CHAPTER II.3

SUPER MAURER-CARTAN EQUATIONS AND THE GEOMETRY OF SUPERSPACE

II.3.1 - Maurer-Cartan equations of supergroups on supergroup manifolds

In Chapter I.3 we considered the dual formulation of Lie algebras in terms of Maurer Cartan equations.

The starting point was the construction, on the Lie group-manifold G, of the left-invariant (alternatively right-invariant 1-forms):

$$\sigma_{(L)} = g^{-1}dg \quad ; \quad \sigma_{(R)} = g\,dg^{-1} \tag{II.3.1}$$

where

$$g = g(t_1,\ldots,t_n) \tag{II.3.2}$$

is any matrix representation of the group element identified by parameters $t_1,\ldots,t_n$ (supposing G of dimension n).

In want of better g can always be taken in the adjoint (regular) representation.

Since $\sigma_{(L)}$ (or $\sigma_{(R)}$) are Lie algebra valued we can expand them along a basis of matrix generators $\{T_A\}$

$$\sigma_{(L)} = \sigma^A_{(L)}T_A \quad ; \quad \sigma_{(R)} = \sigma^A_{(R)}T_A \tag{II.3.3}$$

and by exterior differentiation of (II.3.1) we obtain the Maurer Cartan equations:

$$d\sigma^A_{(L)} + \frac{1}{2}\,C^A_{\bullet BC}\sigma^B_{(L)} \wedge \sigma^C_{(L)} = 0 \tag{II.3.4a}$$

$$d\sigma^A_{(R)} + \frac{1}{2}\,C^A_{\bullet BC}\sigma^B_{(R)} \wedge \sigma^C_{(R)} = 0 \tag{II.3.4b}$$

where $C^A{}_{BC}$ are the structure constants of the Lie algebra spanned by the matrix generators T_A

$$[T_A, T_B] = C_{AB}^{\bullet\bullet C}T_C \tag{II.3.5}$$

The tangent vectors $\vec{T}_A^{(R)}$ and $\vec{T}_A^{(L)}$ are dual respectively to the left and right-invariant 1-forms

$$\sigma^A_{(L)}(\vec{T}^{(R)}_B) = \delta^A_B \tag{II.3.6a}$$

$$\sigma^A_{(R)}(\vec{T}^{(L)}_B) = \delta^A_B \tag{II.3.6b}$$

They are differential operators on the group manifold G and are, respectively, the generators of the right and left translations $L_{(a)}$ and $R_{(a)}$ defined by equations (I.3.4-5). Both $\vec{T}^{(R)}_A$ and $\vec{T}^{(L)}_A$ satisfy the Lie algebra (II.3.5):

$$[\vec{T}^{(R)}_A, \vec{T}^{(R)}_B] = C_{AB}^{\cdot\cdot C}\, \vec{T}^{(R)}_C \tag{II.3.7a}$$

$$[\vec{T}^{(L)}_A, \vec{T}^{(L)}_B] = C_{AB}^{\cdot\cdot C}\, \vec{T}^{(L)}_C \tag{II.3.7b}$$

Furthermore (see Eq. (I.3.23)) they commute among themselves

$$[\vec{T}^{(R)}_A, \vec{T}^{(L)}_B] = 0 \tag{II.3.8}$$

As we saw in Chapter II.1 the elements of the classical and exceptional supergroups are represented by graded matrices whose entries are Grassmann algebra elements rather than real or complex numbers.

In addition we saw that one can introduce the notion of a supermanifold, whose coordinates are Grassmann algebra elements, and one can straightforwardly extend the calculus of exterior forms from manifolds to supermanifolds.

Hence the elements of a supergroup can be regarded as the points of a supergroup manifold and the notion of left-invariant or right-invariant 1-forms (II.3.1) can be canonically extended to supergroups. It suffices to replace the matrix g in (II.3.1) by a graded matrix $\hat{g}$:

$$\hat{\sigma}_{(L)} = \hat{g}^{-1} d\hat{g} \quad ; \quad \hat{\sigma}_{(R)} = \hat{g}\, d\hat{g}^{-1} \tag{II.3.9}$$

Expanding the graded matrix valued 1-form $\hat{\sigma}_{\binom{L}{R}}$ along a basis of matrix generators of the supergroups (see Eqs. (II.2.110-111-112)):

$$\hat{\sigma}_{\binom{L}{R}} = \hat{\sigma}^A_{\binom{L}{R}} T_A = \hat{\sigma}^a_{\binom{L}{R}} t_a + \hat{\sigma}^\alpha_{\binom{L}{R}} t_\alpha \tag{II.3.10}$$

we obtain a natural separation into bosonic and fermionic 1-forms. According to Eq. (II.2.109) we have

$$\hat{\sigma}^a_{\binom{L}{R}} \wedge \hat{\sigma}^b_{\binom{L}{R}} = - \hat{\sigma}^b_{\binom{L}{R}} \wedge \hat{\sigma}^a_{\binom{L}{R}} \tag{II.3.11a}$$

$$\hat{\sigma}^\alpha_{\binom{L}{R}} \wedge \hat{\sigma}^\beta_{\binom{L}{R}} = \hat{\sigma}^\beta_{\binom{L}{R}} \wedge \hat{\sigma}^\alpha_{\binom{L}{R}} \tag{II.3.11b}$$

$$\hat{\sigma}^a_{\binom{L}{R}} \wedge \hat{\sigma}^\beta_{\binom{L}{R}} = - \hat{\sigma}^\beta_{\binom{L}{R}} \wedge \hat{\sigma}^a_{\binom{L}{R}} \tag{II.3.11c}$$

Therefore from the Maurer Cartan equation

$$d\hat{\sigma}_{\binom{L}{R}} + \hat{\sigma}_{\binom{L}{R}} \wedge \hat{\sigma}_{\binom{L}{R}} = 0 \tag{II.3.12}$$

which follows from (II.3.9) upon exterior differentiation, by inserting Eq. (II.3.10) we get

$$d\hat{\sigma}^{a}_{\binom{L}{R}}\, t_a + d\hat{\sigma}^{\alpha}_{\binom{L}{R}}\, t_\alpha + \frac{1}{2}\,(\hat{\sigma}^{a}_{\binom{L}{R}} \wedge \hat{\sigma}^{b}_{\binom{L}{R}}\,[t_a,t_b] +$$

$$+\, 2\hat{\sigma}^{a}_{\binom{L}{R}} \wedge \hat{\sigma}^{\beta}_{\binom{L}{R}}\,[t_a,t_\beta] + \hat{\sigma}^{\alpha}_{\binom{L}{R}} \wedge \hat{\sigma}^{\beta}_{\binom{L}{R}}\,\{t_\alpha,t_\beta\}) = 0 \qquad \text{(II.3.13)}$$

which can be rewritten as:

$$d\hat{\sigma}^{A}_{\binom{L}{R}} + \frac{1}{2}\,\hat{C}^{A}_{\cdot BC}\,\hat{\sigma}^{B}_{\binom{L}{R}} \wedge \hat{\sigma}^{C}_{\binom{L}{R}} = 0 \qquad \text{(II.3.14)}$$

In (II.3.14) $C^{A}{}_{BC}$ are the graded structure constants defined by Eq. (II.2.13).

Once more as in the case of ordinary Lie algebras eq. (II.3.14) can be taken as the definition of the superalgebra. Indeed if we are able to write a super Maurer-Cartan equation (II.3.14) which is integrable:

$$dd\hat{\sigma}^{A} = -\,\hat{C}^{A}_{\cdot BC}\,d\hat{\sigma}^{B} \wedge \sigma^{C} = \hat{C}^{A}_{\cdot BC}\,\hat{C}^{B}_{FG}\,\hat{\sigma}^{F} \wedge \hat{\sigma}^{G} \wedge \hat{\sigma}^{C} \equiv 0 \qquad \text{(II.3.15)}$$

then the constants $C^{A}{}_{BC}$ satisfy the graded Jacobi identities (II.2.14) and define a superalgebra.

II.3.2 - Maurer-Cartan equations of Osp(4/N) and $\overline{\mathrm{Osp}(4/N)}$

As an exercise let us write the Maurer Cartan equations associated to the Osp(4/N) superalgebra (II.2.140) whose Inonü-Wigner

contraction is the N-extended super Poincaré algebra (II.2.19,43,44).

The procedure is very easy. Consider the graded matrix (II.2.135) which represents an element of the Osp(4/N) algebra and let the parameters $(\varepsilon^{ab}, \varepsilon^{a}, \varepsilon^{AB}, \theta^{B})$ be finite rather than infinitesimal.

$$(\varepsilon^{ab},\varepsilon^{a},\varepsilon^{AB},\xi^{B}) \mapsto (\eta^{ab},x^{a},\eta^{AB},\theta^{B}) \qquad \text{(II.3.16)}$$

$$\Lambda(\varepsilon^{ab},\varepsilon^{a},\varepsilon^{AB},\xi^{B}) \mapsto \Lambda(\eta^{ab},x^{a},\eta^{AB},\theta^{B}) \qquad \text{(II.3.17)}$$

A generic element of the Osp(4/N) supergroup can be written as the exponential of Λ:

$$\mathcal{O}(\eta,x,\Theta) = \exp\left[\Lambda(\eta,x,\Theta)\right] \qquad \text{(II.3.18)}$$

and the parameters η, x, Θ can be regarded as the coordinates of the supergroup manifold. The left-invariant 1-form

$$\hat{\sigma}_{L} = \mathcal{O}^{-1}(\eta,x,\Theta)\,d\mathcal{O}(\eta,x,\Theta) \qquad \text{(II.3.19)}$$

is an element of the Osp(4/N) superalgebra and as such it can be written in the form (II.2.135)

$$\hat{\sigma}_{L} = \left(\begin{array}{c|c} -\frac{1}{4}\,\omega^{ab}\gamma_{ab} + \frac{i}{2}\,V^{a}\gamma_{a} & \psi^{B} \\ \hline \bar{\psi}^{A} & \frac{1}{2}\,A^{AB} \end{array}\right) \qquad \text{(II.3.20)}$$

where $\omega^{ab}=-\omega^{ba}$, V^a, $A^{AB}=-A^{BA}$ are bosonic one-forms while ψ^A is a Majorana spinor fermionic 1-form

$$\psi^A = C(\bar{\psi}^A)^T \tag{II.3.21}$$

As a result of its being left-invariant the graded-matrix valued 1-form $\hat{\sigma}_{(L)}$ satisfies Eq. (II.3.15).

Performing the multiplication $\hat{\sigma}_{(L)} \wedge \hat{\sigma}_{(L)}$ we obtain:

$$d\hat{\sigma}_{(L)} + \hat{\sigma}_{(L)} \wedge \hat{\sigma}_{(L)} \equiv$$

$$\equiv \left(\begin{array}{c|c} -\frac{1}{4} R^{ab}\gamma_{ab} + \frac{i}{2} R^a\gamma_a & \rho^B \\ \hline \bar{\rho}^A & \frac{1}{2} R^{AB} \end{array} \right) = 0 \tag{II.3.22}$$

where the 2-forms R^{ab}, R^a, R^{AB}, ρ^A are defined below

$$R^{ab} = d\omega^{ab} - \omega^{ac} \wedge \omega_c^{\cdot b} + V^a \wedge V^b + \frac{1}{2} \bar{\psi}_A \wedge \gamma^{ab}\psi_A \tag{II.3.23a}$$

$$R^a = dV^a - \omega^{ab} \wedge V_b - \frac{i}{2} \bar{\psi}_A \wedge \gamma^a\psi_A \tag{II.3.23b}$$

$$\rho_A = d\psi_A - \frac{1}{4} \omega^{ab} \wedge \gamma_{ab}\psi_A - \frac{i}{2} \gamma_a\psi_A \wedge V^a + \frac{1}{2} A_{AB} \wedge \psi_B \tag{II.3.23c}$$

$$R^{AB} = dA^{AB} + \frac{1}{2} A^{AC} \wedge A^{CB} + 2\, \bar{\psi}^A \wedge \psi^B \tag{II.3.23d}$$

To obtain this result the only ingredient which we have used, besides the multiplication of gamma matrices, is the following Fierz identity

$$\begin{aligned}(\psi_A \wedge \bar{\psi}_B)^{\alpha\beta} &= \frac{1}{4} 1^{\alpha\beta}\bar{\psi}_B \wedge \psi_A + \frac{1}{4} (\gamma_5)^{\alpha\beta}\bar{\psi}_B \wedge \gamma_5\psi_A \\ &+ \frac{1}{4} (\gamma_5\gamma_a)^{\alpha\beta}\bar{\psi}_B \wedge \gamma_5\gamma_a\psi_A + \frac{1}{4} (\gamma_a)^{\alpha\beta}\bar{\psi}_B \wedge \gamma_a\psi_A \\ &- \frac{1}{8} (\gamma_{ab})^{\alpha\beta}\bar{\psi}_B \wedge \gamma^{ab}\psi_A\end{aligned} \tag{II.3.24}$$

plus the observation that $\bar{\psi}_B\, \gamma_5\, \bar{\psi}_A$, $\bar{\psi}_B \wedge \bar{\psi}_A$, $\bar{\psi}_B \wedge \gamma_5\, \gamma_a\, \psi_A$ are anti-symmetric in $(A \leftrightarrow B)$ while $\bar{\psi}_A \wedge \gamma_a\, \psi_B$ and $\bar{\psi}_A \wedge \gamma_{ab}\, \psi_B$ are symmetric in the same indices.

Eq. (II.3.24) will be derived in Chapter (II.8): for the moment the reader should take it as given.

Comparing Eqs. (II.3.20) and (II.3.22) with Eqs. (II.2.135) and (II.2.138) we see that ω^{ab}, V^a, A^{AB}, ψ^A are, respectively, the 1-form coefficients of the Lorentz generators M_{ab}, the translations P_a, the SO(N) generators T_{AB} and of the supersymmetry charges Q_A.

In the previous chapter we prepared for the Inonü Wigner contraction by defining a new basis of generators $(M_{ab})^{new}$, $(T_{AB})^{new}$, $(P_a)^{new}$, $(Q_{A\alpha})^{new}$ related to the old ones by the rescalings (II.2.141).

Through the identification

$$(\sigma^A_{(L)})^{old}(T_A)^{old} = (\sigma^A_{(L)})^{new}(T_A)^{new} \tag{II.3.25}$$

we see that (II.2.141) are equivalent to the following rescalings of the left-invariant 1-forms:

$$(\omega^{ab})^{old} = (\omega^{ab})^{new} \Rightarrow (R^{ab})^{old} = (R^{ab})^{new} \tag{II.3.26a}$$

$$(A^{AB})^{old} = 2\bar{e}(A^{AB})^{new} \Rightarrow (R^{AB})^{old} = 2\bar{e}(R^{AB})^{new} \qquad \text{(II.3.26b)}$$

$$(V^{a})^{old} = 2\bar{e}(V^{a})^{new} \Rightarrow (R^{a})^{old} = 2\bar{e}(R^{a})^{new} \qquad \text{(II.3.26c)}$$

$$(\psi^{A})^{old} = \sqrt{2\bar{e}}\,(\psi^{A})^{new} \Rightarrow (\rho^{A})^{old} = \sqrt{2\bar{e}}\,(\rho^{A})^{new} \qquad \text{(II.3.26d)}$$

In terms of the new quantities, depending on the rescaling parameter $\bar{e}$, the Maurer Cartan equations of Osp(4/N) read:

$$R^{a} \equiv DV^{a} - \frac{i}{2}\bar{\psi}_{A} \wedge \gamma^{a}\psi_{A} = 0 \qquad \text{(II.3.27a)}$$

$$R^{ab} \equiv R^{ab} + 4\bar{e}^{2}V^{a} \wedge V^{b} + \bar{e}\bar{\psi}_{A} \wedge \gamma^{ab}\psi_{A} = 0 \qquad \text{(II.3.27b)}$$

$$\rho_{A} \equiv \mathcal{D}\psi_{A} - i\bar{e}\gamma_{a}\psi_{A} \wedge V^{a} = 0 \qquad \text{(II.3.27c)}$$

$$R^{AB} \equiv F^{AB} + 2\bar{\psi}^{A} \wedge \psi^{B} = 0 \qquad \text{(II.3.27d)}$$

where we have used the definitions

$$R^{ab} \equiv d\omega^{ab} - \omega^{ac} \wedge \omega_{c}^{\cdot b} \qquad \text{(II.3.27e)}$$

$$DV^{a} \equiv dV^{a} - \omega^{ab} \wedge V_{b} \qquad \text{(II.3.27f)}$$

$$F^{AB} \equiv dA^{AB} + \bar{e}A^{AC} \wedge A^{CB} \qquad \text{(II.3.27g)}$$

$$D\psi_{A} \equiv d\psi_{A} - \frac{1}{4}\omega^{ab} \wedge \gamma_{ab}\psi_{A} \qquad \text{(II.3.27h)}$$

$$\mathcal{D}\psi_{A} \equiv D\psi_{A} + \bar{e}A_{AB} \wedge \psi_{B} \qquad \text{(II.3.27i)}$$

The interpretation of Eqs. (II.3.28) is quite evident. R^{ab} is the Riemann curvature 2-form associated to the spin connection ω^{ab}; DV^{a} is the Lorentz covariant derivative of the vierbein V^{a}, while $D\psi_{A}$ is the Lorentz covariant derivative of the spinor form ψ_{A}. F^{AB} is the field strength 2-form of the SO(N) gauge field A_{AB} while $\mathcal{D}\psi_{A}$ is the derivative of ψ_{A} covariantized not only with respect to SO(1,3) but also with respect to SO(N).

In the limit $\bar{e} \to 0$ in which the Osp(4/N) algebra contracts to the N-extended super Poincaré algebra, the Maurer Cartan equations (II.3.27) become

$$R^{a} \equiv DV^{a} - \frac{i}{2}\bar{\psi}_{A} \wedge \gamma^{a}\psi_{A} = 0 \qquad \text{(II.3.28a)}$$

$$R^{ab} \equiv R^{ab} = 0 \qquad \text{(II.3.28b)}$$

$$\rho_{A} \equiv \mathcal{D}\psi_{A} = 0 \qquad \text{(II.3.28c)}$$

$$R^{AB} \equiv dA^{AB} + 2\bar{\psi}^{A} \wedge \psi^{B} = 0 \qquad \text{(II.3.28d)}$$

II.3.3 - Osp(4/N) Maurer-Cartan equations as the structure equations of rigid superspace

The Maurer-Cartan equations (II.3.27) or (II.3.28) acquire a geometrical meaning, which is the starting point for the construction of supersymmetric field-theories (including supergravity), if we restrict the space on which they hold from the supergroup manifold Osp(4/N) to the supercoset manifold

$$(AdS)^{4/4N} = \frac{Osp(4/N)}{SO(1,3) \otimes SO(N)} \qquad (II.3.29)$$

Equation (II.3.29) defines a supermanifold which we call the N-extended anti de Sitter superspace. Its coordinates are the 4 bosonic parameters $\{x^a\}$ associated to the translations generators P_a and the 4N fermionic coordinates $\{\Theta^A\}$ associated to the supersymmetry charges $\{Q_A\}$. The ring of functions on $(AdS)^{4/4N}$ is named the ring of anti de Sitter N-extended superfields

$$\Phi = \Phi(x^a, \Theta^A)$$

Similarly

$$(M)^{4/4N} = \frac{\overline{Osp(4/N)}}{SO(1,3) \otimes SO(N)}$$

defines the supermanifold which we name N-extended Minkowski superspace. Its coordinates are $\{x^a, \Theta^A\}$ as in the previous case: the difference however is the following. The bosonic submanifold of

$(M)^{4/4N}$ is flat Minkowski space while the bosonic submanifold of $(AdS)^{4/4N}$ is anti de Sitter space which has constant negative curvature. The ring of functions on $(M)^{4/4N}$ is by definition the ring of Minkowski N-extended superfields.

We construct the explicit form of the left-invariant 1-forms (V^a, ψ_A, ψ_A, ω^{ab}, A^{AB}) extending to the supercosets the techniques described in Chapter I.6 for the ordinary cosets. First we note that a convenient parametrization of the supercoset (II.3.29) is the following one:

$$\mathcal{O}(x^a, \Theta^A) = \mathcal{O}_F(\Theta)\mathcal{O}_B(x) \qquad (II.3.30)$$

where $\mathcal{O}_F(\Theta)$ is a parametrization of the coset

$$Osp(4/N)/Sp(4) \otimes SO(N) \qquad (II.3.31)$$

whose coordinates are purely fermionic and $\mathcal{O}_B(x)$ is a parametrization of the coset

$$\frac{Sp(4) \otimes SO(N)}{SO(3,1) \otimes SO(N)} \qquad (II.3.32)$$

whose coordinates are purely bosonic and which coincides with anti de Sitter space. That (II.3.30) is a parametrization of the coset (II.3.29) follows from a simple argument. Let $\hat{g}$ be an arbitrary Osp(4/N) element. Acting with $\hat{g}$ on $\mathcal{O}(x,\Theta)$ we get

$$\hat{g}\mathcal{O}(x,\Theta) = \hat{g}\mathcal{O}_F(\Theta)\mathcal{O}_B(x) = \mathcal{O}_F(\Theta')g(\Theta)\mathcal{O}_B(x) \qquad (II.3.33)$$

where $g(\Theta) \in Sp(4) \otimes SO(N)$ is the compensator of the transformation $\hat{g}$ on the coset element $\mathcal{O}_F(\Theta)$.

On the other hand since $\mathcal{O}_B(x)$ is a parametrization of the coset (II.3.32) we have:

$$g(\Theta)\mathcal{O}_B(x) = \mathcal{O}_B(x')h(\Theta,x) \qquad \text{(II.3.34)}$$

where $h(\Theta,x) \in SO(1,3) \otimes SO(N)$ is the compensator of the transformation $g(\Theta)$ on the coset element $\mathcal{O}_B(x)$.

Therefore we get

$$\hat{g}\mathcal{O}(x,\Theta) = \mathcal{O}(x',\Theta')h(x,\Theta) \qquad \text{(II.3.35)}$$

which is the correct behaviour for a parametrization of the coset (II.3.29).

We construct $\mathcal{O}_B(x)$ and $\mathcal{O}_F(\Theta)$ separately. Recalling Eqs. (II.2.139), (II.3.20) and (II.3.26) we set

$$\mathcal{O}_B(x) = \exp\left(\begin{array}{c|c} \frac{i}{2}\, t^a\gamma_a 2\bar{e} & 0 \\ \hline 0 & 0 \end{array}\right) \qquad \text{(II.3.36)}$$

where the relation between the parameters t^a and the coordinates x^a is still to be established. Using the simple relation

$$(t^a\gamma_a)^2 = t^a t_a \, \mathbb{1}_{(4x4)} \qquad \text{(II.3.37)}$$

we obtain

$$\exp(i\bar{e}t^a\gamma_a) = \cos(\sqrt{t^2}\,\bar{e}) + \frac{it^a\gamma_a}{\sqrt{t^2}}\sin(\bar{e}\sqrt{t^2}) \qquad \text{(II.3.38)}$$

By means of the position

$$\frac{2\bar{e}\,x^a}{\sqrt{1+4e^2|x|^2}} = \frac{t^a}{\sqrt{t^2}}\sin(\bar{e}\sqrt{t^2}) \qquad \text{(II.3.39)}$$

which defines the coordinate x^a we get

$$\mathcal{O}_B(x) = \left(\begin{array}{c|c} \dfrac{1+2\bar{e}\,i\,x^a\gamma_a}{\sqrt{1+4\,e^2|x|^2}} & 0 \\ \hline 0 & 1 \end{array}\right) \qquad \text{(II.3.40)}$$

The inverse matrix $\mathcal{O}_B^{-1}(x)$ is easily seen to be identical with $\mathcal{O}_B(-x)$:

$$\mathcal{O}_B^{-1}(x) = \mathcal{O}_B(-x) = \left(\begin{array}{c|c} \dfrac{1+2\bar{e}\,i\,x^a\gamma_a}{\sqrt{1+4\,\bar{e}^2|x|^2}} & 0 \\ \hline 0 & 1 \end{array}\right) \qquad \text{(II.3.41)}$$

Next we construct $\mathcal{O}_F(\Theta)$:

$$\mathcal{O}_F(\Theta) = \exp\left(\begin{array}{c|c} 0 & \xi^B \\ \hline \bar{\xi}^A & 0 \end{array}\right) \qquad \text{(II.3.42)}$$

The exponentiation of this off-diagonal graded matrix can be formally performed in the same way as in Chapter I.6 we exponentiated the off-diagonal bosonic matrix:

$$i\mathbb{K} = \left(\begin{array}{c|c} 0 & B \\ \hline B^T & 0 \end{array}\right) \qquad (II.3.43)$$

Comparing with Eqs. (I.6.43) we can write

$$\mathcal{O}_F(\Theta) = \left(\begin{array}{c|c} (1 + \Theta^M \bar{\Theta}^M)^{\frac{1}{2}} & \Theta^B \\ \hline \bar{\Theta}^A & (\delta^{AB} + \bar{\Theta}^A \Theta^B)^{\frac{1}{2}} \end{array}\right) \qquad (II.3.44)$$

where the square root of the matrices

$$\delta^{\alpha\beta} + M^{\alpha\beta} \equiv \delta^{\alpha\beta} + \Theta^{\alpha}_M \bar{\Theta}^{\beta}_M = (1 + M)^{\alpha\beta} \qquad (II.3.45a)$$

$$\delta^{AB} + N^{AB} \equiv \delta^{AB} + \bar{\Theta}^A \Theta^B = (1 + N)^{AB} \qquad (II.3.45b)$$

is defined by the power series expansion

$$\sqrt{1 + x} = 1 + \frac{1}{2} x - \frac{1}{8} x^2 + \frac{1}{16} x^3 - \frac{5}{128} x^4 + \ldots \qquad (II.3.46)$$

What changes with respect to the case of ordinary cosets is that being the Θ^A.s anticommuting the series stops at a certain point, all the subsequent terms being identically zero. In particular since

there are 4N-different Θ^A.s and since both M and N are quadratic in Θ.s the power series expansion (II.3.46) stops at the 2N-power.

In the N=8 case which is the highest of physical relevance we arrive at the 16-th power. In general we can write

$$(1 + \Theta^M \bar{\Theta}^M)^{\frac{1}{2}} = \mathbb{1} + \frac{1}{2} \Theta^M \bar{\Theta}^M - \frac{1}{8} \Theta^{M_1} \bar{\Theta}^{M_1} \Theta^{M_2} \bar{\Theta}^{M_2} + \ldots$$
$$- \frac{1\cdot1\cdot3\cdot5\cdot7\ldots(2N+1)}{2\cdot4\cdot6\cdot8\cdot10\ldots(4N)} \Theta^{M_1} \bar{\Theta}^{M_1} \ldots \Theta^{M_{2N}} \bar{\Theta}^{M_{2N}} \qquad (II.3.47a)$$

$$(1 + \bar{\Theta}^{A_1} \Theta^B)^{\frac{1}{2}} = \delta^{A_1 B} + \frac{1}{2} \bar{\Theta}^{A_1} \Theta^B - \frac{1}{8} \bar{\Theta}^{A_1} \Theta^{A_2} \bar{\Theta}^{A_2} \Theta^B + \ldots$$
$$- \frac{1\cdot1\cdot3\cdot5\cdot7\ldots(2N+1)}{2\cdot4\cdot6\cdot8\cdot10\ldots(4N)} \bar{\Theta}^A \Theta^{A_2} \bar{\Theta}^{A_2} \ldots \bar{\Theta}^{A_{2N}} \Theta^B \qquad (II.3.47b)$$

Recalling the rescaling prescription (II.3.26), it is convenient to rewrite Eq. (II.3.44) as follows:

$$\mathcal{O}_F(\Theta) = \left(\begin{array}{c|c} (1 + 2\bar{e}\,\Theta_M \bar{\Theta}_M)^{\frac{1}{2}} & \sqrt{2\bar{e}}\,\Theta^B \\ \hline \sqrt{2\bar{e}}\,\Theta^A & (\delta_{AB} + 2\bar{e}\,\bar{\Theta}_A \Theta_B)^{\frac{1}{2}} \end{array}\right) \qquad (II.3.48)$$

and we get

$$\mathcal{O}_F^{-1}(\Theta) = \mathcal{O}_F(-\Theta) = \left(\begin{array}{c|c} (1 + 2\bar{e}\,\Theta^M\bar{\Theta}^M)^{\frac{1}{2}} & \sqrt{2\bar{e}}\,\Theta^B \\ \hline -\sqrt{2\bar{e}}\,\bar{\Theta}^A & (\delta^{AB} + 2\bar{e}\,\bar{\Theta}^A\Theta^B)^{\frac{1}{2}} \end{array}\right) \tag{II.3.49}$$

We are now in a position to calculate the left-invariant 1-forms:

$$\sigma_L = \mathcal{O}^{-1}(x,\Theta)d\mathcal{O}(x,\Theta) = \mathcal{O}_B^{-1}(x)\mathcal{O}_F^{-1}(\Theta)d\mathcal{O}_F(\Theta)\mathcal{O}_B(x) + $$
$$+ \mathcal{O}_B^{-1}(x)d\mathcal{O}_B(x) \tag{II.3.50}$$

Comparing Eq. (II.3.20) with the definition of V^a, ψ^A, ω^{ab}, A^{AB}

$$\sigma_{(L)} = \left(\begin{array}{c|c} -\frac{1}{4}\,\omega^{ab}\gamma_{ab} + i\,\bar{e}\,\gamma_a V^a & \sqrt{2\bar{e}}\,\psi^B \\ \hline \sqrt{2\bar{e}}\,\bar{\psi}^A & \bar{e}\,A^{AB} \end{array}\right) \tag{II.3.51}$$

we obtain

$$\psi_A = \left(\frac{1 - 2\bar{e}\,i\,x^a\gamma_a}{\sqrt{1+4\,\bar{e}^2|x|^2}}\right)(1 + 2\bar{e}\,\Theta_M\bar{\Theta}_M)^{\frac{1}{2}}d\Theta_A \tag{II.3.52a}$$

$$V^a = -\frac{i}{4\bar{e}}\,\mathrm{Tr}\left\{\gamma^a\left[\frac{1-2i\,\bar{e}\,\gamma^b x_b}{\sqrt{1+4\,\bar{e}^2|x|^2}}\,d\left(\frac{1+2i\,\bar{e}\,\gamma^c x_c}{\sqrt{1+4\,\bar{e}^2|x|^2}}\right)\right]\right\}$$
$$-\frac{i}{4\bar{e}}\,\mathrm{Tr}\left\{\gamma^a\,\frac{1-2i\,\bar{e}\,\gamma^b x_b}{1+4\,\bar{e}^2|x|^2}\left[(1+2\,\bar{e}\,\Theta_M\bar{\Theta}_M)^{\frac{1}{2}}d(1+2\,\bar{e}\,\Theta_M\bar{\Theta}_M)^{\frac{1}{2}}\right.\right.$$
$$\left.\left.-2\,\bar{e}\,\Theta_M\,d\,\Theta_M\right](1+2i\,\bar{e}\,\gamma^c x_c)\right\} \tag{II.3.52b}$$

$$\omega^{ab} = \frac{1}{2}\,\mathrm{Tr}\left\{\gamma^{ab}\left[\frac{1-2i\,\bar{e}\,\gamma^b x_b}{\sqrt{1+4\,e^2|x|^2}}\,d\left(\frac{1+2i\,\bar{e}\,\gamma^c x_c}{\sqrt{1+4\,e^2|x|^2}}\right)\right]\right\}$$
$$+\frac{1}{2}\,\mathrm{Tr}\left\{\gamma^{ab}\,\frac{1-2i\,\bar{e}\,\gamma^b x_b}{1+4\,e^2|x|^2}\right](1+2\,\bar{e}\,\Theta_M\bar{\Theta}_M)^{\frac{1}{2}}d(1+2\,\bar{e}\,\Theta_N\bar{\Theta}_N)^{\frac{1}{2}}$$
$$-2\,\bar{e}\,\Theta_N\,d\,\bar{\Theta}_N\Big\} \tag{II.3.52c}$$

$$A^{AB} = \frac{1}{\bar{e}}\left[(\delta^{AB}+2\,\bar{e}\,\bar{\Theta}^A\Theta^C)^{\frac{1}{2}}d(\delta^{CB}+2\,\bar{e}\,\bar{\Theta}^C\Theta^B) - 2\,\bar{e}\,\bar{\Theta}^A\,d\,\Theta^B\right] \tag{II.3.52d}$$

As we see the spinor 1-form ψ^A contains only the differential $d\Theta^A$ of the fermionic coordinate Θ^A, while V^a contains both dx^a and $d\Theta$-terms.

The dx^a-part of V^a is the vierbein of an anti de Sitter space whose Riemann tensor is

$$R^{ab}_{\cdot\cdot mn} = -\,4\bar{e}^2\,\delta^{ab}_{mn} \tag{II.3.53}$$

Indeed on the submanifold $\Theta^A \equiv 0 (\Longrightarrow d\Theta^A = 0)$ we have:

$$\psi^A = A^{AB} = 0 \quad ;$$

$$V^a = \overset{o}{V}{}^a_b(x)\,dx^b \quad ; \quad \omega^{ab} = \overset{o}{\omega}{}^{ab}_c(x)\,dx^c \tag{II.3.54}$$

and the Maurer Cartan equations (II.3.27) reduce to:

$$D\overset{o}{V}{}^a = 0 \tag{II.3.55a}$$

$$R^{ab} = -4\,\bar{e}^2\,\overset{o}{V}{}^a \wedge \overset{o}{V}{}^b \tag{II.3.55b}$$

which are the structure equations of an anti de Sitter space with the Riemann tensor (II.3.53). This justifies our claim that the bosonic submanifold of the supercoset manifold (II.3.29) is anti de Sitter space. With a similar argument we can show that the bosonic submanifold of (II.3.31) is Minkowski space, characterized by a vanishing Riemann tensor:

$$R^{ab}{}_{mn} = 0 \tag{II.3.56}$$

It suffices to perform the limit $\bar{e} \to 0$ on the Eqs. (II.3.52) and on the Maurer-Cartan Eqs. (II.3.27).

Actually in the contraction limit $\bar{e} \to 0$ the left-invariant 1-forms ψ^A, V^a, ω^{ab}, A^{AB} take the very simple form

$$\psi^A = d\,\theta^A \tag{II.3.57a}$$

$$V^a = 2dx^a + \frac{i}{2}\,\bar{\theta}^A\gamma^a\,d\,\theta^A \tag{II.3.57b}$$

$$\omega^{ab} = 0 \tag{II.3.57c}$$

$$A^{AB} = d\,\theta^A\theta^B - \bar{\theta}^A\,d\,\theta^B \tag{II.3.57d}$$

which allows an immediate verification of the Maurer Cartan equations (II.3.28).

The $(4 \oplus 4N)$ 1-forms $\{V^a, \psi^A\}$ are the components of an anholonomic cotangent frame on the N-extended superspace (either Minkowski or anti de Sitter).

Given any function on $M^{4/4N}$ ($AdS^{4/4N}$), namely any superfield $\phi(x,\theta)$ its exterior differential can be written as:

$$d\phi(x,\theta) = \Phi_a V^a + \bar{\Lambda}_A\psi^A \tag{II.3.58}$$

where the superfields $\Phi_a(x,\theta)$ are named the <u>inner components</u> of $d\phi$ while the superfields $\Lambda_A(x,\theta)$ are christened the <u>outer components</u> of the same.

Calling D_a and $\bar{D}_{A\alpha}$ the tangent vectors dual to $(V^a, \psi_{A\alpha})$:

$$V^a(D_b) = \delta^a_b \quad ; \quad V^a(\bar{D}_{A\alpha}) = 0 \tag{II.3.59a}$$

$$\psi^A(D_b) = 0 \quad ; \quad \psi^{A\alpha}(\bar{D}_{B\beta}) = \delta^A_B\delta^\alpha_\beta \tag{II.3.59b}$$

we can write

$$\phi_a(x,\theta) = D_a\phi(x,\theta) \tag{II.3.60a}$$

$$\bar{\Lambda}_{A\alpha}(x,\Theta) = \bar{D}_{A\alpha}\phi(x,\Theta) \tag{II.3.60b}$$

In the case (II.3.57) of Minkowski superspace the explicit form of D_a and $\bar{D}_{A\alpha}$ is easily obtained. Recalling equations (II.2.104) we can write

$$D_a = \frac{1}{2}\,\partial_a = \frac{\partial}{\partial x^a} \Longrightarrow \Phi_a(x,\Theta) = \frac{1}{2}\,\Phi_a\;(x,\Theta) \tag{II.3.61a}$$

$$\bar{D}_{A\alpha} = \frac{\partial}{\partial\Theta^{A\alpha}} + \frac{i}{2}\,(\bar{\Theta}_A\gamma^a)\,\frac{\partial}{\partial x^a} \Longrightarrow \bar{\Lambda}_{A\alpha} = \frac{\partial}{\partial\Theta^{A\alpha}}\,\Phi + \frac{i}{2}\,\bar{\Theta}_A\gamma^a\Phi_a \tag{II.3.61b}$$

where we have used

$$d\Theta^{A\alpha}\left(\frac{\partial}{\partial\Theta^{B\beta}}\right) = \delta^A_B\delta^\alpha_\beta \qquad ; \qquad d\bar{\Theta}^{A\alpha}\left(\frac{\partial}{\partial\bar{\Theta}^{B\beta}}\right) = \delta^A_B\delta^\alpha_\beta \tag{II.3.62}$$

The differential operators D_a and $D_{A\alpha}$ are named <u>the invariant derivatives of superspace</u>. They should not be confused with the generators $\{P_a,\ \bar{Q}_{A\alpha}\}$ of the supersymmetry algebra (II.2.142). In the next section we discuss why.

<u>II.3.4</u> - <u>Killing vectors on superspace, that is the generators of the supersymmetry algebra of superisometries</u>

The reason why D_a and $\bar{D}_{A\alpha}$ should not be taken for the translation and supersymmetry generators of the supersymmetry algebra (II.2.142) is best understood by recalling that superspace is a coset rather than a group manifold. On a group-manifold we have two sets of tangent vectors, both satisfying the Lie algebra: the generators of right-translations (dual to the left-invariant 1-forms) and the generators of the left-translations (dual to the right-invariant 1-forms).

On a coset manifold, instead, the symmetry between left and right is broken by the very fact that G/H can be chosen to be either a right or a left coset space. In this book we have adopted the convention that we consider every coset manifold to be a right-coset space and this is the reason why we restrict our attention to left-invariant 1-forms. Under these conditions what happens is that the generators of left-translations (which are not dual to the left-invariant 1-forms!) become the Killing vectors of the coset manifold and satisfy the complete Lie algebra. On the other hand the generator of the right-translations are now restricted only to the directions of the vielbein (in our case the V^a and ψ^α directions) and in general are not required to satisfy any algebra. The D_a and $\bar{D}_{A\alpha}$ vectors are the remnants on the coset of the full set of right-translations existing only on the supergroup manifold. The Killing vectors $(P_a, \bar{Q}_{A\alpha}, M_{ab}, T_{AB})$, instead, have still to be constructed.

To obtain their form it suffices to apply the techniques of Chapter I.6 and calculate the action of a Lie algebra element on the matrix $\mathcal{O}(x,\Theta)$ parametrizing the coset.

Recalling Eq. (I.6.72) we can set

$$\mathbb{T}_A\mathcal{O}(x,\Theta) = \vec{\mathbb{K}}_A\mathcal{O}(x,\Theta) - \mathcal{O}(x,\Theta)\,\mathbb{T}_i W^i_A(x,\Theta) \tag{II.3.63}$$

where $\mathbb{T}_A$ is any of the generators $(P_a, \bar{Q}_{A\alpha}, M_{ab}, T_{AB})$, $\vec{\mathbb{K}}_A$ is the associated Killing vector, $\mathbb{T}_i$ is either M_{ab} or T_{AB} and $W^i_A(x,\ \Theta)$ is the $SO(1,3) \otimes SO(N)$ compensator. Utilizing the fundamental representation (II.2.135) by explicit evaluation of (II.3.63) one can work out the form of the differential operators $(P_a, \bar{Q}_{A\alpha}, M_{ab}, T_{AB})$. In the $(\bar{e}\neq 0)$ case this is quite a bit of work which we do not feel the need to do explicitly. In the contracted case $(\bar{e}=0)$ we can attempt a direct evaluation of the Killing vectors $(\vec{P}_a, \vec{\bar{Q}}_A, \vec{M}_{ab}, \vec{T}_{AB}) = \{\vec{K}_A\}$ relying on their alternative definition as isometries of the vielbein defined by Eqs. (II.3.57a-b). Hence we write:

$$\ell_{\vec{K}_A} V^a = W_A^{ab} V^b \tag{II.3.64a}$$

$$\ell_{\vec{K}_A} \psi^B = -\frac{1}{4} W_A^{ab} \gamma_{ab} \psi^B \tag{II.3.64b}$$

where ℓ denotes the Lie derivative $W_A{}^{ab}$ is a suitable SO(1,3) compensator and furthermore we impose that the $\{\vec{K}_A\}$ satisfy the appropriate super algebra (II.2.142) (with $\bar{e}=0$). (We note that in the limit $\bar{e}\to 0$ the group SO(N) degenerates into a bunch of N U(1).s whose action is zero on everything).

To solve this problem we make the ansatz

$$\vec{P}_a = \alpha \partial_a \tag{II.3.65a}$$

$$\vec{\bar{Q}}_{A\alpha} = \beta \frac{\partial}{\partial\theta^{A\alpha}} + \gamma \bar{\theta}_{A\beta} \gamma^a_{\beta\alpha} \partial_a \tag{II.3.65b}$$

$$M_{ab} = \delta(x_a\partial_b - x_b\partial_a) + \eta\bar{\theta}_A \gamma_{ab} \frac{\partial}{\partial\bar{\theta}^A} \tag{II.3.65c}$$

$$T_{AB} = 0 \tag{II.3.65d}$$

where α, β, γ, δ, η are numerical coefficients to be determined.

Considering first the commutation relations (II.2.142) we obtain

$$\delta = \frac{1}{2} \qquad \eta = \frac{1}{4} \tag{II.3.66a}$$

$$2\,\beta\gamma = i\alpha \tag{II.3.66b}$$

Then we impose the invariance conditions of the vielbeins, that is Eqs. (II.3.64). We begin with P_a and $\bar{Q}_{A\alpha}$ and we assume $W_a^{ab} = W_{A\alpha}^{ab} = 0$

The definition of the Lie derivative $(\ell_t = t\rfloor d + d\, t\rfloor)$ extends trivially to supermanifolds and hence we find:

$$\ell_{P_b} V^a = P_b\rfloor dV^a + d(P_b\rfloor V^a) = P_b\rfloor \frac{i}{2} d\bar{\theta}^A \gamma^a d\theta^A + d(\alpha\delta^a_b) = 0 \tag{II.3.67}$$

Similarly

$$\ell_{P_b} \psi^{A\alpha} = P_b\rfloor d\psi^{A\alpha} + d(P_b\rfloor \psi^{A\alpha}) = 0 \tag{II.3.68a}$$

$$\ell_{\bar{Q}_{B\beta}} \psi^{A\alpha} = \bar{Q}_{B\beta}\rfloor d\psi^{A\alpha} + d(\bar{Q}_{B\beta}\rfloor d\theta^{\alpha A}) = d(\delta^A_B \delta^\alpha_\beta) = 0 \tag{II.3.68b}$$

On the other hand:

$$\ell_{\bar{Q}_{B\beta}} V^a = \underline{\bar{Q}_{B\beta}}\rfloor dV^a + d(\underline{\bar{Q}_{B\beta}}\rfloor V^a) =$$

$$= i\beta\, d\bar{\Theta}^A\gamma^a \underline{\frac{\partial}{\partial\theta^{B\beta}}}\rfloor d\Theta^A + (2\gamma\, d\bar{\Theta}^B\gamma^a - \frac{i}{2}\,\beta d\bar{\Theta}^B\gamma^a)_\beta$$

$$= (\frac{i}{2}\,\beta + 2\gamma)\,(d\bar{\Theta}^B\gamma^a)_\beta \tag{II.3.69}$$

Hence in order for $\bar{Q}_{B\beta}$ to be a Killing vector we must have

$$\gamma = -\frac{i}{4}\,\beta \tag{II.3.70}$$

In the above equations note that we used the commutation rule:

$$\underline{\partial/\partial\theta}\rfloor(d\bar{\Theta}\wedge\omega) = \underline{\partial/\partial\theta}\rfloor d\bar{\Theta}\wedge\omega + d\bar{\Theta}\wedge\underline{\partial/\partial\theta}\rfloor\omega \tag{II.3.71}$$

which is correct since both the 1-form $d\Theta$ and the vector $\partial/\partial\Theta$ are of the fermionic type. In general for graded differential forms and vectors we have:

$$\underline{t^{(c)}}\rfloor\omega^{(a)}_{(p)}\wedge\omega^{(b)}_{(q)} = \underline{t^{(c)}}\rfloor\omega^{(a)}_{(p)}\wedge\omega^{(b)}_{(q)} + (-)^{ac+p}\omega^{(a)}_{(p)}\wedge\underline{t^{(c)}}\rfloor\omega^{(b)}_{(q)} \tag{II.3.72}$$

where (c), (a), (b) are the gradings and (p) and (q) the degrees of the differential forms.

Calculating the Lie derivative of V^a and ψ^A along M_{ab} we can determine the compensating function W^{cd}_{ab}. We find

$$\ell_{M_{ab}} V^c = \underline{M_{ab}}\rfloor(\frac{i}{2}d\bar{\Theta}^A\gamma^c d\Theta^A) + d(\underline{M_{ab}}\rfloor V^c) =$$

$$= \frac{i}{4}\,\bar{\Theta}^A(\gamma^{ab}\gamma^c)d\Theta^A + d(2x_{[a}\delta^c_{b]} - \frac{i}{8}\,\bar{\Theta}^A\gamma^{ab}\gamma^c\Theta^A) \tag{II.3.73}$$

In the term under exterior derivative the only surviving current is

$$\bar{\Theta}_A\gamma^{abc}\Theta_A$$

Indeed the matrix $\gamma^{abc} = \text{const}\ \varepsilon^{abcd}\gamma_5\gamma_d$ is antisymmetric and can sit in between two anticommuting Θ.s while γ^a is symmetric and gives a vanishing contribution in between Θ.s (see Chapter II.8). Taking this into account we get:

$$\ell_{M_{ab}} V^c = \frac{i}{4}\,\bar{\Theta}^A\gamma^{ab}\gamma^c d\Theta^A + 2dx_{[a}\delta^c_{b]} - \frac{i}{4}\,\bar{\Theta}^A\gamma^{abc}d\Theta^A =$$

$$= \frac{i}{8}\,\bar{\Theta}^A[\gamma^{ab},\gamma^c]d\Theta^A + 2dx_{[a}\delta^c_{b]} = V_{[a}\delta^c_{b]} \Rightarrow$$

$$\Rightarrow W^{cd}_{ad} = -\,\delta^{cd}_{ab} \tag{II.3.74}$$

Summarizing: Minkowski or anti de Sitter N-extended superspaces are supermanifolds whose supervielbein (V^a, ψ^A) admits a group of isometries isomorphic to the $\overline{Osp(4/N)}$ or $Osp(4/N)$ supergroup. The isometries are generated by the Killing vectors discussed above. The Maurer Cartan equations (II.3.28) or (II.3.27) are to be reinterpreted as the structure equations of the supermanifold, (ρ^A, R^a) being the supertorsion and (R^{ab}, R^{AB}) the supercurvatures.

The vectors D_a, $\bar{D}_{A\alpha}$ dual to the supervielbein are the left-invariant generators of the right translations on the supermanifold. (Their covariant Lie derivative along all the Killing vectors is zero). Any constraint imposed on a superfield $\Phi(x,\Theta)$ by means of the differential operator D_a and $\bar{D}_{A\alpha}$ has an invariant character with respect to the superisometries: typically a supersymmetric field equation is obtained by applying to $\phi(x,\Theta)$ some operator $\Box$ constructed with D_a and $\bar{D}_{A\alpha}$.

In Table II.3.I we summarize the explicit form of all the relevant forms and operators in Minkowski superspace. The reader will in particular note that we have

$$\{\bar{D}_{A\alpha},\bar{D}_{B\beta}\} = i(C\gamma^a)_{\alpha\beta}\delta_{AB}D_a = - i(C\gamma^a)_{\alpha\beta}\delta_{AB}P_a$$

$$\{\bar{Q}_{A\alpha},\bar{Q}_{B\beta}\} = i(C\gamma^a)_{\alpha\beta}\delta_{AB}P_a$$

$$\{\bar{Q}_{A\alpha},\bar{D}_{B\beta}\} = 0 \qquad \text{(II.3.75)}$$

The fact that the anticommutator of the $\bar{D}_{A\alpha}$.s is almost identical to that of the $\bar{Q}_{A\alpha}$.s is often a source of confusion with respect to which operator should be named the supersymmetry generator. We hope to have clarified the matter. The $\bar{D}_{A\alpha}$.s which used to generate the right-supersymmetries on the group-manifold are not Killing vectors of superspace! Rather they generate fermionic translations which are invariant under supersymmetry!

Finally let us introduce some names. The 1-form V^a is the vierbein while the 1-form ψ^A is christened the gravitino 1-form. The reason is simple: with a procedure similar to the one discussed in Chapter I.3, on superspace we can introduce new sets of vielbeins ($V^{a'}$, $\psi^{A'}$) and connections ($\omega^{ab'}$, $A^{AB'}$) which do not satisfy the Maurer-Cartan equations (II.3.27) (the soft forms). They describe dynamical fields for which we are going to write action principles (supergravity theories). While V^a is associated to a spin 2-particle, (the graviton) ψ^A turns out to be associated to N spin 3/2 particles (the gravitinos).

When the curvatures R^{ab}, R^a, ρ^A, R^{AB} are non zero, the closure of the original Maurer equations is reflected into the Bianchi identities:

$$DR^a + R^{ab} \wedge V_b - i\bar{\psi}^A \wedge \gamma^a\psi_A = 0 \qquad \text{(II.3.76a)}$$

$$DR^{ab} - 8\bar{e}^2R^{[a} \wedge V^{b]} + 2\bar{e}\bar{\psi}_A \wedge \gamma^{ab}\rho_A = 0 \qquad \text{(II.3.76b)}$$

$$D\rho_A - \bar{e}R_{AB} \wedge \psi_B - i\bar{e}\gamma_a\psi_A \wedge R^a - \frac{1}{4}R^{ab} \wedge \gamma_{ab}\psi_A = 0 \qquad \text{(II.3.76c)}$$

$$DR_{AB} + 4\,\bar{\psi}_A \wedge \rho_B = 0 \qquad \text{(II.3.76d)}$$

which in the Poincaré limit $\bar{e} \to 0$ become

$$DR^a + R^{ab} \wedge V_b - i\bar{\psi}^A \wedge \gamma^a\rho_A = 0 \qquad \text{(II.3.77a)}$$

$$DR^{ab} = 0 \qquad \text{(II.3.77b)}$$

$$D\rho_A - \frac{1}{4}R^{ab} \wedge \gamma_{ab}\psi_A = 0 \qquad \text{(II.3.77c)}$$

$$dR_{AB} + 4\,\bar{\psi}_{[A} \wedge \rho_{B]} = 0 \qquad \text{(II.3.77d)}$$

As we shall see the Bianchi identities play a crucial role in the construction of the supergravity models.

TABLE II.3.1

Minkowski N-extended superspace

A) Maurer Cartan equations of Osp(4/N)

$$R^a = DV^a - \frac{i}{2}\bar{\psi}^A \wedge \gamma^a \psi^A = 0$$

$$R^{ab} = R^{ab} = 0$$

$$\rho_A = D\psi_A = 0$$

$$R^{AB} = dA^{AB} + 2\,\bar{\psi}^A \wedge \psi^B = 0$$

B) Explicit form of the vielbein and connections

$$V^a = 2dx^a + \frac{i}{2}\bar{\Theta}^A \wedge \gamma^a d\Theta^A \quad ; \quad \psi^A = d\,\Theta^A$$

$$\omega^{ab} = 0 \quad ; \quad A^{AB} = d\,\bar{\Theta}^A\Theta^B - \bar{\Theta}^A d\,\Theta^B$$

C) Invariant derivatives in superspace

$$D_a = \frac{1}{2}\frac{\partial}{\partial x^a} \quad ; \quad \bar{D}_{A\alpha} = \frac{\partial}{\partial\Theta^{A\alpha}} + \frac{i}{4}(\bar{\Theta}^A\gamma^a)_\alpha \frac{\partial}{\partial x^a}$$

D) Explicit form of the Killing vectors

$$P_a = -\frac{1}{2}\frac{\partial}{\partial x^a} \qquad T_{AB} = 0$$

$$\bar{Q}_{A\alpha} = \frac{\partial}{\partial\Theta^{A\alpha}} - \frac{i}{4}(\bar{\Theta}^A\gamma^a)_\alpha \frac{\partial}{\partial x^a}$$

$$M_{ab} = \frac{1}{2}(x_a\partial_b - x_b\partial_a) + \frac{1}{4}(\bar{\Theta}_A\gamma_{ab}\frac{\partial}{\partial\bar{\Theta}^A})$$

E) Explicit form of the W-compensators

$$W_a^{cd} = W_{A\alpha}^{cd} = W_{AB}^{cd} = 0 \quad ; \quad W_{ab}^{cd} = -\delta_{ab}^{cd}$$

F) Invariance equations

$$\ell_{\begin{pmatrix}P_a\\ \bar{Q}_{\alpha A}\\ T_{AB}\end{pmatrix}} V^a = \ell_{\begin{pmatrix}P_a\\ \bar{Q}_{\alpha A}\\ T_{AB}\end{pmatrix}} \psi^A = 0 \Rightarrow [P_a, D_b] = [\bar{Q}_{A\alpha}, D_b] =$$

$$= [T_{AB}, D_b] = [P_a, \bar{D}_{A\alpha}] = [T_{AB}, \bar{D}_{A\alpha}] =$$

$$= \{\bar{Q}_{B\beta}, \bar{D}_{A\alpha}\} = 0$$

$$\ell_{M_{ab}} V^c = V_{[a}\delta^c_{b]} \quad ; \quad \ell_{M_{ab}}\psi_A = \frac{1}{4}\gamma_{ab}\psi_A$$