$$

commute. In particular, part 2) of Corollary 4.13 implies that the map f is a stable equivalence on the underlying simplicial R-mdoule spectra. This detail allows us to check (see the argument in [14]) that the composite

$$A \otimes S^1 \xrightarrow{\eta \otimes 1} U_R V_R A \otimes S^1 \xrightarrow{j_* \otimes 1} U_R(V_R A)_s \otimes S^1$$

is a stable equivalence of simplicial R-module spectra.

Now use Lemma 4.11 to show that the composite $j_* \eta$ is a stable equivalence of simplicial R-module spectra if and only if the map $j_* \eta \otimes S^1$ is a stable equivalence of simplicial R-module spectra. □

Proof of Theorem 4.10. We use the layer filtration $L_n A$ for a cofibrant simplicial R-module spectrum A. There are natural stable equivalences

$$\Sigma^\infty A^n[-n] \to L_n A$$

and

$$\Sigma^\infty A^n[-n] \otimes S^n \to \Sigma^\infty A^n.$$

The functors at both ends of the map

$$A \xrightarrow{\eta} U_R V_R A \xrightarrow{j_*} U_R(V_R A)_s$$

preserve stable weak equivalences between cofibrant simplicial R-module spectra, so Lemma 4.15 shows that proving the statement of Theorem 4.10 for $L_n A$ reduces to proving the statement for $A = \Sigma^\infty B$, but this is Lemma 4.14.

The final step is to show that the composite map

$$\varinjlim L_n A \xrightarrow{\eta} U_R V_R(\varinjlim L_n A) \xrightarrow{j_*} U_R(V_R(\varinjlim L_n A))_s$$

is a stable equivalence. This follows from the previous assertions and the observation that there is a stable equivalence

$$\varinjlim (V_R L_n A)_s \to (V_R(\varinjlim L_n A))_s.$$

□

Lemma 4.16. *Suppose that the maps $A \to B$ and $C \to D$ are stable cofibrations of $\mathbf{Spt}^\Sigma(s\,\mathrm{Mod}_R)$. Then the induced map*

$$(B \otimes_\Sigma C) \cup_{A \otimes_\Sigma C} (A \otimes_\Sigma D) \to B \otimes_\Sigma D \tag{4.3}$$

is a stable cofibration, which is also a stable equivalence if either i or j is a stable equivalence.

Proof. Suppose that $A_1 \to B_1$ and $A_2 \to B_2$ are cofibrations of pointed simplicial presheaves. Then the map

$$(F_n B_1 \otimes_\Sigma F_m A_2) \cup (F_n A_1 \otimes_\Sigma F_m B_2) \to F_n B_1 \otimes_\Sigma F_m B_2 \tag{4.4}$$

can be identified with the map induced by applying the functor F_{n+m} to the map of free simplicial R-modules which is induced by the cofibration

$$(B_1 \wedge A_2) \cup (A_1 \wedge B_2) \to B_1 \wedge B_2$$

of pointed simplicial presheaves (compare [7, Prop.2.2.6]). It follows that the map (4.4) is a stable cofibration, and insofar as all stable cofibrations are built by pushouts and filtered colimits of maps of this form up to retraction, all maps (4.3) are stable cofibrations.

The cokernel of the map (4.3) is the object $B/A \otimes_\Sigma D/C$, where both B/A and D/C are cofibrant. To prove the second claim, it suffices to show that $E \otimes_\Sigma E'$ is stably trivial if E and E' are both cofibrant and E is stably trivial.

It is a consequence of Lemmas 4.5 and 4.6 that there is a stable equivalence of presheaves of symmetric spectra

$$u(E \otimes_\Sigma F_n \tilde{R} K) \simeq uE \wedge VS[-n] \wedge K,$$

and smashing with $VS[-n] \wedge K$ preserves level equivalences in the category of presheaves of symmetric spectra. We can therefore replace uE by a cofibrant, stably trivial presheaf of symmetric spectra, and then the monoidal structure for the model structure of presheaves of symmetric spectra implies that $u(E \otimes_\Sigma F_n K)$ is stably trivial. The object $E \otimes_\Sigma E'$ is a retract of a filtered colimit of pushouts of cofibrations involving objects of the form $E \otimes_\Sigma F_n \tilde{R}K$, so that $E \otimes_\Sigma E'$ is stably trivial. □

5 Higher Tor functors

5.1 The unstable case

Suppose that C and D are ordinary (ie. positively graded) chain complexes of R-modules, and recall that the classical tensor product $C \otimes D$ is the chain complex defined in degree n by

$$(C \otimes D)_n = \bigoplus_{i+j=n} C_i \otimes D_j,$$

with boundary $\partial : (C \otimes D)_n \to (C \otimes D)_{n-1}$ defined on the summand $C_i \otimes D_j$ by

$$\partial(x \otimes y) = \partial(x) \otimes y + (-1)^i x \otimes \partial(y).$$

It is also typical to refer to this object as the total complex $\mathrm{Tot}(C \otimes D)$ of a bicomplex, also denoted by $C \otimes D$, where

$$(C \otimes D)_{i,j} = C_i \otimes D_j$$

and having horizontal (respectively vertical) boundary specified on $C_i \otimes D_j$ by

$$\partial_h(x \otimes y) = \partial(x) \otimes y, \quad \partial_v(x \otimes y) = (-1)^i x \otimes \partial(y).$$

Suppose that A is a simplicial R-module. Then the *Moore complex* MA for A has n-chains specified by $MA_n = A_n$, and has boundary defined by

$$\partial(x) = \sum_{i=0}^{n} (-1)^i d_i(x).$$

Suppose that A and B are simplicial R-modules. Then the bisimplicial R-module $A \otimes B$ is specified by

$$(A \otimes B)_{i,j} = A_i \otimes B_j$$

and has horizontal and vertical simplicial structure maps defined in the obvious way. The *diagonal* simplicial R-module $d(A \otimes B)$ has

$$d(A \otimes B)_n = A_n \otimes B_n.$$

In the language developed here, the generalized Eilenberg-Zilber theorem [3, p.205] specializes to the assertion that there is a natural chain homotopy equivalence

$$Md(A \otimes B) \simeq \mathrm{Tot}(MA \otimes MB).$$

The Moore chains MA on a simplicial R-module are naturally homotopy equivalent to the *normalized chain complex* NA associated to A, by the same argument as for simplicial abelian groups [3, Th.IV.2.5]. Recall that the normalized chains are defined in degree n by

$$NA_n = \cap_{i=0}^{n-1} \ker d_i$$

and $\partial(x) = (-1)^n d_n(x)$ defines the boundary. Then there is an obvious inclusion $NA \subset MA$, and this inclusion induces an isomorphism $NA \cong MA/DA$ of simplicial R-modules, where DA is the subcomplex of MA which is generated by degenerate simplices in each degree.

If A and B are simplicial R-modules, the chain equivalences $NA \subset MA$ and $NB \subset MB$ induce a natural homology isomorphism

$$\mathrm{Tot}(NA \otimes NB) \to \mathrm{Tot}(MA \otimes MB),$$

by a standard spectral sequence argument. It follows that the are natural quasi-isomorphisms

$$Nd(A \otimes B) \subset Md(A \otimes B) \simeq \mathrm{Tot}(MA \otimes MB) \leftarrow \mathrm{Tot}(NA \otimes NB) \qquad (5.1)$$

Suppose that C and D are chain complexes. Then in view of the natural equivalences appearing in (5.1), there are natural identifications up to quasi-isomorphism of the form

$$\mathrm{Tot}(C \otimes D) \cong \mathrm{Tot}(N\Gamma C \otimes N\Gamma D) \simeq Nd(\Gamma C \otimes \Gamma D).$$

It is therefore harmless to identify the chain complex tensor product $\mathrm{Tot}(C \otimes D)$ with the simplicial abelian group tensor product $\Gamma C \otimes \Gamma D = d(\Gamma C \otimes \Gamma D)$ in the derived category. It also follows [3, Cor.IV.2.7] that there are natural isomorphisms

$$\pi_r(\Gamma C \otimes \Gamma D) \cong H_r(C \otimes D). \qquad (5.2)$$

Suppose that A and B are simplicial R-modules, and define $\mathrm{Tor}_n(A, B)$ *as a homology sheaf* by

$$\mathrm{Tor}_r(A, B) = \pi_r(X \otimes Y)$$

where the maps $X \to A$ and $Y \to B$ are cofibrant models for A and B, in the sense that both maps are local weak equivalences and X and Y are cofibrant. Note that there is a natural sheaf isomorphism

$$\mathrm{Tor}_0(A, B) \cong \pi_0 A \otimes \pi_0 B$$

on account of the identificiations above, where the indicated tensor product is in the sheaf category.

The monoidal model structure for the category of simplicial R-modules is used to show that $\mathrm{Tor}_r(A, B)$ is invariant up to isomorphism of the choices of X and Y, since the closed model structure of Lemma 1.3 on the category of simplicial abelian presheaves is monoidal with respect to the tensor product. More generally, we have the following:

Lemma 5.1. *Suppose that X is a cofibrant simplicial R-module. Then the functor $A \mapsto X \otimes A$ is exact, and preserves weak equivalences.*

Proof. The small object argument implies that X is a retract of a simplicial R-module Y, where Y is a transfinite filtered colimit

$$Y = \varinjlim V_i$$

such that $V_0 = RK_0$ is free on a simplicial presheaf K_0, and there are pushouts of simplicial R-modules

$$\begin{array}{ccc} RK_i & \longrightarrow & V_i \\ {\scriptstyle j_*}\downarrow & & \downarrow \\ RK_{i+1} & \longrightarrow & V_{i+1} \end{array}$$

where j_* is induced by a cofibration $j : K_i \to K_{i+1}$ of simplicial presheaves. Then j_* is a split monomorphism in each choice of section and degree, so all sequences

$$0 \to RK_i \otimes A \to (RK_{i+1} \otimes A) \oplus (V_i \otimes A) \to V_{i+1} \otimes A \to 0$$

are exact sequences of simplicial R-modules, in the presheaf category. The functors $A \mapsto RK \otimes A$ are plainly exact, so it suffices to show that they preserve quasi-isomorphisms.

This last claim is proved with a spectral sequence argument, which is based on the observation that there is a sheaf isomorphism

$$H_r(RK_n \otimes A) \cong L(RK_n) \otimes H_r A$$

since tensoring with a free module is exact. Here, L denotes the associated sheaf functor. □

Corollary 5.2. *Suppose that X is a cofibrant simplicial R-module. Then for all $n \geq 0$ the functor $A \mapsto A \otimes X_n$ is exact. There is a natural sheaf isomorphsm*

$$\pi_r(A \otimes X_n) \cong (\pi_r A) \otimes L(X_n),$$

Proof. The exactness is a consequence of the proof of Lemma 5.1. It follows that the canonical map

$$NA \otimes X_n \to N(A \otimes X_n)$$

is an isomorphism of chain complexes, and that the sheaf map

$$(H_r NA) \otimes L(X_n) \to H_r(NA \otimes X_n)$$

is an isomorphism. □

Corollary 5.3. *Suppose that X is a cofibrant simplicial R-module. Then the functor $A \mapsto A \otimes NX_n$ is exact on simplicial R-modules, for all $n \geq 0$. There is a natural sheaf isomorphism*

$$\pi_r(A \otimes NX_n) \cong (\pi_r A) \otimes L(NX_n).$$

Proof. Following the proof of [3, Th. IV.2.1], write

$$N_j X_n = \cap_{i=0}^{j} \ker(d_i) \subset X_n$$

for $0 \leq j < n$. In particular, $NX_n = N_{n-1}X_n$; write $N_{-1}X_n = X_n$. There are exact sequences

$$0 \to N_{j+1}X_n \to N_j X_n \xrightarrow{d_{j+1}} N_j X_{n-1} \to 0$$

and the face map d_{j+1} is split by the degeneracy $s_{j+1} : N_j X_{n-1} \to N_j X_n$. All functors $A \mapsto N_{-1}X_n \otimes A$ are exact by the Corollary 5.2, so that all maps $A \mapsto N_j X_n \otimes A$ are exact by induction on j. It follows that the canonical map

$$NA \otimes NX_n \to N(A \otimes NX_n)$$

is an isomorphism of chain complexes. □

Example 5.4. Suppose that R is a commutative ring with identity, so that the category Mod_R has enough projectives. Then every chain complex C of R-modules has a projective resolution, in the sense that there is a homology isomorphism $P \to C$, where P is a chain complex of projective R-modules — this is usually shown by using the Eilenberg-Cartan resolution of C. Then there is an analogue of Lemma 5.1 for chain complexes of R-modules: if P is a chain complex of projectives, then the functor $A \mapsto A \otimes \Gamma P$ on simplicial R-modules is exact and preserves quasi-isomorphisms. Suppose that C and D are chain complexes, and choose projective resolutions $P \to C$ and $Q \to D$. Choose cofibrant resolutions $X \to \Gamma P$ and $Y \to \Gamma Q$. Then the composites

$$X \to \Gamma P \to \Gamma C, \quad Y \to \Gamma Q \to \Gamma D$$

are cofibrant resolutions for ΓC and ΓD respectively, and there are weak equivalences of simplicial R-modules

$$X \otimes Y \xrightarrow{\simeq} \Gamma P \otimes Y \xrightarrow{\simeq} \Gamma P \otimes \Gamma Q$$

by Lemma 5.1 and by the corresponding observation about projective resolutions. Finally,

$$\pi_r(\Gamma P \otimes \Gamma Q) \cong H_r(P \otimes Q) = \mathrm{Tor}_r(C, D)$$

so that

$$\pi_r(X \otimes Y) \cong \mathrm{Tor}_r(C, D).$$

In other words, the definition of the Tor_* functors given above for simplicial R-modules on a presheaf of rings R specializes to the definition we know in the case of chain complexes of R-modules when R is an ordinary ring.

Corollary 5.5. *Suppose that A and B are simplicial R-modules, and that $X \to A$ is a cofibrant resolution of A. Then there is a sheaf isomorphism*

$$\mathrm{Tor}_r(A, B) \cong H_r(X \otimes B).$$

Proof. Suppose that $Y \to B$ is a cofibrant resolution of B. Then

$$\mathrm{Tor}_r(A, B) = H_r(X \otimes Y),$$

while the map $X \otimes Y \to X \otimes B$ is a weak equivalence since X is cofibrant. □

Corollary 5.6. *Suppose that A is a simplicial R-module and that*

$$0 \to B_1 \to B_2 \to B_3 \to 0$$

is a short exact sequence of simplicial R-modules. The there is a long exact sequence of sheaves

$$\cdots \to \mathrm{Tor}_1(A, B_3) \to \pi_0 A \otimes \pi_0 B_1 \to \pi_0 A \otimes \pi_0 B_2 \to \pi_0 A \otimes \pi_0 B_3 \to 0$$

Proof. Suppose that $X \to A$ is a cofibrant resolution of A. Then the sequence

$$0 \to X \otimes B_1 \to X \otimes B_2 \to X \otimes B_3 \to 0$$

is exact since X is cofibrant, and $\pi_r(X \otimes B_i) \cong \mathrm{Tor}_r(A, B_i)$. □

The skeletal filtration $\mathrm{sk}_n A$ of a simplicial R-module corresponds to the bad truncation $\overline{T}_n NA$ of the normalized chain complex NA at level n. In general, the *bad truncation* $\overline{T}_n C$ of a chain complex C is defined by

$$\overline{T}_n C_j = \begin{cases} C_j & \text{if } j \leq n, \\ 0 & \text{if } j > n. \end{cases}$$

The truncation $\overline{T}_n C$ is "bad" because it does not preserve homology isomorphisms. The good news is that $C = \cup_n \overline{T}_n C$, and there are short exact sequences of chain complexes

$$0 \to \overline{T}_n C \to \overline{T}_{n+1} C \to C_{n+1}[-n-1] \to 0$$

Here, $C_{n+1}[-n-1]$ is the chain complex consisting of a copy of C_{n+1} concentrated in degree $n+1$.

Lemma 5.7. *Suppose that A and B are simplicial R-modules. Then there is a convergent spectral sequence of sheaves of R-modules, with*

$$E_2^{p,q} = \mathrm{Tor}_p(A, \pi_q B) \Rightarrow \mathrm{Tor}_{p+q}(A, B)$$

where $\pi_q B$ is identified with the simplicial R-module $K(\pi_q B, 0)$.

Proof. We can suppose that A and B are cofibrant. Then the functor $? \otimes B$ is exact, and so the skeletal filtration on A gives exact sequences

$$0 \to \operatorname{sk}_{n-1} A \otimes B \to \operatorname{sk}_n A \otimes B \to \Gamma(NA_n[-n]) \otimes B \to 0$$

The spectral sequence arising from this filtration of $A \otimes B$ has E_1-term

$$E_1^{p,q} = \pi_{p+q}(\Gamma(NA_p[-p]) \otimes B) \cong H_{p+q}(NA_p[-p] \otimes NB) \cong NA_p \otimes H_q NB.$$

The last isomorphism follows from the fact (Corollary 5.3) that tensoring with NA_p is exact since A is cofibrant. It follows that there are isomorphisms

$$E_2^{p,q} \cong H_p(NA \otimes H_q NB) \cong \operatorname{Tor}_p(A, \pi_q B)$$

as required. □

Remark 5.8. The invariants $\operatorname{Tor}_n(A, \pi_q B)$ in the E_2-term of the spectral sequence of Lemma 5.7 can be computed with a second application of the same result: there is a spectral sequence

$$E_2^{r,s} = \operatorname{Tor}_r(\pi_s A, \pi_q B) \Rightarrow \operatorname{Tor}_{r+s}(A, \pi_q B).$$

5.2 The stable case

Suppose that X and Y are spectra, and let $X \wedge_n Y$ be the following particular choice of *naive smash product*:

$$(X \wedge_n Y)^{2n} = X^n \wedge Y^n, \ \ (X \wedge_n Y)^{2n+1} = X^{n+1} \wedge Y^n,$$

and the bonding maps are specified by:

$$\begin{cases} S^1 \wedge X^n \wedge Y^n \xrightarrow{\sigma \wedge 1} X^{n+1} \wedge Y^n \\ S^1 \wedge X^{n+1} \wedge Y^n \xrightarrow{\tau \wedge 1} X^{n+1} \wedge S^1 \wedge Y^n \xrightarrow{1 \wedge \sigma} X^{n+1} \wedge Y^{n+1} \end{cases}$$

Suppose that Z is a symmetric spectrum, and choose $\theta_i \in \Sigma_i$ for $i \geq 1$. Then there is a spectrum $\theta_*(UZ)$ with level spaces $\theta_*(UZ)^n = Z^n$, and having bonding maps σ_θ given by the composites

$$S^1 \wedge Z^n \xrightarrow{\sigma} Z^{n+1} \xrightarrow{\theta_{n+1}} Z^{n+1}$$

There is a natural isomorphism of spectra $\nu_\theta : UZ \to \theta_* UZ$ which is defined by finding elements $\nu_{\theta,n} \in \Sigma_n$ inductively. We require that $\nu_{\theta,n}$ is the identity in levels 0 and 1. Then there is a commutative diagram

$$\begin{array}{ccc} S^1 \wedge Z^n & \xrightarrow{\sigma} & Z^{n+1} \\ {\scriptstyle 1 \wedge \nu_{\theta,n}} \downarrow & & \downarrow {\scriptstyle 1 \oplus \nu_{\theta,n}} \\ S^1 \wedge Z^n & \xrightarrow{\sigma} & Z^{n+1} \\ & {\scriptstyle \sigma_\theta} \searrow & \downarrow {\scriptstyle \theta_{n+1}} \\ & & Z_{n+1} \end{array}$$

Then

$$\nu_{\theta,n+1} = \theta_{n+1}(1 \oplus \nu_{\theta,n}) = \theta_{n+1}(1 \oplus \theta_n)(1 \oplus \theta_{n-1}) \cdots (1 \oplus \theta_2)$$

Suppose now that X and Y are symmetric spectra. Then there are commutative diagrams

$$\begin{array}{ccc} S^1 \wedge X^n \wedge Y^n & \xrightarrow{\sigma \wedge 1} & X^{n+1} \wedge Y^n \\ \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle c} \\ S^1 \wedge (X \wedge Y)^{2n} & \xrightarrow[\sigma]{} & (X \wedge Y)^{2n+1} \end{array}$$

and

$$\begin{array}{ccccc} S^1 \wedge X^{n+1} \wedge Y^n & \xrightarrow{\tau \wedge 1} & X^{n+1} \wedge S^1 \wedge Y^n & \xrightarrow{1 \wedge \sigma} & X^{n+1} \wedge Y^n \\ \downarrow{\scriptstyle 1 \wedge c} & & & & \downarrow{\scriptstyle c} \\ S^1 \wedge (X \wedge Y)^{2n+1} & \xrightarrow[\sigma]{} & (X \wedge Y)^{2n+2} & \xrightarrow[c_{1,n+1} \oplus 1]{} & (X \wedge Y)^{2n+2} \end{array}$$

Here, $c : X^p \wedge Y^q \to (X \wedge Y)^{p+q}$ is the canonical map, and $c_{1,n+1} \in \Sigma_{n+2}$ is the shuffle map which moves the number 1 past the numbers $2, \dots, n+2$, in order.

It follows that the canonical maps $c : X^p \wedge Y^q \to (X \wedge Y)^{p+q}$ determine a natural map of spectra

$$c : UX \wedge_n UY \to \theta_* U(X \wedge Y),$$

where

$$\theta_i = \begin{cases} 1 & \text{if } i = 2n+1, \\ c_{1,n+1} \oplus 1 & \text{if } i = 2n+2. \end{cases}$$

Lemma 5.9. *The map c_* given by the composite*

$$X \wedge_n Y \to UVX \wedge_n UVY \xrightarrow{c} \theta_* U(VX \wedge VY) \xrightarrow{Uj} \theta_* U(VX \wedge VY)_s$$

is a stable equivalence if the spectra X and Y are cofibrant.

The proof of Lemma 5.9 requires the following:

Lemma 5.10. *Suppose that X is a stably fibrant symmetric spectrum and that K is a pointed simplicial set. Then any stably fibrant model $j : X \wedge K \to (X \wedge K)_s$ is a stable equivalence of the underlying spectra.*

Proof. The map $j : X \wedge S^1 \to (X \wedge S^1)_s$ is adjoint to the composite

$$X \xrightarrow{\eta} \Omega(X \wedge S^1) \xrightarrow{j_*} \Omega(X \wedge S^1)_s$$

which is a stable equivalence [7, Th. 3.1.14]. This map is a level equivalence since X is stably fibrant, so it is a stable equivalence of the underlying spectra. Its adjoint j is therefore a stable equivalence of the underlying spectra.

The fact that the map $j : X \wedge S^1 \to (X \wedge S^1)_s$ is a stable equivalence of the underlying spectra implies that all maps

$$j : X \wedge S^n \to (X \wedge S^n)_s$$

are stable equivalences of underlying spectra. In effect, the question of whether or not j has this property is insensitive to the model for S^n up to homotopy type, so that we can use $S^n = S^1 \wedge \cdots \wedge S^1$ and iterate the construction for S^1.

Fibre sequences and cofibre sequences coincide in the homotopy category of symmetric spectra. In particular, if

$$A \subset B \xrightarrow{p} B/A$$

is a level cofibre sequence of symmetric spectra and F is the stable homotopy fibre of p, then the canonical map $A \to F$ is a stable equivalence (a prototype proof appears in [14, Cor. 3.10]).

Suppose that K is a pointed simplicial set, and consider the cofibre sequence

$$\mathrm{sk}_{n-1} K \to \mathrm{sk}_n K \to \bigvee_{NK_n} S^n$$

Form the diagram

$$\begin{array}{ccccc} X \wedge \mathrm{sk}_{n-1} K & \longrightarrow & X \wedge \mathrm{sk}_n K & \longrightarrow & X \wedge (\bigvee S^n) \\ \downarrow & & \downarrow{\scriptstyle i} & \nearrow{\scriptstyle p} & \\ F & \longrightarrow & Z & & \end{array}$$

where p is a stable fibration, and i is a stable cofibration and a stable equivalence of symmetric spectra. Then the induced map $X \wedge \mathrm{sk}_{n-1} K \to F$ is a stable equivalence of symmetric spectra and F is stably fibrant; this map can inductively be assumed to be a stable equivalence of the underlying spectra, so that the map i is a stable equivalence of underlying spectra. Now form the comparison diagram

$$\begin{array}{ccccc} F & \longrightarrow & Z & \xrightarrow{p} & X \wedge (\bigvee S^n) \\ \downarrow & & \downarrow{\scriptstyle i'} & & \downarrow{\scriptstyle j} \\ F' & \longrightarrow & Z' & \xrightarrow[p']{} & (X \wedge (\bigvee S^n))_s \end{array}$$

of stable fibre sequences of symmetric spectra, where p' is a stable fibration and i' is a stable cofibration and stable weak equivalence. Then the map j is a stable equivalence of underlying spectra, and the induced map $F \to F'$ is a stable hence level equivalence of fibres. It follows that the map $i' : Z \to Z'$ is a stable equivalence of underlying spectra. Note finally that the composite map

$$X \wedge \mathrm{sk}_n K \xrightarrow{i} Z \xrightarrow{i} Z'$$

is a stably fibrant model for $X \wedge \mathrm{sk}_n K$ in the category of symmetric spectra.

If F is a finite pointed set and X is stably fibrant, then the canonical map

$$X \wedge F_+ = \bigvee_F X \to \prod_F X$$

is a stable equivalence of the underlying spectra, and the product $\prod_F X$ is a stably fibrant symmetric spectrum. Any filtered colimit

$$Y = \varinjlim_i Y_i$$

of stably fibrant symmetric spectra Y_i is an Ω-spectrum and is thus stably fibrant. It follows that any stably fibrant model

$$X \wedge \mathrm{sk}_0 K \to (X \wedge \mathrm{sk}_0 K)_s$$

is a stable equivalence of the underlying spectra. □

Proof of Lemma 5.9. Both models for the smash product preserve stable equivalences and filtered colimits in Y, and one can show that the map

$$c_* : X \wedge_n (Y \wedge K) \to \theta_* U(VX \wedge V(Y \wedge K))$$

is isomorphic to the map

$$c_* \wedge 1 : (X \wedge_n Y) \wedge K \to \theta_* U(VX \wedge VY) \wedge K$$

for any pointed simplicial set K. There is also a map f making the diagram

$$\begin{array}{ccc} (VX \wedge VY) \wedge K & \xrightarrow{j \wedge 1} & (VX \wedge VY)_s \wedge K \\ {\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle f} \\ (VX \wedge V(Y \wedge K)) & \xrightarrow[j]{} & (VX \wedge V(Y \wedge K))_s \end{array}$$

commute, since j and $j \wedge 1$ can be chosen to be stably trivial cofibrations of symmetric spectra. It follows that f is a stably fibrant model, and is therefore a stable equivalence of underlying spectra, by Lemma 5.10

A layer filtration argument thus implies that it suffices to show that the maps

$$c_* : X \wedge_n S[-p] \to \theta_* U(VX \wedge V(S[-p]))_s$$

are stable equivalences for all $p \geq 0$.

It also follows that the map $c_* : X \wedge_n Y \to \theta_* U(VX \wedge VY)_s$ is a stable equivalence if and only if the map $c_* : X \wedge_n (Y \wedge S^1) \to \theta_* U(VX \wedge V(Y \wedge S^1))_s$ is a stable equivalence. It therefore suffices to show that the map $c_* : X \wedge_n S \to \theta_* U(VX \wedge S)_s$ is a stable equivalence.

A similar analysis applied to the cofibrant spectrum X implies that it is enough to show that the map

$$c_* : S \wedge_n S \to \theta_*(S \wedge S)_s$$

is a stable equivalence of spectra.

There is an identification

$$S \wedge_n S = \theta_* S,$$

and a canonical isomorphism

$$m : S \wedge S \xrightarrow{\cong} S.$$

The composite

$$S \wedge_n S \to \theta_*(S \wedge S)_s \xrightarrow{m_*} \theta_* S_s$$

can be identified with the twist $\theta_* j$ of the spectrum map underlying the stably fibrant model $j : S \to S_s$ of the symmetric sphere spectrum, which induces a stable equivalence of the underlying spectra. □

Corollary 5.11. *Suppose that X is a stably fibrant presheaf of symmetric spectra and that K is a pointed simplicial presheaf on a small Grothendieck site $\mathcal{C}$. Then any stably fibrant model $X \wedge K \to (X \wedge K)_s$ in the category of presheaves of symmetric spectra is a stable equivalence of the underlying presheaves of spectra.*

Proof. Forget about the topology for a moment: the object X is pointwise (ie. sectionwise) stably fibrant, and any pointwise stably fibrant model $X \wedge K \to Z$ is a pointwise stable equivalence of presheaves of spectra, by Lemma 5.10. Putting the topology back in, one sees that there is a level equivalence $\alpha : Z \to Z_i$ where Z_i is an injective fibrant model for Z [13, Th. 2]. Then Z is a presheaf of Ω-spectra, so that Z_i is a stably fibrant presheaf of symmetric spectra. The composite

$$X \wedge K \to Z \xrightarrow{\alpha} Z_i$$

is a stable equivalence of the underlying presheaves of spectra. □

Suppose that A and B are simplicial R-module spectra, and let $A \otimes_n B$ be the following particular choice of *naive tensor product*: $(A \otimes_n B)^{2n} = A^n \otimes B^n$, $(A \otimes_n B)^{2n+1} = A^{n+1} \otimes B^n$, and the following are the bonding maps:

$$\begin{cases} S^1 \otimes A^n \otimes B^n \xrightarrow{\sigma \otimes 1} A^{n+1} \otimes B^n \\ S^1 \otimes A^{n+1} \otimes B^n \xrightarrow{\tau \otimes 1} A^{n+1} \otimes S^1 \otimes B^n \xrightarrow{1 \otimes \sigma} A^{n+1} \otimes B^{n+1} \end{cases}$$

Suppose that C is a symmetric simplicial R-module spectrum, and choose $\theta_i \in \Sigma_i$ for $i \geq 1$. Then there is a simplicial R-module spectrum $\theta_*(U_R C)$ with level spaces $\theta_*(U_R C)^n = C^n$, and having bonding maps σ_θ given by the composites

$$S^1 \otimes C^n \xrightarrow{\sigma} C^{n+1} \xrightarrow{\theta_{n+1}} C^{n+1}$$

There is a natural isomorphism of simplicial R-module spectra

$$\nu_\theta : U_R C \to \theta_* U_R C$$

which is defined by finding elements $\nu_{\theta,n} \in \Sigma_n$ inductively. We require that $\nu_{\theta,n}$ is the identity in levels 0 and 1. Then there is a commutative diagram

$$\begin{array}{ccc}
S^1 \otimes C^n & \xrightarrow{\sigma} & C^{n+1} \\
{\scriptstyle 1\otimes\nu_{\theta,n}}\downarrow & & \downarrow{\scriptstyle 1\oplus\nu_{\theta,n}} \\
S^1 \otimes C^n & \xrightarrow{\sigma} & C^{n+1} \\
& {\scriptstyle \sigma_\theta}\searrow & \downarrow{\scriptstyle \theta_{n+1}} \\
& & C_{n+1}
\end{array}$$

Then

$$\nu_{\theta,n+1} = \theta_{n+1}(1 \oplus \nu_{\theta,n}) = \theta_{n+1}(1 \oplus \theta_n)(1 \oplus \theta_{n-1}) \cdots (1 \oplus \theta_2)$$

Suppose now that A and B are symmetric simplicial R-module spectra. Then there are commutative diagrams

$$\begin{array}{ccc}
S^1 \otimes A^n \otimes B^n & \xrightarrow{\sigma\otimes 1} & A^{n+1} \otimes B^n \\
{\scriptstyle c}\downarrow & & \downarrow{\scriptstyle c} \\
S^1 \otimes (A \otimes_\Sigma B)^{2n} & \xrightarrow[\sigma]{} & (A \otimes_\Sigma B)^{2n+1}
\end{array}$$

and

$$\begin{array}{ccccc}
S^1 \otimes A^{n+1} \otimes B^n & \xrightarrow{\tau\otimes 1} & A^{n+1} \otimes S^1 \otimes B^n & \xrightarrow{1\otimes\sigma} & A^{n+1} \otimes B^n \\
{\scriptstyle 1\otimes c}\downarrow & & & & \downarrow{\scriptstyle c} \\
S^1 \otimes (A \otimes_\Sigma B)^{2n+1} & \xrightarrow[\sigma]{} & (A \otimes_\Sigma B)^{2n+2} & \xrightarrow[c_{1,n+1}\oplus 1]{} & (A \otimes_\Sigma B)^{2n+2}
\end{array}$$

It therefore follows that the canonical maps $c : A^p \otimes B^q \to (A \otimes_\Sigma B)^{p+q}$ determine a natural map of simplicial R-module spectra

$$c : U_R A \otimes_n U_R B \to \theta_* U_R(A \otimes_\Sigma B),$$

where

$$\theta_i = \begin{cases} 1 & \text{if } i = 2n+1, \\ c_{1,n+1} \oplus 1 & \text{if } i = 2n+2. \end{cases}$$

Lemma 5.12. *The map c_* given by the composite*

$$A \otimes_n B \to U_R V_R A \otimes_n U_R V_R B \to \theta_* U_R(V_R A \otimes_\Sigma V_R B) \xrightarrow{U_R j} \theta_* U_R(V_R A \otimes_\Sigma V_R B)_s$$

is a stable equivalence if the simplicial R-module spectra A and B are cofibrant.

Lemma 5.13. *Suppose that A is a stably fibrant symmetric simplicial R-module spectrum and that K is a pointed simplicial presheaf. Then any stably fibrant model $j : A \otimes K \to (A \otimes K)_s$ is a stable equivalence of the underlying simplicial R-module spectra.*

Proof. Suppose that $j : A \otimes K \to (A \otimes K)_s$ is a stably fibrant model in the category of symmetric simplicial R-module spectra. Then the underlying map $j_* : u(A \otimes K) \to u(A \otimes K)_s$ is a stable equivalence of presheaves of symmetric spectra, and the objects uA and $u(A \otimes K)_s$ are stably fibrant. The canonical map $\gamma : uA \wedge K \to u(A \otimes K)$ is a stable equivalence of the underlying presheaves of spectra by Lemma 4.5, so that the composite

$$uA \wedge K \xrightarrow{\gamma} u(A \otimes K) \xrightarrow{j_*} u(A \otimes K)_s$$

is a stably fibrant model for $uA \wedge K$. It follows from Corollary 5.11 that this composite is a stable equivalence of the underlying presheaves of spectra, and hence that j is a stable equivalence of the underlying simplicial R-module spectra. ☐

Proof of Lemma 5.12. Both models for the tensor product preserve stable equivalences and filtered colimits in B, and one can show that the map

$$c_* : A \otimes_n (B \otimes K) \to \theta_* U_R(V_R A \otimes_\Sigma V_R(B \otimes K))$$

is isomorphic to the map

$$c_* \otimes 1 : (A \otimes_n B) \otimes K \to \theta_* U_R(V_R A \otimes_\Sigma V_R B) \otimes K$$

for any pointed simplicial set K. There is also a map f making the diagram

$$\begin{array}{ccc}
(V_R A \otimes_\Sigma V_R B) \otimes K & \xrightarrow{j \otimes 1} & (V_R A \otimes_\Sigma V_R B)_s \otimes K \\
\cong \downarrow & & \downarrow j \\
(V_R A \otimes_\Sigma V_R(B \otimes K)) & \xrightarrow[j]{} & (V_R A \otimes_\Sigma V_R(B \otimes K))_s
\end{array}$$

commute, since j and hence $j \otimes 1$ can be chose to be a stably trivial cofibration of symmetric simplicial R-module spectra. It follows that f is a stably fibrant model, and is therefore a stable equivalence of the underlying simplicial R-module spectra, by Lemma 5.13.

It therefore follows from a layer filtration argument that it suffices to show that the maps

$$c_* : A \otimes_n S[-p] \to \theta_* U_R(V_R A \otimes_\Sigma V_R(S[-p]))_s$$

are stable equivalences for all $p \geq 0$.

It also follows that the map $c_* : A \otimes_n B \to \theta_* U_R(V_R A \otimes_\Sigma V_R B)_s$ is a stable equivalence if and only if the map $c_* : A \otimes_n (B \otimes S^1) \to \theta_* U_R(V_R A \otimes_\Sigma$

$V_R(B \otimes S^1))_s$ is a stable equivalence. It therefore suffices to show that the map $c_* : A \otimes_n S \to \theta_* U_R(V_R A \otimes_\Sigma S)_s$ is a stable equivalence.

A similar analysis applied to the cofibrant simplicial R-module spectra A shows that it is enough to show that the map

$$c_* : S \otimes_n S \to \theta_*(S \otimes_\Sigma S)_s$$

is a stable equivalence of simplicial R-module spectra.

There is an identification of spectrum objects

$$S \otimes_n S = \theta_* S,$$

and a canonical isomorphism of symmetric spectrum objects

$$m : S \otimes_\Sigma S \xrightarrow{\cong} S.$$

The composite

$$S \otimes_n S \to \theta_*(S \otimes_\Sigma S)_s \xrightarrow{m_*} \theta_* S_s$$

can be identified with the twist $\theta_* j$ of the simplicial R-module spectrum map underlying the stably fibrant model $j : S \to S_s$ of the symmetric simplicial R-module sphere spectrum, which induces a stable equivalence of the underlying simplicial R-module spectra. □

In general, if A and B are symmetric spectrum objects, we take cofibrant resolutions $X \to A$ and $Y \to B$, and define the sheaves of R-modules $\mathrm{Tor}_i(A, B)$ by

$$\mathrm{Tor}_i(A, B) = \pi_i U_R(X \otimes_\Sigma Y)_s.$$

In particular, if Z and W are cofibrant spectrum objects, then Lemma 5.12 implies that there is an isomorphism

$$\mathrm{Tor}_i(V_R Z, V_R W) \cong \pi_i(Z \otimes_n W)$$

so that Tor_i can be defined by stable homotopy groups of naive tensor products.

Lemma 5.14. *Suppose that A is a simplicial R-module and that B is a simplicial R-module spectrum. Then there is an isomorphism*

$$\mathrm{Tor}_i(V_R B, \Sigma^\infty A) \cong \varinjlim_n \mathrm{Tor}_{i+n}(B^n, A).$$

Proof. We can assume that the simplicial R-module A and the spectrum object B are cofibrant. There is an isomorphism of symmetric spectrum objects

$$V_R B \otimes_\Sigma \Sigma^\infty A \cong V_R(B \otimes A),$$

where $B \otimes A$ is the spectrum object with $(B \otimes A)^n = B^n \otimes A$. The spectrum object $B \otimes A$ is cofibrant, so that the composite

$$B \otimes A \xrightarrow{\eta} U_R V_R(B \otimes A) \to U_R(V_R(B \otimes A))_s$$

is a stable equivalence, by the proof of Theorem 4.10. There is an isomorphism

$$\pi_i(B \otimes A) \cong \varinjlim_n \pi_{i+n}(B^n \otimes A).$$

□

Corollary 5.15. *Suppose that A and B are simplicial R-modules. Then there is an isomorphism*

$$\mathrm{Tor}_i(\Sigma^\infty B, \Sigma^\infty A) \cong \begin{cases} \mathrm{Tor}_i(B, A) & \text{for } i \geq 0, \\ 0 & \text{for } i < 0. \end{cases}$$

The skeletal filtration $\mathrm{sk}_n A$ of a simplicial R-module spectrum A corresponds to the bad truncation $\overline{T}_n NA$ of the normalized chain complex NA at level n. Explicitly, we define $\mathrm{sk}_n A$ by

$$\mathrm{sk}_n A = \Gamma \overline{T}_n NA.$$

There is a natural equivalence

$$A \simeq \Gamma NA = \cup_n \Gamma \, \mathrm{sk}_n A,$$

and there are short exact sequences

$$0 \to \mathrm{sk}_n A \to \mathrm{sk}_{n+1} A \to \Gamma NA_{n+1}[-n-1] \to 0$$

Suppose that A and B are simplicial R-module spectra, and write

$$\mathrm{Tor}_i(A, B) = \mathrm{Tor}_i(V_R X, V_R Y),$$

where $X \to A$ and $Y \to B$ are choices of cofibrant models for A and B respectively. In view of Lemma 5.12, there is an isomorphism

$$\mathrm{Tor}_i(A, B) \cong \pi_i(X \otimes_n Y).$$

If D is a presheaf of R-modules, write

$$\mathrm{Tor}_i(A, D) = \mathrm{Tor}_i(A, K(D, 0)).$$

Lemma 5.16. *Suppose that A and B are simplicial R-module spectra. Then there is a spectral sequence of sheaves of R-modules, with*

$$E_2^{p,q} = \mathrm{Tor}_p(A, \pi_q B) \Rightarrow \mathrm{Tor}_{p+q}(A, B).$$

The spectral sequence converges if A is bounded below in the sense that $\pi_j A = 0$ if $j < m$ for some m.

Proof. We can suppose that A and B are cofibrant. Then the functor $? \otimes_n B$ is exact and preserves stable equivalences. It follows that the skeletal filtration on A gives exact sequences

$$0 \to \mathrm{sk}_{m-1} A \otimes_n B \to \mathrm{sk}_m A \otimes_n B \to \Gamma(NA_m[-m]) \otimes_n B \to 0$$

The spectral sequence arising from this filtration of the naive tensor product

$$A \otimes_n B \simeq (\Gamma NA) \otimes_n B$$

has E_1-term

$$E_1^{p,q} = \pi_{p+q}(\Gamma(NA_p[-p]) \otimes_n B) \cong NA_p \otimes \pi_q B.$$

The last isomorphism is a consequence of the fact (Corollary 5.3) that tensoring with NA_p is exact since A is cofibrant. It follows that there are isomorphisms

$$E_2^{p,q} \cong H_p(NA \otimes \pi_q B) \cong \mathrm{Tor}_p(A, \pi_q B)$$

as required.

If A is bounded below, then there is a cofibrant model $\Sigma^\infty C[-m] \to A$, where C is a cofibrant simplicial R-module. Then there is a quasi-isomorphism $NC[-m] \simeq N\Sigma^\infty C[-m]$, and a comparison of filtrations

$$\Gamma \overline{T}_i NC[-m] \otimes_n B \to \mathrm{sk}_i A \otimes_n B$$

which induces an isomorphism of spectral sequences at the E_2 level. Then $\overline{T}_i NC[-m] = 0$ for $i < m$, so the filtration $\{\Gamma \overline{T}_i NC[-m]\}$ is bounded below. ∎

Remark 5.17. As in Remark 5.8 the the invariants $\mathrm{Tor}_n(A, \pi_q B)$ in the E_2-term of the spectral sequence of Lemma 5.16 can be computed with a second application of the same result: there is a spectral sequence

$$E_2^{r,s} = \mathrm{Tor}_r(\pi_s A, \pi_q B) \Rightarrow \mathrm{Tor}_{r+s}(A, \pi_q B).$$

We finish by noting that the Tor functors respect shift in the expected way. There are isomorphisms

$$\mathrm{Tor}_i(A, B \otimes S^1) \cong \mathrm{Tor}_{i-1}(A, B) \tag{5.3}$$

for all symmetric spectrum objects A and B. In effect, we may as well assume that A and B are cofibrant since tensoring with S^1 preserves stable equivalences of symmetric spectrum objects. Then there is a map

$$(A \otimes_\Sigma B)_s \otimes S^1 \to ((A \otimes_\Sigma B)_s \otimes S^1)_s$$

which is a stable equivalence of the underlying spectra by Lemma 5.13, and the object $((A \otimes_\Sigma B)_s \otimes S^1)_s$ is a stably fibrant model for

$$A \otimes_\Sigma (B \otimes S^1) \cong (A \otimes_\Sigma B) \otimes S^1.$$

It follows that there are isomorphisms

$$\mathrm{Tor}_i(A, B[-1]) \cong \mathrm{Tor}_{i+1}(A, B). \tag{5.4}$$

To see this, observe that there is an isomorphism

$$A \otimes_\Sigma B[-1] \cong A \otimes_\Sigma B \otimes_\Sigma S[-1],$$

and tensoring with $S[-1]$ preserves level equivalences. Thus, we can assume that A and B are cofibrant, and apply the isomorphism (5.3) in conjunction with the stable equivalence $S[-1] \wedge S^1 \simeq S$.

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