# $Spin^c$-MANIFOLDS

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## 1. Introduction

$Spin^c$-structures on manifolds are a complex analogue to the more common notion of spin structures on manifolds. They have been known since the 1960's (see **[A-B-S]**), but they had no real importance (as far as I can tell), until the recent announcement of the Seiberg-Witten equations for 4-manifolds in **[W]**. These equations promise to vastly simplify the study of smooth 4-manifolds, and their definition requires the presence of a $spin^c$-structure. In this paper I will review the definition of $spin^c$-structures on manifolds from both a geometric and algebraic point of view, and prove their existence in some important cases. I will conclude by looking at how they appear in the formulation of the Seiberg-Witten equations.

## 2. Geometric formulation of $Spin^c_n$

In one sense, *spin* and $spin^c$ structures are just generalizations of orientations. Consider a smooth manifold $M^n$ with tangent bundle $TM$. This vector space bundle gives rise to a principal $O(n)$-bundle of frames, which we denote $P_O(TM)$. Recall that the manifold is said to be *orientable* if this bundle can be reduced to an $SO(n)$-bundle $P_{SO}(TM)$, making the fibers connected. This means that any trivialization of the bundle over the (disconnected) 0-skeleton of $M$ can be extended to a trivialization over the (connected) 1-skeleton. The next step is to make the fiber simply connected (where possible). This will mean that a trivialization over the 1-skeleton of $M$ can be extended over the 2-skeleton. Recalling that, for $n \geq 3$, $\pi_1(SO(n)) = \mathbb{Z}_2$, we define $Spin_n$ to be the double cover of $SO(n)$. For $n \geq 3$, this is the universal (i.e. simply- connected) cover; in the exceptional cases we have $Spin_2 = S^1$ and $Spin_1 = S^0$. We then say that the manifold is *spin* if the bundle $P_{SO}(TM)$ has a double cover by a principal $Spin_n$-bundle $P_{Spin}(TM)$.

To find the complex analogue, we replace $SO(n)$ by the group $SO(n) \times U(1)$, and consider its double cover. With this in mind, we define:

$$Spin^c_n = (Spin_n \times U(1))/\{\pm(1,1)\} = Spin_n \times_{\mathbb{Z}_2} U(1)$$

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This is the desired double cover of $SO(n) \times U(1)$ via the map $[A, \lambda] \mapsto [p(A), \lambda^2]$, where $p$ is the double cover of $SO(n)$ by $Spin_n$. Finally, we define M to be $spin^c$ if given the bundle $P_{SO}(TM)$, there are principal bundles $P_{U(1)}(TM)$ and $P_{Spin^c}(TM)$ with a $spin^c$-equivariant bundle map:

$$\xi : P_{Spin^c}(TM) \longrightarrow P_{SO}(TM) \times P_{U(1)}(TM).$$

This definition of $Spin_n^c$ leads to a very nice geometric criterion for the existence of a $spin^c$-structure ([**K2**]). Since $U(1) = SO(2)$, there is a natural map $SO(n) \times U(1) \to SO(n+2)$ which extends (via Whitney sum) to a map of bundles. We can define $Spin_n^c$ as the pullback by this map of the covering map $Spin_{n+2} \to SO(n+2)$:

$$\begin{array}{ccc} Spin_n^c & \longrightarrow & Spin_{n+2} \\ \downarrow & & \downarrow \\ SO(n) \times U(1) & \longrightarrow & SO(n+2) \end{array}$$

Therefore, a $spin^c$-structure on $TM$ consists of a complex line bundle $L$ and a *spin*-structure on $TM \oplus L$. We can restate this as:

**Theorem 1.** *A manifold $M$ is $spin^c$ (i.e. $TM$ has a $spin^c$-structure) $\Leftrightarrow$ there is a complex line bundle $L$ over $M$ such that $TM \oplus L$ has a spin-structure.*

So $M$ is $spin^c$ if the obstruction to extending a trivialization of the tangent bundle over the 2-skeleton can be removed by adding a complex line bundle.

## 3. Examples of $Spin^c$-manifolds

We start with examples of manifolds which have canonical $Spin^c$-structures.

**Theorem 2.** *If $M$ is a spin manifold, then $M$ has a canonical $spin^c$-structure.*

Proof: We simply extend the spin structure by taking the fiber product with the trivial $U(1)$-bundle $U_1$, letting

$$P_{Spin^c}(TM) = P_{Spin}(TM) \times_{M,\mathbb{Z}_2} U_1. \square$$

**Theorem 3.** *If $M$ has an almost complex structure, then $M$ has a canonical $spin^c$-structure.*

Proof: Let $j : U(k) \to SO(2k)$ denote the natural homomorphism. Then we can define a homomorphism $g : U(k) \to SO(2k) \times U(1)$ by $g(A) = (j(A), det(A))$. Although $j$ does not lift to $Spin_{2k}$, $g$ *does* lift to $Spin_{2k}^c$. Denote this lift $\gamma$. An almost complex structure on $M$ means $TM$ can be viewed as a complex vector bundle, and so $M$ has an unitary frame bundle $P_{U(n)}(TM)$. We now construct the desired $Spin^c$ bundle as an associated bundle:

$$P_{Spin^c}(TM) = P_{U(n)}(TM) \times_\gamma Spin^c_{2k}. \ \square$$

In fact, we can give another, more algebraic, general criterion for whether a manifold has a $Spin^c$-structure:

**Theorem 4.** *An orientable manifold $M$ can be given a $Spin^c$-structure $\Leftrightarrow$ the second Stiefel-Whitney class $w_2(M)$ is the mod 2 reduction of an integral class.*

PROOF: Recall that a manifold $M$ has a *spin*-structure $\Leftrightarrow$ the second Stiefel-Whitney class $w_2(M)$ is 0 (see **[L-M]** and **[K2]**). So we apply our geometric criterion from the last section, which says that $M$ can be given a $spin^c$-structure $\Leftrightarrow$ there is a complex line bundle $L$ such that $TM \oplus L$ is spin, which means $w_2(TM \oplus L)$ is 0. But, since the Stiefel-Whitney classes are stable, we have:

$$w_2(TM \oplus L) = w_2(TM) + w_2(L) + w_1(TM)w_1(L) = 0$$

Both these bundles are orientable, so the first Stiefel-Whitney classes are both 0, which means $w_2(TM) + w_2(L) = 0$. Since these are mod 2 classes, $w_2(TM) = w_2(L)$. $w_2(L)$ has an integral lift, the first Chern class of the line bundle, so $w_2(TM) = w_2(M)$ also has an integral lift, which proves the theorem in one direction. To go the other way, we can follow the same argument backwards, since if $w_2(TM)$ lifts to an integral class $e$, we can always find a complex line bundle with first Chern class $e$, which will be the line bundle we need for our $spin^c$-structure. $\square$

In particular, by **[M]**, this means that any orientable four manifold can be given a $Spin^c$-structure, which will be crucial to the formulation of the Seiberg-Witten equations.

## 4. Classification of $spin^c$-structures of a manifold

We will classify $spin^c$-structures by using classifying spaces, an important tool from algebraic topology. Our discussion here follows [**?**]. We start with a basic definition:

DEFINITION: A *classifying space* for a group $G$ is a CW-complex $BG$ and principal $G$-bundle $EG$ over $BG$ such that given any space $X$ and a principal $G$-bundle $E$ over $X$, there is a map $f : X \to BG$ such that $E = f^*(EG)$.

It is not hard to show that $BG$ is unique up to homotopy equivalence. From our definition and discussion of $Spin_n^c$ we have the following commutative diagram of groups, with rows and columns exact:

$$\begin{array}{ccccc}
\mathbb{Z}_2 & \subset & U(1) & \rightarrow & U(1) \\
\| & & \downarrow & & \downarrow \\
\mathbb{Z}_2 & \subset & Spin_n^c & \rightarrow & SO(n) \times U(1) \\
 & & \downarrow & & \downarrow \\
 & & SO(n) & = & SO(n)
\end{array}$$

This diagram induces a similar commutative diagram of classifying spaces (by, for example, Milgram's construction of the classifying space in **[P]**). Therefore, we can view $BSpin_n^c$ as a bundle over $BSO(n)$ with fiber $BU(1)$.

Now we view the tangent bundle of a manifold $M$ as a map $\eta : M \rightarrow BSO(n)$. A *spin*$^c$-structure on the tangent bundle is then a lift of this map to $BSpin_n^c$, giving a commutative diagram:

$$\begin{array}{ccc}
BU(1) & \rightarrow & BSpin_n^c \\
 & \nearrow & \downarrow \\
M & \stackrel{\eta}{\rightarrow} & BSO(n)
\end{array}$$

**Theorem 5.** *The set of lifts of $\eta$ is in bijective correspondence with [M,BU(1)].*

PROOF: Let $h_p$ denote the homeomorphism from $BU(1)$ to the fiber of $BSpin_n^c$ over the point $p \in BSO(n)$. Given a map $\lambda \in [M, BU(1)]$, define the lift $\eta_\lambda$ by $\eta_\lambda(x) = h_{\eta(x)} \circ \lambda(x)$. This is clearly an injective map from $[M, BU(1)]$ into the set of lifts; it is also surjective, since two different lifts will have to disagree on at least one fiber. □

Since $[M, BU(1)]$ is just the set of complex line bundles over M, which are classified by their first Chern class, the theorem implies that the set of lifts (and hence the *spin*$^c$-structures on $M$) is in correspondence with the second cohomology group $H^2(M;\mathbb{Z})$. (Alternatively, we note from **[P]** that $BU(1) = BS^1 = \mathbb{CP}^\infty$. Since $\mathbb{CP}^\infty = K(\mathbb{Z}, 2)$, the Eilenberg-Maclane space, this means $[M, BU(1)] = [M, K(\mathbb{Z}, 2)] = H^2(M;\mathbb{Z})$, by **[K1]**.) We can combine this group structure with the correspondence to define a simply transitive group action of $[M, BU(1)] = H^2(M;\mathbb{Z})$ on the set of lifts:

$$\gamma \cdot \eta_\lambda = \eta_{\gamma\cdot\lambda}$$
$$\gamma, \lambda \in H^2(M;\mathbb{Z})$$

We also want to consider our geometric criterion identifying a *spin*$^c$-structure on $M$ with a complex line bundle $L$ over $M$ and a *spin*-structure on $TM \oplus L$. The first question is whether the *spin*$^c$-structure determines the complex line bundle in this description. The answer is "Yes." From the commutative diagram of groups drawn above, we can induce the following commutative diagram:

$$\begin{array}{ccccc} & & BSpin_n^c & & \\ & \overset{\mu}{\nearrow} & & \overset{pr}{\searrow} & \\ M & & \downarrow & & B(SO(n)\times U(1)) \\ & \overset{\eta}{\searrow} & & \swarrow & \\ & & BSO(n) & & \end{array}$$

where the map $\mu : M \to BSpin_n^c$ is a lift of the map $\eta : M \to BSO(n)$, and the maps on the right-hand side of the diagram are projections induced from our commutative diagram of groups. So the lift $\mu$ of $\eta$ canonically gives us a lift $pr \circ \mu : M \to B(SO(n) \times U(1))$. This lift is the complex line bundle desired.

We can also ask the question in reverse: does the complex line bundle determine the $spin^c$-structure? Here, the answer is unsurprisingly "No." Recall from the proof of Theorem 4 in Section 3 that we must have $w_2(TM) = w_2(L) = c_1(L)$ mod 2. Hence there are strictly less than $|H^2(M;\mathbb{Z})|$ possible line bundles, so these cannot determine the $|H^2(M;\mathbb{Z})|$ $spin^c$-structures in a one-to-one fashion. The question now becomes: given a complex line bundle, how many different $spin^c$ -structures are associated with that bundle?

As a first approximation, we compute the number of *spin*-structures on $TM \oplus L$. As above, the *spin*-structures on $TM \oplus L$ correspond to lifts of a map $\eta : M \to BSO(n+2)$ to $BSpin_{n+2}$, so we have a diagram:

$$\begin{array}{ccc} B\mathbb{Z}_2 & \rightarrow & BSpin_{n+2} \\ & \nearrow & \downarrow \\ M & \xrightarrow{\eta} & BSO(n+2) \end{array}$$

Exactly as in the previous theorem, we find that the set of lifts is in bijective correspondence with $[M, B\mathbb{Z}_2]$. **[P]** proves that $B\mathbb{Z}_2 = \mathbb{RP}^\infty$. But $\mathbb{RP}^\infty$ is just the Eilenberg-Maclane space $K(\mathbb{Z}_2, 1)$, so we have $[M, B\mathbb{Z}_2] = [M, K(\mathbb{Z}_2, 1)] = H^1(M;\mathbb{Z}_2)$ (the last equality is proved in [**K1**]). Therefore, the set of *spin*-structures on $TM \oplus L$ corresponds to $H^1(M;\mathbb{Z}_2)$.

While each of these *spin*-structures pulls back to a different lift from $B(SO(n)\times U(1))$ to $BSpin_n^c$, they are not all different when considered as lifts from $BSO(n)$ to $BSpin_n^c$. We will not completely answer the question of when they are or are not different, but we will show:

**Theorem 6.** *Two lifts which differ by the action of an element in $H^1(M;\mathbb{Z}_2)$ which comes from $H^1(M;\mathbb{Z})$ give the same $spin^c$-structure, assuming the complex line bundles are the same.*

PROOF: As above, we have that $H^1(M;\mathbb{Z}) = [M, K(\mathbb{Z},1)] = [M, S^1]$. It will clearly suffice to show that a lift corresponding to an element in $H^1(M;\mathbb{Z}_2)$ which comes from $H^1(M;\mathbb{Z})$ gives the same $spin^c$- structure as the lift corresponding to the 0 element. Such a lift would factor through $S^1$ in each fiber; i.e. the image of the lift in each fiber $B\mathbb{Z}_2 = \mathbb{RP}^\infty$ lies in the canonical copy of $S^1$ embedded in $\mathbb{RP}^\infty$ as $\mathbb{RP}^1$. However, when we view $BSpin_n^c$ as a bundle over $BSO(n)$, the fiber is $BU(1) = \mathbb{CP}^\infty$, which is simply-connected. Therefore the copies of $S^1$ can all be homotoped to a point in these fibers (simultaneously, since the homotopy is the same in each fiber), which means the lift is the same as the 0-lift. □

Hence, the number of $spin^c$-structures on $M$ associated with each complex line bundle over $M$ is at most

$$|H^1(M;\mathbb{Z}_2) \mathrm{modulo those elements coming from} H^1(M;\mathbb{Z})|.$$

## 5. A Description of $Spin_n^c$ via Clifford modules

In this section I will give a much more algebraic formulation of the groups $Spin_n$ and $Spin_n^c$. This formulation will give us information about the structure of these groups which is very useful in studying vector bundles. However, before diving into a sea of algebra, I will try to give some geometrical motivation, following **[K2]**.

Recall that an element of the orthogonal group $O(n)$ can always be written as a product of reflections $\rho_i$ across hyperplanes through the origin. Each such reflection is determined by a unit normal $v_i$ to the hyperplane; note that $v_i$ and $-v_i$ determine the same reflection. So we can write an element of $O(n)$ as a "product" $[v_1 \cdot v_2 \cdot \cdots \cdot v_k]$, where each equivalence class contains a product and its negative, and $0 \leq k \leq n$. Then the double cover of $O(n)$ is just the group of signed products, which is called $Pin_n$ (a play on $SO(n)$ and $Spin_n$ which stuck). We will define the *Clifford algebra* $C\ell_n$ so that it contains $Pin_n$ in a natural way.

DEFINITION: Given a real vector space $V$ with an inner product $Q$, the *Clifford algebra* $C\ell(V,Q)$ is the quotient algebra $\mathcal{T}(\mathcal{V})/\mathcal{I}(\mathcal{V})$, where $\mathcal{T}(\mathcal{V})$ is the tensor algebra $\bigotimes V$, and $\mathcal{I}(\mathcal{V})$ is the ideal generated by elements of the form $v \otimes v - Q(v,v)$.

To increase the resemblance to our geometric motivation (and to make things easier to write) we will usually write products as $vw$ rather than $v \otimes w$. The relation given in the definition can be rewritten as $vw + wv = 2Q(v,w)$. These relations have a particularly nice form when we consider an orthonormal basis $\{e_1, \ldots, e_n\}$ for $V$, and assume that $Q$

is positive definite. Then we have that $e_ie_j = -e_je_i$ and $e_ie_i = 1$. From these, we can see that a basis for $C\ell(V,Q)$ is $\{e_I = e_{i_1} \dots e_{i_k}$ where $i_1 < i_2 < \cdots < i_k$, and $0 \leq k \leq n\}$ (when $k = 0$ we get the identity $1 = e_\emptyset$). Therefore, the dimension of $C\ell(V,Q)$ is $2^n$, where $n$ is the dimension of $V$.

$C\ell(V,Q)$ has a natural $\mathbb{Z}_2$-grading $C\ell(V,Q) = C\ell^0(V,Q) \oplus C\ell^1(V,Q)$ where the first term is generated by products of an even number of elements of $V$, and the second is generated by products of an odd number of elements of $V$. We consider the multiplicative group of units in the Clifford algebra, denoted $C\ell^\times(V,Q)$. This group has a natural representation in the Clifford algebra, called the *adjoint* representation:

$$Ad : C\ell^\times(V,Q) \longrightarrow Aut(C\ell(V,Q))$$

$$Ad(\varphi)(x) = \varphi x \varphi^{-1}$$

If $v \in V$ with $Q(v,v) \neq 0$, then $v$ is a unit ($v^{-1} = -v/Q(v,v)$), and $Ad(v)$ preserves the inner product $(Q(Ad(v)(w), Ad(v)(w)) = Q(w,w))$; so $Ad$ restricts to a representation of $P(V,Q) = \{v \in V \text{ s.t. } Q(v,v) \neq 0\}$ in $O(V,Q) = \{\lambda \in GL(V) \text{ preserving } Q\}$. Now we define:

$$Pin(V,Q) \subset P(V,Q) \text{ is the subgroup generated by } v \in V \text{ with } Q(v,v) = \pm 1$$

$$Spin(V,Q) = Pin(V,Q) \cap C\ell^0(V,Q)$$

We can show that these groups (for a real vector space) are double covers of $O(V,Q)$ and $SO(V,Q)$ respectively, so this agrees with our geometric definition of the spin groups.

We are particularly interested in the case when $V = \mathbb{R}^n$, and $Q$ is the usual positive definite inner product (dot product). Then we define $C\ell_n = C\ell(V,Q)$, $Spin_n = Spin(V,Q)$, etc. We now define the groups $Spin_n^c$ as before:

$$Spin_n^c = Spin_n \times_{\mathbb{Z}_2} U(1)$$

We associate with $C\ell_n$ a *volume element* $\omega = e_1 e_2 \cdots e_n$, where $\{e_1, \dots, e_n\}$ is an orthonormal basis for $\mathbb{R}^n$ (with a given orientation). $\omega$ is independent of the choice of this basis (in $C\ell_n$), and we have the relation:

$$\omega^2 = (-1)^{n(n+1)/2}$$

Similarly, we consider the case when $V$ is a complex vector space and define $\mathbb{C}\ell_n$ to be $C\ell_n \otimes \mathbb{C}$. Notice that $Spin_n^c \subset \mathbb{C}\ell_n$. Again, we define a volume element $\omega_{\mathbb{C}} = \imath^{[(n+1)/2]}\omega$. In this case, we find the square of the volume element is always 1.

These volume elements give us useful decompositions of vector spaces which have $C\ell_n$- representations.

Definition: A $C\ell_n - module$ is a real vector space $W$ together with a representation $\rho : C\ell_n \to Hom_{\mathbb{R}}(W,W)$. We often denote $\rho(\varphi)(w)$ by

$\varphi \cdot w$, and call this operation *Clifford multiplication.* Similarly, in the complex case we define $\mathbb{C}\ell_n$-modules.

If $W$ is a $C\ell_n$-module, and $\omega^2 = 1$, then we get a decomposition $W = W^+ \oplus W^-$ into the eigenspaces of $\omega$, so $W^\pm = (1/2)(1 \pm \omega)W$. In the complex case, the square of the volume element is always 1, so the decomposition always exists.

We say that the representation $\rho$ is *reducible* if $W$ can be written $W_1 \oplus W_2$, where $\rho(\varphi)(W_i) \subseteq W_i$ for every $\varphi \in C\ell_n$. Otherwise, we call the representation *irreducible.* We call two representations $\rho_j : C\ell_n \to Hom(W_j, W_j)$ *equivalent* if there is a linear isomorphism $F : W_1 \to W_2$ such that $F \circ \rho_1(\varphi) \circ F^{-1} = \rho_2(\varphi)$ for every $\varphi \in C\ell_n$. There is a well-understood classification of Clifford algebras (see **[L-M]**) which gives us the following fact:

**Theorem 7.** *The number of inequivalent irreducible real representations of $C\ell_n$ is 2 if $n+1 \equiv 0 \pmod 4$, and 1 otherwise. The number of inequivalent irreducible complex representations of $\mathbb{C}\ell_n$ is 2 if n is odd and 1 if n is even.*

Finally, we will introduce one more type of bundle - the *spinor bundles* of a manifold:

Definition: If the manifold $M$ has a spin structure $\xi : P_{Spin}(TM) \to P_{SO}(TM)$, a *real spinor bundle* is an associated bundle $S(M) = P_{Spin}(TM) \times_\mu W$, where $W$ is a left module for $C\ell_n$ and $\mu : Spin_n \to SO(W)$ is the representation given by Clifford multiplication by elements of $Spin_n \subset C\ell_n^0$. Similarly, we define a *complex spinor bundle*, with $W$ a complex left module for $\mathbb{C}\ell_n = C\ell_n \otimes \mathbb{C}$.

We easily generalize this definition to $spin_c$-manifolds by defining the spinor bundle $S(M) = P_{Spin^c}(TM) \times_\triangle V$, where $V$ is a complex $C\ell_n$-module, and $\triangle : Spin_n^c \to GL(V)$ is the restriction of the $C\ell_n$ representation to $Spin_n^c \subset C\ell_n \otimes \mathbb{C}$. If this representation is irreducible, we say that the spinor bundle is *fundamental.* So by the theorem above, there is one fundamental spinor bundle if n is even, and two if n is odd. However, in the odd case the two bundles are equivalent when restricted to $Spin_n^c$, so in fact there is always a unique fundamental spinor bundle, which we denote $S(M)$. Since we are in the complex case, we can use the volume element $\omega_{\mathbb{C}}$ to decompose $S(M)$ into two bundles $S^\pm(M) = (1/2)(1 \pm \omega_{\mathbb{C}})S(M)$. We will use these bundles in the next section to define the Seiberg-Witten equations.

## 6. The Seiberg-Witten equations

To define the Seiberg-Witten equations, we specialize to the case of orientable 4-manifolds, following **[T]** and **[A]**. We know, from section 3, that any orientable 4-manifold has a $spin^c$-structure. We also know, from the classification of Clifford algebras in **[L-M]**, that $\mathbb{C}\ell_4 = \mathbb{C}(4)$, the algebra of $4 \times 4$ complex matrices. The unique irreducible complex representation is the natural representation of this group on $\mathbb{C}^4$, so the fundamental spinor bundle $S(M)$ is a $\mathbb{C}^4$-bundle, which splits (as described in section 4) into two $\mathbb{C}^2$-bundles $S^{\pm}(M)$. By restricting this representation to the natural copy of $\mathbb{R}^4$ lying inside $\mathbb{C}\ell_4$, Clifford multiplication gives us a map $c$ from the cotangent bundle $T^*(M)$ into the skew-adjoint endomorphisms of $S(M) = S^+(M) \oplus S^-(M)$ (skew-adjoint because of the relation $vv = -Q(v,v)$). $c$ induces the following map by duality:

$$\sigma : S^+(M) \otimes T^*(M) \to S^-$$

$$\sigma(s \otimes v) = p_-(c(v)(s,0))$$

where $p_-$ is the projection $S(M) \to S^-(M)$.

We will construct the fundamental spinor bundles $S^{\pm}(M)$ explicitly as associated bundles to representations. First, we recall the following Lie group isomorphisms:

$$Spin_4 = SU(2) \times SU(2)$$

$$SO(4) = (SU(2) \times SU(2))/\{\pm 1\}$$

$$Spin_4^c = (SU(2) \times SU(2) \times U(1))/\{\pm 1\}$$

These give us two natural actions of $SO(4)$ on $\mathbb{R}^3$:

$$\lambda_{\pm} : SO(4) \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

$$\lambda_+ : ([p,q], x) \longmapsto Im(px)$$

$$\lambda_- : ([p,q], x) \longmapsto Im(qx)$$

where we are identifying $SU(2) = S^3$ with the unit quaternions, and $\mathbb{R}^3$ with the imaginary quaternions. The associated $\mathbb{R}^3$-bundles to these representations are isomorphic to $\Lambda_+$ (the self-dual two-forms) and $\Lambda_-$ (the anti-self-dual two-forms) respectively. We extend these actions to actions of $Spin_4^c$ on the quaternions $\mathbb{H}$:

$$s_{\pm} : Spin_4^c \times \mathbb{H} \longrightarrow \mathbb{H}$$

$$s_+ : ([p,q,\lambda], x) \longmapsto px\lambda^{-1}$$

$$s_- : ([p,q,\lambda], x) \longmapsto qx\lambda^{-1}$$

We view the associated $\mathbb{R}^4$-bundles to these actions as $\mathbb{C}^2$-bundles, and by **[A]** these are the spinor bundles $S^+(M)$ and $S^-(M)$, respectively. Then we have a pairing:

$$(,) : S^+(M) \otimes S^+(M)^* \longrightarrow \Lambda_+$$

which is the equivariant extension of the map on fibers given by:

$$(,) : x \otimes y \longmapsto Im(xiy)$$

where the bundle of imaginary quaternions is identified with $\Lambda_+$ as before.

Our penultimate step is to introduce the complex line bundle $L = det(S^+(M))$, together with a connection $A$. Together with the riemannian connection on $T^*(M)$, $A$ induces a covariant derivative $\nabla_A$ on $S^+(M)$ which maps sections of $S^+(M)$ to sections of $S^+(M) \otimes T^*(M)$. We define the Dirac operator $D_A$ as the composition of this map with $\sigma$:

$$D_A : \Gamma(S^+(M)) \to \Gamma(S^-(M))$$
$$D_A(s)(m) = \sigma(\nabla_A(s)(m))$$

We are now ready to state the Seiberg-Witten equations. The data for these equations is a pair $(A, \psi)$ where $A$ is a connection on $L$ and $\psi$ is a section of $S^+(M)$, and we let $F_A^+$ denote the self-dual part of the curvature of $A$:

$$D_A(\psi) = 0$$
$$F_A^+ = (\psi, \psi^*)$$

The Seiberg-Witten invariant is given by properly counting the solutions to these equations, as described in **[T]**. Taubes also states the fundamental theorem:

**Theorem 8.** *If $M$ is a compact, oriented, connected 4-manifold with $b_2^+ > 1$, then the Seiberg-Witten invariant $SW$ is a map from the space of spin$^c$-structures on $M$ to the integers $\mathbb{Z}$ which depends only on the underlying smooth structure of M.*

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