GRAPH COHOMOLOGY — AN OVERVIEW AND SOME COMPUTATIONS

DROR BAR-NATAN AND BRENDAN D. MCKAY

This is a DRAFT.

Abstract. This paper is not abstract.

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This paper is introductory.

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Figure 1. Multiple edges, a loop, and "turning the handle".

3. The Graph Cohomology Zoo

Let us start by fixing what we mean by the words "graph" and "graph isomorphism":

Definition 3.1. A graph G is a set $F = F_G$ of "flags" (to be thought of as vertices with half-edges emenating from them), together with a partition $V = V_G$ of F (the "vertices") and a partition $E = E_G$ of F into pairs (the "edges").

Definition 3.2. A isomorphism of graphs is a bijective map of the set of flags of one graph to the set of flags of another, that carries (in the natural sense) the set of vertices and the set of edges of the first graph to the set of vertices and the set of edges of the second graph. Similarly one may speak of "an automorphism" of a given graph.

Thus a graph may have loops and multiple edges (Figure 1). It is fully labeled (its flag set is labeled), but it is not directed. And while loops are not directed, the "turning the handle" automorphism (Figure 1) of a loop is ragarded as non-tirivial.

We will not stop to define other classical graph theoretical notions such as vartex and edges colorings, directed graphs, paths, cycles, connectivity, etc. There is no difficulty in transporing these standard notions to our context.

3.1. The Basic Example.

Definition 3.3. A "graph with an anti-symmetric set of edges", or an "ASE-graph", is a triple (s, G, O_E), where $s \in \{\pm 1\}$ is a sign, G is a graph, and O_E is an ordering of the set E_G of edges of G (a bijection between E_G and an initial segment of the natural numbers), regarded up to the following relation:

$$(s, G, O_E) \sim ((-1)^\pi s, G, \pi O_E),$$

where π is any reordering of the edges of G, and $(-1)^\pi$ denotes the signature of the permutation π. We will denote a triple $(1, G, O_E)$ simply by (G, O_E), and sometimes abuse the notation and denote it simply by G.

Note that an isomorphism of graphs allows one to identify an ordering of the edge set of one of the graphs with an ordering of the edge set of the other graph, and so there is a well defined notion of "an isomorphism between ASE-graphs".

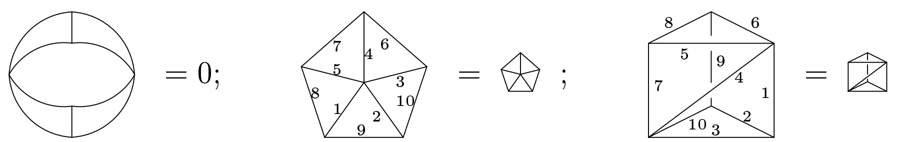
Definition 3.4. Let ${}^b\tilde{C}$ be the space of formal R-linear combinations of isomorphism classes of ASE-graphs G satisfying:

- G has no multiple edges and no loops.
- All vertices of G have valencies 3 or more.

Let ${}^bC$ be the quotient of ${}^b\tilde{C}$ by the relation $(-1, G, O_E) = -(G, O_E)$.

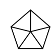
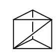
The elements of ${}^bC$ can be thought of as "unlabeled graphs with an anti-symmetric set of edges". Notice that definitions 3.3 and 3.4 imply that graphs that have an automorphism that induces an odd permutation on their set of edges vanish in ${}^bC$.

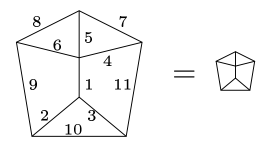
Example 3.5. . The first graph shown below vanishes in ${}^b\mathcal{C}$, because the obvious 180°-degree rotation switches five pairs of edges, and hence acts as an odd permutation on the set of edges. The other two graphs, a pentagonal wheel with spokes and a triangular prizm with one diagonal inserted (shown with an explicit edge-ordering and a small scale icon), do not vanish in ${}^b\mathcal{C}$, because all their automorphisms act on their edges by even permutation (exercise!).



Definition 3.6. Unless otherwise noted, we set the "degree" of a graph G to be $n(G) = |E_G| - |V_G|$, and the "excess" to be $k(G) = 2|E_G| - 3|V_G|$. In some of the cases below these definitions will be slightly modified.

The degree and the excess induce a decomposition ${}^b\mathcal{C} = \bigoplus_{k,n\geq 0} {}^b\mathcal{C}^k_n$, where ${}^b\mathcal{C}^k_n$ is the homogenuous excess k and degree n part of ${}^b\mathcal{C}$.

Example 3.7. The space ${}^{bc}\mathcal{C}^2_4$ is spanned by the graphs  and  of Example 3.5, while the space ${}^{bc}\mathcal{C}^1_4$ is spanned by the graph shown on the right.



Definition 3.8. Let $d : {}^b\mathcal{C} \to {}^b\mathcal{C}$ be the linear operator defined on generators (G, O_E) by

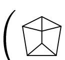
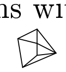
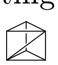
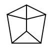
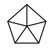
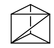
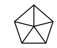
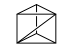
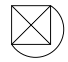
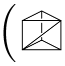
$$d(G, O_E) = \sum_{e\in E_G} \pm(G\backslash e, O_E\backslash e), \tag{1}$$

where:

- $G\backslash e$ is the contraction at e of the graph G.
- By convention, elements are removed from an anti-symmetric set only at the first position. This means that when, say, the jth element is removed from some anti-symmetric set O_E, one has to first move the jth element over the $j-1$ proceeding elements, at the cost of a sign, $(-1)^{j-1}$. Specifically, the sign left unspecified in (1) is $(-1)^{O_E(e)-1}$, where $O_E(e)$ is the serial number of e in O_E.

Strictly speaking, the image of d may lie outside of its target space ${}^b\mathcal{C}$, when an edge contraction leads to a graph that has a double edge. We simply drop such contractions from the definition of d, whenever they occur. Alternatively, we could have allowed graphs with multiple edges in the definition of ${}^b\mathcal{C}$, but then note that such graphs always have a sign-reversing automorphism (flipping two "parallel" edges), and so they vanish anyway modulo the defining relations of ${}^b\mathcal{C}$, and their inclusion does not change a thing.

There is no difficulty in showing that d is well defined, and that it maps ${}^b\mathcal{C}^k_n$ to ${}^b\mathcal{C}^{k+1}_n$.

Example 3.9. In computing $d\Big($  $\Big)$ only the contractions of edges 1, 9, and 11 (numbering as in Example 3.7) contribute; all other contractions lead to diagrams with multiple edges. Contracting edge 1, we clearly get  . Contracting edge 11, we get  , which is isomorphic to  by the isomorphism given by the edge numbering used in examples 3.5 and 3.7. Contracting edge 9 we get same same answer as for edge 11. So we find that (with the given edge orderings), $d\Big($  $\Big) =$  $+\,2$  . All edge contractions of  yield graphs with multiple edges, and so $d\Big($  $\Big) = 0$. Finally, only one edge contraction of  yields a graph with no multiple edges, the contraction of the 'far back' edge, numbered 9 in Example 3.5. But the result of that contraction is  , which is 0 in ${}^b\mathcal{C}$ because it's flip-over-the-diagonal automorphism induces an odd permutation of the edges. So $d\Big($  $\Big) = 0$ too.

Proposition 3.10. *$d^2 = 0$, and hence* $\operatorname{im} d \subset \ker d$.

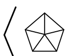
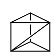
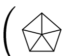
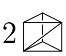
Definition 3.11. Basic Graph Cohomology is the space

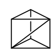
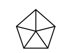
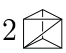
$$ {}^bH = \ker d / \operatorname{im} d. $$

Basic Graph Cohomology can be decomposed as a direct sum ${}^bH = \bigoplus_{k,n\geq 0} {}^bH_n^k$, where ${}^bH_n^k$ is the degree n and excess k graph cohomology, defined by

$$ {}^bH_n^k = \ker d|_{{}^b\mathcal{C}_n^k} / \operatorname{im} d|_{{}^b\mathcal{C}_n^{k-1}}. $$

Example 3.12. Examples 3.5, 3.7 and 3.9 imply that

$${}^{bc}H_4^2 = \Big\langle$$  ,  $$\Big\rangle \Big/ \Big($$  $+\,2$  $$\Big).$$

I.e., $\dim {}^{bc}H_4^2 = 1$, it is generated by  , and as cohomology classes,  $= -2$  .

This is the simplest example of graph cohomology. All other examples arise as various subcomplexes and/or quotient complexes of twists and/or decorations of this example.

The simplest modification one can make to the above definitions is to restrict everywhere to connected graphs, calling the resulting complex ${}^{bc}\mathcal{C}$ and its cohomology ${}^{bc}H$. Clearly, the computation of ${}^{bc}H$ is equidifficult with the computation of ${}^bH$, as the computation of ${}^bH$ can proceed in an independent manner on different connected components. Slightly more formally, one can show that ${}^bH$ is the symmetric algebra of ${}^{bc}H$, in the $\mathbb{Z}_2$-graded sense.

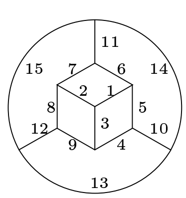
Habitat. While simplest to define, Basic Graph Cohomology does not appear in nature.

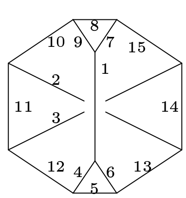
Results. At present, very little is known about ${}^{bc}H_n^k$. The only dimensions we have computed are in Table 1. The data in that table is displayed using the folowing format for each pair (n, k):

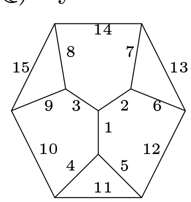
(2)

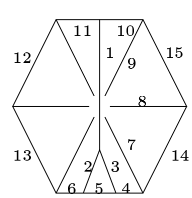
$\mathbf{\dim {}^{bc}H_n^k}$	$\dim {}^{bc}\mathcal{C}_n^k$
$\dim \ker d\|_{{}^{bc}\mathcal{C}_n^k} / \dim \operatorname{im} d\|_{{}^{bc}\mathcal{C}_n^{k-1}}$	

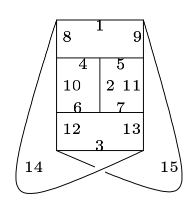
Example 3.13. ${}^{bc}H_5^0$ is generated (over $\mathbb{Q}$) by

$\frac{2}{3}$ 

$+$ 

$+\,\frac{4}{3}$ 

$+\,2$ 

$+$ 

.

Problems. ${}^bH$ is simpler than its twist H, defined below. Why is it that H is related to so many things while ${}^bH$ is related to none? What is ${}^bH$?

	$n=4$	$n=5$	$n=6$	$k=7$	$n=8$	$n=9$
$k=0$	0	**1** 7 1/0	**0** 29 0 / 0	**0** 214 0 / 0	**0** 2496 0 / 0	**1** 30307 1 / 0
$k=1$	**0** 1 0/0	**0** 13 6 / 6	**0** 109 29/29	**0** 1261 214/214	**?** 16134 ? / 2496	**?** 226296 ? / 30306
$k=2$	**1** 2 2/1	**0** 12 7 / 7	**0** 186 80/80	**1** 2926 1048/1047	?	?
$k=3$		**0** 6 5/5	**0** 170 106/106	**0** 3491 1878/1878	?	?
$k=4$		**0** 1 1/1	**1** 75 65/64	**0** 2328 1613/1613	?	?
$k=5$			**0** 10 10/10	**0** 879 716/715	**?** 38906 27533/?	?
$k=6$				**0** 179 163/163	**1** 13867 11374/11373	?
$k=7$				**0** 16 16/16	**0** 2742 2493/2493	?
$k=8$					**0** 262 249/249	?
$k=9$					**0** 14 13/13	?
$k=10$					**0** 1 1/1	?

Table 1. Dimensions of ${}^{bc}H^k_n$.

3.2. **The Fundamental Example.** We don't know of any direct use of the basic graph cohomology in other parts of mathematics. Let us now discuss the "Fundamental Example"; a certain twist of the original complex, that seems to be related to a variety of other subjects.

The Fundamental Example is simply a different choice of signs in equation (1), for which Proposition 3.10 still holds, and thus for which Definition 3.11 makes sense. There are several ways to describe the new choice of signs. We show two of them below, and leave their equivalence as an exercise.

Definition 3.14. The "oriented loop space" description: In addition to asserting that the set of edges of a graph G is anti-symmetric as in Definition 3.3, assert also that the $(|E|-|V|+1)$-dimensional vector space of closed directed cycles in G, commonly denoted $H_1(G)$ by topologists, is oriented. Here is a more complete description:

- Define an ASEC-graph (Anti-Symmetric Edges and Cycles) to be a quadruple (s, G, O_E, B), where s, G, and O_E are as before and B is a basis of $H_1(G)$, modulo the relation

$$(s, G, O_E, B) \sim ((-1)^\pi (\operatorname{sign}\det T)s, G, \pi O_E, TB).$$

  Here π and $(-1)^\pi$ are as in Definition 3.3, and T is any automorphism of $H_1(G)$. Notice that a isomorphism of graph $G \to G'$ induces an isomorphism $H_1(G) \to H_1(G')$, and so the notion of "isomorphic ASEC-graphs" makes sense.

- Define C as in Definition 3.4, only this time using ASEC-graphs and allowing multiple edges.
- Define $d : C \to C$ as in equation (1), noting that $H_1(G)$ and $H_1(G\backslash e)$ are cannonically isomorphic and thus the extra baggage B can be loaded on equation (1) at no extra cost. (Note also that while graphs with multiple edges do not necessary vanish in the new context, their contractions that have loops necessarily do vanish, and hence can be ignored).
- Finally define H and H_n^k as in Definition 3.11, but using C instead of ${}^b\!C$.

Definition 3.15. The "anti-symmetric flag and vertex set" description:

- Define an ASFV-graph to be a quadruple (s, G, O_F, O_V) with s a sign, G a graph, and O_F and O_V orderings of the flag set and the vertex set of G respectively, modulo the relation
$$(s, G, O_F, O_V) \sim ((-1)^\pi(-1)^\sigma s, G, \pi O_F, \sigma O_V).$$
Here π and σ are reorderings of the flag set and of the vertex set of G respectively.
- Isomorphisms of ASFV-graphs are easily defined, and this allows to define C as in Definition 3.4, only this time using ASFV-graphs and allowing multiple edges.
- Define $d : C \to C$ as in equation (1). This time the specification of the signs and orderings is a bit more complicated, though. The idea remains the same: when elements are added or removed from an anti-symmetric set, the operations are performed "at the start" of the set. Precisely, contracting the edge e of a graph G involves removing two flags $f_{1,2}$ and the two corresponding vertices $v_{1,2}$ (with v_i lying on f_i), and adding a new vertex v, the result of combining v_1 and v_2. In the case when $f_{1,2}$ are the first two elements of O_F and $v_{1,2}$ are the first two elements of O_V, namely when $O_F = (f_1, f_2, f_3, \dots)$ and $O_V = (v_1, v_2, v_3, \dots)$, we will set $O_F\backslash e = (f_3, \dots)$ and $O_V\backslash e = (v, v_3, \dots)$, and take the sign in equation (1) to be $+1$. By a preliminary reordering of O_F and O_V and at the cost of some signs, we can always get to the case just described. If the original placement of $f_{1,2}$ in O_F is $j_{1,2}$ and the original placement of $v_{1,2}$ in O_V is $k_{1,2}$, that sign cost is $(-1)^{j_1+j_2+k_1+k_2}\operatorname{sign}(j_1 - j_2)\operatorname{sign}(k_1 - k_2)$.
- Finally define H and H_n^k as in Definition 3.11, but using C instead of ${}^b\!C$.

Exercise 3.16. Show that Definition 3.14 and Definition 3.15 are equivalent.

One may define ${}^c\!C$ and ${}^c\!H$ by restricting everything to connected graphs. As before, H is the symmetric algebra over ${}^c\!H$ is the $\mathbb{Z}_2$-graded sense.

Habitat. H^0, also known as $(\mathcal{A}(\emptyset))^\star$, enumerates finite-type invariants of integral homology spheres [Oh, LMO, Le, BGRT1]. $(H^0)^\star$, also known as $\mathcal{A}(\emptyset)$, enumerates numerical invariants of metrized Lie algebras [B-N1, BGRT2]. It is reasonable to guess that H^1 is related to the integrability question for finite-type invariants of integral homology spheres [Hu, B-N5]. According to [Ko], H^k enumerates invariants of k-parameter families of integral homology spheres.

Results. The dimensions of H_n^0 were computed up to $n = 8$ in [B-N1], and then up to $n = 11$ in [Kn], using the relationship of H_n^0 with $\mathcal{A}$. The results are shown in Table 2. In addition, we have computed some dimensions of ${}^c\!H_n^k$ for $k \geq 0$. The results are shown in Table 3.

n	0	1	2	3	4	5	6	7	8	9	10	11
$\dim {}^c\!H^0_n$	0	1	1	1	2	2	3	4	5	6	8	9
$\dim H^0_n$	1	1	2	3	6	9	16	25	42	50	90	146

Table 2. Some dimensions from [B-N1] and from [Kn].

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$k=0$	**1** 1 1/0	**1** 2 1/0	**1** 4 1/0	**2** 14 2 / 0	**2** 54 2 / 0	**3** 298 3 / 0	**4** 2130 4 / 0
$k=1$		**0** 1 1/1	**0** 3 3/3	**0** 19 12/12	**0** 131 52/52	**0** 1162 295/295	**0** 12138 2126/2126
$k=2$			**0** 0 0/0	**0** 15 7 / 7	**0** 205 79/79	**0** 2688 867/867	**?** 36170 ?/10012
$k=3$			**0** 1 0/0	**0** 19 8 / 8	**1** 288 127/126	**1** 4316 1822/1821	?
$k=4$			**0** 1 1/1	**0** 15 11/11	**0** 250 161/161	**0** 4365 2494/2494	?
$k=5$				**0** 4 4/4	**0** 107 89/89	**0** 2646 1871/1871	?
$k=6$					**0** 20 18/18	**0** 989 775/775	**?** 35324 24836/?
$k=7$					**0** 3 2/2	**0** 267 214/214	**0** 13703 10488/10488
$k=8$					**0** 1 1/1	**0** 61 53/53	**0** 3877 3215/3215
$k=9$						**0** 8 8/8	**0** 735 662/662
$k=10$							**0** 78 73/73
$k=11$							**0** 6 5/5
$k=12$							**0** 1 1/1

Table 3. Dimensions of ${}^c\!H^k_n$, using the same format as in (2).

3.3. **Graph Cohomology for graphs with a fixed skeleton.** A *skeleton* is not-necessarily-connected graph S, with or without some extra information: vertex or edge coloring, and orientations on some or all of the edges. A graph with skeleton S is a graph G with an embedded picture of S in it — an injection of the vertices of S into the vertices of G and a choice of a path in G between the images in G of any two vertices in S that are connected by an edge, so that the resulting paths are disjoint except at their endpoints. For later convenience, we also require that the univalent vertices of S remain univalent in G. Some examples are in Figure 2. The degree of a graph with skeleton G is its degree as a plain graph minus the degree of S, and similarly, the excess of G is the excess excess it has beyond

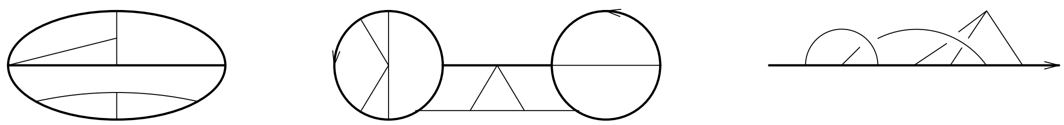


Figure 2. A graph with skeleton $\ominus$) and whose degree is 4 and excess is 1, a graph with skeleton ↺—↺ whose degree and excess are 7 and 3, and a graph with skeleton $\longrightarrow$, degree 4, and excess 0. In all cases the skeleton is emphasized with thicker lines.

the excess of its skeleton. (In other words, we simply shift the previous definitions of degree and excess so that the degree and excess of S itself both vanish).

We repeat the definition of ASEC-graphs in the current context and extend the notion of graph isomorphism to ASEC-graphs with skeleton S in the natural manner: we say that two such graphs are isomorphic if there is an ASEC-graph isomorphism between them that carries the skeleton of one onto the skeleton of the other, preserving the skeleton colorings and orientations if any are present. Given that, we make the analogs of Definitions 3.4 and 3.8 in this context:

Definition 3.17. Let S be some fixed skeleton, and let $\tilde{C}(S)$ be the space of formal linear combinations of isomorphism classes of ASEC-graphs with skeleton S that have no non-skeletal loops (loops that are not a part of the skeleton), and no vertices with valency less than 3 unless they are already in the skeleton. (The Examples in Figure 2 all satisfy these conditions). As in Definition 3.4, let $C(S)$ be the quotient of $\tilde{C}(S)$ by the relation $(-1, G, O_E) = -(G, O_E)$.

Definition 3.18. Define dG, as before, to be a sum over edge contractions signed just as in Definition 3.8, only skipping all contractions that produce a graph outside of $\tilde{C}(S)$ (for example, if a certain edge contraction change connects teo parts of the skeleton that were not connected before, it is not performed).

The newly defined map d is still a differential ($d^2 = 0$), and hence we can define $H(S)$ and $H_n^k(S)$ as before.

As the skeleton is always present, the appropriate notion of connectedness here is S-connectedness: A graph with skeleton S is S-connected if it is connected in the usual sense when S is collapsed to a single point. (Thus S itself is always S-connected). Using S-connectedness we can define ${}^{c}C(S)$ and ${}^{c}H(S)$. It is not difficult to check that $H(S)$ is the free H module generated by ${}^{c}H(S)$ in the $\mathbb{Z}_2$-graded sense.

Habitat. ${}^{c}H_n^0(S)$ enumerates finite-type invariants of embeddings of S in a ball in $\mathbb{R}^3$, so that the univalent vertices of S are at fixed positions on the boundary of the ball [St, KT]. $(H_n^0(S))^\star$ enumerates numerical invariants of Lie algebras with a representation for each edge of the skeleton and an invariant tensor in the tensor product of the representation spaces incident to each vertex of the skeleton. ${}^{c}H_n^1(S)$ is likely to be related to integrability questions [Hu, B-N5] for finite-type invariants of S, and ${}^{c}H_n^k(S)$ for general k is likely to be related to invariants of k-parameter families of embeddings of S.

3.3.1. *Paths.* One of the most interesting special cases of the above discussion of skeletons is when the skeleton S is $\uparrow_X$, the disjoint union of $|X|$ directed edges colored bijectively by some finite set of colors X.

n	0	1	2	3	4	5	6	7	8	9	10	11	12
$\dim {}^c\!H^0_n(\uparrow)$	1	1	2	3	6	10	19	33	60	104	184	316	548

Table 4. Some dimensions from [B-N1] and from [Kn]

	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$c=1$	1	2	3	6	10	19	33
$c=2$	3	9	23	60	148	366	884
$c=3$	6	28	111	413	1,461	5,027	16,924
$c=4$	10	69	394	2,035	9,849	45,680	205,612
$c=5$	15	145	1,130	7,781	49,455	297,622	1,722,724
$c=6$	21	272	2,778	24,632	198,981	1,506,218	10,875,542

Table 5. Some dimensions from [B-N3]. For $n = 7$ these dimensions were only computed over a large finite field.

Habitat. ${}^c\!H^0_n(\uparrow_X)$ enumerates finite-type invariants of X-marked pure tangles [B-N2, BGRT1]. In particular, the case where X is a singleton is the first and most studied type of finite type invariants — the case of Vassiliev invariants of long knots [B-N4]. (In this case X is ususally suppresed from the notation and $\uparrow_X$ is simply denoted $\uparrow$. $({}^c\!H^0_n(\uparrow_X))^\star$, also called $\mathcal{A}(\uparrow_X)$, is a combinatorial model of the ad-invariant elements of tensor powers of universal enveloping algebras of metrized Lie algebras. In particular, $\mathcal{A}(\uparrow)$ is a combinatorial model of the center of universal enveloping algebras of metrized Lie algebras, and as such it has quite a lot of sturcture. See e.g. [B-N1].

Results. The dimensions of ${}^c\!H^0_n(\uparrow)$ were computed up to $n = 9$ in [B-N1], and then up to $n = 12$ in [Kn], using the relationship of ${}^c\!H^0_n(\uparrow)$ with $\mathcal{A}(\uparrow)$. The results are reproduced in Table 4. For $c = |X| > 1$, some dimensions were computed in [B-N3]. The results are reproduced in Table 5. It turns out that these numbers depend polynomially on c. These polynomials are determined by the numbers in Table 5, and are printed (to the extent that they are known) in [B-N3].

3.3.2. *Cycles.* Strictly speaking, an oriented circle with no base point is not a graph (it is a "closed edge" with no vertices), and hence not a skeleton falling under the definitions of Section 3.3. But there is no difficulty in extending the definitions there to this special case, and thus in defining $H^k_n(\circlearrowleft_X)$, the graph cohomology spaces for graphs with "skeleton" a disjoint union of circles colored bijectively by the colors in some finite set X.

Habitat. ${}^c\!H^0_n(\circlearrowleft_X)$ enumerates finite-type invariants of X-marked links. The case where X is a singleton is equivalent to the case of ${}^c\!H^0_n(\uparrow)$, as long knots are equivalent to 1-component links, that is, to knots. $({}^c\!H^0_n(\circlearrowleft_X))^\star$, also called $\mathcal{A}(\circlearrowleft_X)$, is a combinatorial model of the ad-invariant elements of tensor powers of the coinvariant quotients of universal enveloping algebras of metrized Lie algebras. As in the equivalent case of $\mathcal{A}(\uparrow)$, much is known about $\mathcal{A}(\circlearrowleft)$.

Results. For X a singleton and $k = 0$, the results are the same as in Table 4. Other than that, very little is known.

Problems. For X a singleton and $k > 0$, what is the relationship between ${}^c\!H_n^k(\circlearrowleft_X)$ and ${}^c\!H_n^k(\uparrow_X)$? We feel that they must differ by something easily computable.

3.4. **Univalent vertices.** A rather simple modification to the definition of C, or, equally well, to the definition of $C(S)$, is to consider graphs G that in addition to the previous features also have some fixed number u of univalent vertices, or some fixed numbers $u_1, u_2, \dots$ of colored univalent vertices, colored by distinct colors $c_1, c_2, \dots$ (u_1 of color c_1, u_2 of c_2, …). The new univalent vertices are never allowed to lie on the skeleton, if a skeleton is present. The differential d is modified only so as to preserve the number of univalent vertices; it is defined by the same summation as in (1), only that the edges that connect a univalent vertex to the rest of the graph do not participate in the summation. We denote the resulting graph complex by $C(S; *_{c_1}^{u_1} *_{c_2}^{u_2} \dots)$. If there's no skeleton, we omit it from the notation. If there's only one color, we omit it from the notation and simply write $C(S; *^u)$. When we omit some or all of the u_i's from the notation, it means that we are not constraining the number of univalent vertices of some or all of the colors. In other words, $C(*) = \oplus_{u=0}^{\infty} C(*^u)$. We can now define $H_n^k(S; *_{c_1}^{u_1} *_{c_2}^{u_2} \dots)$ in the usual way.

One can come up with several reasonable notions of connectivity for graphs with univalent vertices. Let us the disscuss the two notions that arise in applications:

- We can use the usual notion of connectivity for graphs with skeleton, as in Section 3.3, and call the resulting Graph Cohomology ${}^c\!H(S; *_{c_1}^{u_1} *_{c_2}^{u_2} \dots)$. We find that $H(S; *_{c_1} *_{c_2} \dots)$ is the free $H(*_{c_1} *_{c_2} \dots)$ module generated by ${}^c\!H(S; *_{c_1} *_{c_2} \dots)$ in the $\mathbb{Z}_2$-graded sense, and that $H(*_{c_1} *_{c_2} \dots)$ is the free $\mathbb{Z}_2$-graded generated by ${}^c\!H(*_{c_1} *_{c_2} \dots)$.
- We say that a graph G is *weakly connected* if it becomes connected when all the univalent vertices in it, as well as the skeleton if a skeleton is present, are collapsed to a single newly-created vertex ∞. Equivalently, if every connected component of G (in the ususal sense) contains at least one univalent vertex or at least one component of the skeleton. We denote the resulting Graph Cohomology by ${}^w\!H(S; *_{c_1}^{u_1} *_{c_2}^{u_2} \dots)$. Clearly, $H(S; *_{c_1}^{u_1} *_{c_2}^{u_2} \dots)$ is the free H module generated by ${}^w\!H(S; *_{c_1}^{u_1} *_{c_2}^{u_2} \dots)$ in the $\mathbb{Z}_2$-graded sense.

Habitat. As is often the case, only the excess 0 case has a natural habitat in mathematics, at least in as much as we know now. $({}^w\!H^0(*_{c_1} *_{c_2} \dots))^\star$ is the space $\mathcal{B}$ of uni-trivalent graphs with colored legs that appears in [B-N1, B-N2]. Given a metrized Lie algebra $\mathfrak{g}$, $({}^c\!H^0(*_{c_1} *_{c_2} \dots))^\star$ is related to the space of invariant elements of $(\operatorname{sym}\mathfrak{g}) \otimes (\operatorname{sym}\mathfrak{g}) \otimes \dots$ and to the space of functions on $\mathfrak{g} \oplus \mathfrak{g} \oplus \dots$ [BGRT1, BGRT2].

Results. The electronic publication [B-N3] contains the dimensions of many spaces ${}^c\!H(*_{c_1}^{u_1} *_{c_2}^{u_2} \dots)$, and in the case of a single color c, a few more dimensions are in [Kn]. In Table 6 we reproduce some of the single color results.

3.5. **Trees.**

Habitat.

Results.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\dim {}^c\!H^0_n(*^2)$	1	1	1	1	2	2	3	4	5	6	8	9
$\dim {}^c\!H^0_n(*^4)$				1	1	2	3	4	6	8	10	13
$\dim {}^c\!H^0_n(*^6)$						1	2	3	5	8	11	15
$\dim {}^c\!H^0_n(*^8)$								1	2	4	8	12
$\dim {}^c\!H^0_n(*^{10})$										1	2	5
$\dim {}^c\!H^0_n(*^{12})$												1

Table 6. Some dimensions from [B-N3] and from [Kn]

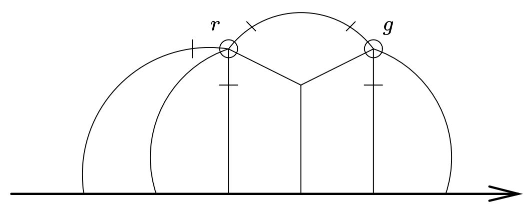


Figure 3. A $(\circledast^3_r \circledast^2_g)$-graph with skeleton $\longrightarrow$, degree 8 and execess 4. We mark the distinguished vertices by surrounding them with small circles, and the special edges emenating from them by crossing them with short tags.

3.6. **Link Relations.** The following variation is somewhat artificial. The only justification for its inclusion here is that its excess 0 case appears in nature as the "link relation" of [BGRT2, Me]. The idea is that we want to allow univalent vertices, like in Section 3.4, but this time they participate in the game in a more active way — we allow to contract an edge that leads to a univalent vertex, but some provisions apply. It is easier to describe everything in a precise way by introducing a distintuished vertex with special properties, and by attaching all the univalent vertices to it. If there's more than one color of univalent vertices, we will similarly introduce several colored distintuished vertices, one for each color of the univalent vertices.

Definition 3.19. A $(\circledast^{u_1}_{c_1} \circledast^{u_2}_{c_2} \dots)$-graph is a graph with distinguished vertices colored c_1, c_2, $\dots$, together with a marking of precisely u_i of the edges emenating from the c_i-distinguished vertex as "special", for each i. (In particular, the valency of the c_i-distinguished vertex must be $\geq u_i$). When additional structure is present (a skeleton, Section 3.4-style univalent vertices), we require that it is disjoint from the currently distinguished vertices. We declare the local degree at a distinguished vertex to be its valency, and the local excess at a distinguished vertex to be the number of unmarked edges emenating from it. An example is in Figure 3.

Habitat.

Results.

3.7. **Directed Edges.**

Habitat.

Results.

3.8. Terminus Free Graphs.

Habitat.

Results.

3.9. Acrobats.

3.9.1. *Acrobat Towers.*

Habitat.

Results.

3.9.2. *Acrobat Jungles.*

Habitat.

Results. **Acknowledgement:**

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Institute of Mathematics, The Hebrew University, Giv'at-Ram, Jerusalem 91904, Israel
E-mail address: drorbn@math.huji.ac.il

Computer Science Department, Australian National University, Canberra ACT 0200, Australia
E-mail address: bdm@cs.anu.edu.au