Why is the Hall Conductance Quantized? Solution of an open math-phys problem.

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Abstract

In the last decade the theory of the Quantum Hall effect for interacting electrons in the thermodynamic limit was put on firm mathematical foundations. The solution rests on advances made by Hastings and Michalakis[2, 10], Giuliani, Mastropietro and Porta, [8], Bachmann, De Roeck and Fraas [4] and others [7, 6, 9, 10]. Here is a brief outline of the main results and of one emergent insight.

1 Introduction

Almost 20 years ago M. Aizenman initiated a page of "Open problems in mathematical physics" on the IAMP web site [1]. In 1999, Ruedi Seiler and myself advertised on this page the problem to put the theory of the Integer Quantum Hall effect for interacting electrons in the thermodynamic limit on solid mathematical foundations.

The problem was dormant for about 10 years. In 2009 an ArXiv article of Matt Hastings and Spyridon Michalakis, [1], described a central pillar of the solution. The paper, which also introduced new methods and techniques, is not quite bedtime reading. Six years passed before it got published [2].

Alessandro Giuliani, Vieri Mastropietro and Marcello Porta [8] derived closely related results using the methods of multi-scale analysis and Ward identities.

Both Hastings *et al.* and Giuliani *et al.* took Kubo formula for granted. Putting Kubo's formula on firm mathematical foundation is one of B. Simon 15 problems [3]. Sven Bachmann, Wojciech de Roeck and Martin Fraas accomplished this for gapped systems in [4].

This note attempts to briefly review these results and describe one non-technical emergent insight.

[1] The page that does not exists anymore on the IAMP site.

2 Background

Before describing the new results it is worthwhile to recall what was known about the quantum Hall effect when the problem was posed in 1999.

- In 1983 Thouless *et al.* showed that Kubo formula for the Hall conductance of non-interacting Fermion in a periodic potential can be identified with the Chern number of a line bundle over the Brillouin zone. David Thouless received the 2016 Nobel prize in Physics for this discovery.

- In 1985 Niu, Thouless and Wu and independently Avron and Seiler, described a theory of adiabatic quantum transport for general interacting Hamiltonians in an external gauge field A, of finite systems. The two fundamental periods of the gauge field: (ϕ_1, ϕ_2), are identified with two Aharonov-Bohm flux tubes that thread the physical system. The charge transport due to an increase of one of the fluxes by period is given by the corresponding period of the adiabatic curvature. The Chern number associated with the bundle of ground states over (ϕ_1, ϕ_2) is the period of the adiabatic charge transport. It was believed that for large systems the curvature and charge transport are both flux independent, but a proof was missing.

- In 1988 Jean Bellissard extended the results of Thouless *et al.* to non-interacting Fermions in random and quasi-periodic potentials for infinite systems. Kubo's formula was identified with a suitable Fredholm index related to Chern numbers in Non-commutative geometry. In 1990, Avron, Seiler and Simon reformulated Bellissard finding in terms of the relative index of projections.

In the works of Thouless *et al.* and Bellissard *et al.* the Hall conductances of infinite two dimensional systems was identified with a topological invariant. However, these results only applied to non-interacting Fermions. The results of Thouless, Niu and Wu and Avron, Seiler are complementary in that they apply for general interacting Hamiltonians and do not rely on Kubo's formula, but the methods used only apply to finite systems and can not handle large systems in the thermodynamic limit.

3 Hastings and Michalakis: Quantized adiabatic curvature

Omitting technicalities, and stated somewhat cavalierly, Hastings and Michalakis proved [2]

Theorem 3.1 *Let $H(A,\ell)$ be a many body Hamiltonian acting on the Fock space associated with $\mathbb{Z}_\ell^2$, with charge conserving short range interactions and gauge field A. Denote by (ϕ_1,ϕ_2) the "twists" associated with the periods of A for the two fundamental loops (γ_1,γ_2) of the torus, (see Fig. 1). Let P denote the projection of the (non-degenerate) ground state separated by a gap. Then the adiabatic curvature ω over (ϕ_1,ϕ_2)*

$$\omega = i\,Tr\,\Omega, \quad \Omega = P\,dP\wedge dP\,P$$

is quantized in multiples of $(2\pi)^{-1}\,d\phi_1\wedge d\phi_2$, up to an error that decays faster than any power of ℓ.

The proof rests on two new tools introduced into the theory of the Quantum Hall Effect : A generator of transport in Range P with good localization properties, and the Lieb-Robinson bound. The first tool was invented in 2004 by Hastings who also pioneered the second tool in other contexts.

Bachmann *et al.* [6] gave a short proof of the constancy of the curvature under the simplifying assumption of a gap for *all* fluxes. Below I shall attempt to give a brief and informal account of the key underlying non-technical insight behind both [2] and [6]. In the following I shall refer to ω as the scalar curvature and to Ω as the operator valued curvature.

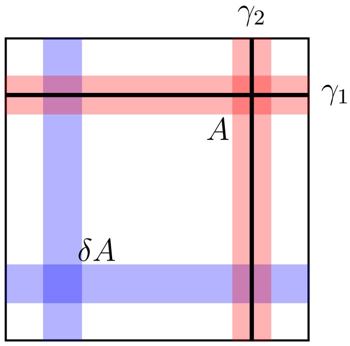

Figure 1: The square represents the torus in coordinate space. The fundamental periods γ_j are the black lines. The red stripes describe A near the fundamental period and determine the fluxes, aka twists, ϕ_j. The variation δA is supported on the blue stripes. The operator valued curvature Ω is localized near the intersection of the blue stripes which is far from the fundamental periods γ_j.

Using the language of continuous gauge fields, the fluxes, or "twists", are given by

$$\phi_j = \oint_{\gamma_j} A \tag{1}$$

For a given magnetic field dA, the scalar curvature ω may then be viewed as a function of ϕ_j. This follows from gauge covariance of the projection P. The gap condition implies that $\omega(\phi, \ell)$ is a smooth 2-form on the torus and the associated Chern number is a well defined integer. The remarkable thing that we want to explain is that the (scalar) curvature $\omega(\phi, \ell)$ approaches a constant 2-form, independent of ϕ, when the systems gets large. Since the total curvature is quantized, the quantization of the scalar curvature, promised by the theorem, follows.

Unlike ω, the operator valued curvature Ω is gauge dependent and can not be viewed as a function of the fluxes. It is a function of the gauge field A and its variation δA. The variation enters because dP involved the variation of the flux through $\delta\phi_j = \oint_{\gamma_j} \delta A$. The variation is constrained by $d\delta A = 0$, so the magnetic field acting on the system remains the same, and only the fluxes are varied. This is reflected in Fig. 1 where the variation is not localized to the intersection of γ_j with the blue stripes, but rather spreads to the entire blue stripes.

The variations dP, is related to the variation dH, which is localized near δA. Since Ω is the wedge product of the variations dP, it is localized near the intersection of the variations δA (provided P and dP are localized).

Since there is gauge freedom in choosing δA it now remains to make a good choice. The two blue stripes in Fig. 1 represent a choice which localizes Ω near the blue square where the blue stripes intersect. Ω is now localized far from the periods γ_j that determine the fluxes ϕ_j. It follows that the local part of Ω is essentially independent of A near the periods γ_j. The value of the fluxes does not matter.

This argument uses the localization properties of P and dP. Note that one may replace dP in the definition of Ω by the commutator equation for dK:

$$dP = i[dK, P]$$

which allows to write the curvature as

$$\Omega = -P\, dK \wedge dK\, P$$

Hastings gave an explicit expression for dK whose localization properties mimic those of dH.

Lieb-Robinson's bounds are used to rigorously control the commutators in Ω and show that Ω inherits the localization properties of dK.

Building on earlier works with Bravyi and Michalakis, Hastings recently proved the existences of a gap for certain weakly interacting systems [10], putting the icing on the cake.

3.1 Ward identities and cluster expansion

Using very different methods, Giuliani, Mastropietro and Porta [8] proved the following theorem, glibly stated:

Theorem 3.2 *Kubo formula for the Hall conductance for a gapped non-interacting translation invariant Fermionic systems equals the conductance of the corresponding weakly interacting systems in the thermodynamic limit.*

This show the universality of the quantum Hall conductivity in the sense that it is independent of the interaction strength (for weak interactions) and provides the value of the Hall coefficient.

The main ingredients of the proof are:

- Construction and proof of analyticity of the ground state Euclidean correlations, uniformly in the system size, via *fermionic cluster expansion* methods.
- Proof of the Wick rotation for the ground state Kubo conductivity (i.e., the Kubo conductivity is equal to its Euclidean counterpart).
- Proof that all the terms in the perturbation series in powers of the interaction series for the Euclidean Kubo conductivity, vanish identically. This follows from a combination of Ward Identities with Schwinger-Dyson equations.

Giuliani et. al. [9] also proved the quantization and universality of the Hall coefficient in the gapless case of the the weakly interacting Haldane model.

4 Kubo's formula: Bachmann, den Roeck and Fraas

In 1984 B. Simon formulated a list of 15 problems in Mathematical Physics. Problem 4 concerns transport theory. This is what he says:

"There are [also] serious foundation questions in quantum transport. A basic formula in condensed matter physics is the Kubo formula for conduction .... Not only are the usual derivation suspect but van Kempen among others has seriously questioned its validity on Physical grounds.

Probelm 4B: Either justify Kubo's formula is a quantum model or else find an alternate theory of conductivity."

Linear response theory involves taking limits: The thermodynamic limit of a large system, the linear response limit of weak perturbation and also the limit of adiabatic switching of the perturbation. Proving Kubo's formula requires controlling the limits and taking them in the correct order where the thermodynamic limit is taken first.

Bachmann *et al.* [4, 5] proved the validity of Kubo formula for for gapped, translation invariant, interacting spin systems with short range interactions. In particular, they proved the commutativity of the thermodynamic and linear response limits. They rely on the tools introduced by Hastings: The Lieb-Robinson bounds and the generator of evolution dK of the previous section.

Bachmann *et al.* [4, 5] also adapted adiabatic theory to the setting of macroscopic systems. The usual, Schrödinger picture of following the quantum state does not work for thermodynamic systems because tiny local errors accumulate as the system gets large and make the approximate state a poor approximant. The way out is to focus instead on the Heisenberg picture of adiabatic evolution of *local* observable. The adiabatic framework treats rigorously the "adiabatic switching" which is central in linear response and prove Kubo's formula for static perturbations.

In subsequent works Teufel *et al.* [7] and [6] extended the results in [4, 5] to adiabatic transport in the Quantum Hall Effect . This involves, among other things, extending the theory from spins to Fermions, and to currents generated by time dependent gauge fields. The strategy of [4, 5] works in the Quantum Hall Effect for, as we have seen, the curvature is associated with a local observable, lying in the intersection of the blue stripes.

The results of [4, 5, 6] and [7] complement those of Hastings *et al.* and Giuliani *et al.* who took Kubo formula for granted.

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References

[1] M.B. Hastings and S. Michalakis. . *Quantization of Hall conductance for interacting electrons on a torus without averaging assumpions* , arXiv:0911.4706 [math-ph] (2009)

[2] M.B. Hastings and S. Michalakis. *Quantization of Hall conductance for interacting electrons on a torus.* Commun. Math. Phys., 334:433–471, (2015).

[3] Simon, Barry. *Fifteen problems in mathematical physics.* Perspectives in mathematics, Birkhäuser, Basel 423 (1984).

[4] S. Bachmann, W. De Roeck, and M. Fraas. *The adiabatic theorem for many-body quantum systems.* arXiv preprint arXiv:1612.01505, 2016.

[5] S. Bachmann, W. De Roeck, and M. Fraas. *The adiabatic theorem and linear response theory for extended quantum systems.* arXiv preprint arXiv:1705.02838, 2017.

[6] Bachmann, Sven and Bols, Alex and De Roeck, Wojciech and Fraas, Martin *Quantization of conductance in gapped interacting systems*, arXiv preprint arXiv:1707.06491, 2017

[7] D. Monaco and S. Teufel. *Adiabatic currents for interacting electrons on a lattice.* arXiv preprint arXiv:1707.01852, 2017.

[8] A. Giuliani, V. Mastropietro, M. Porta: Universality of the Hall Conductivity in Interacting Electron Systems, Comm. Math. Phys. 349, 1107 (2017)

[9] A. Giuliani, I. Jauslin, V. Mastropietro, M. Porta: Topological phase transitions and universality in the Haldane-Hubbard model, Phys. Rev. B 94, 205139 (2016)

[10] M.B. Hastings. *The stability of free Fermi Hamiltonians.* arXiv preprint arXiv:1706.02270v2, 2017