Univalent polymorphism

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Based on:

1. BvdB and Ieke Moerdijk. Exact completion of path categories and algebraic set theory – Part 1: Exact completion of path categories. *Journal of Pure and Applied Algebra*, Volume 222, Issue 10, October 2018, Pages 3137–3181.
2. BvdB. Path categories and propositional identity types. To appear in *ACM Transactions on Computational Logic (TOCL)*. arXiv:1604.06001, 2016.
3. BvdB. Univalent polymorphism. arXiv:1803.10113, 2018.

Overview

Section 1

Path categories

Path category

A *path category* is a category $\mathcal{C}$ equipped with two classes of maps, called *fibrations* and *equivalences*, respectively. A fibration which is also an equivalence will be called *trivial* (or *acyclic*). If $X \to PX \to X \times X$ is a factorisation of the diagonal on X as an equivalence followed by a fibration, then PX is a *path object* for X.

Axioms

1. Isomorphisms are fibrations and fibrations are closed under composition.
2. $\mathcal{C}$ has a terminal object 1 and $X \to 1$ is always a fibration.
3. The pullback of a (trivial) fibration along any other map exists and is again a (trivial) fibration.
4. Isomorphisms are equivalences and equivalences satisfy 6-for-2.
5. Every object X has at least one path object.
6. Trivial fibrations have sections.

Examples

- Fibrant objects in every model category in which every object is cofibrant (for example, simplicial sets)
- Cubical sets *à la* BCH or CCHM with path types
- The syntactic category associated to Martin-Löf's type theory with intensional identity types

In fact, if we formulate Martin-Löf's type theory with "propositional identity types" (meaning that the computation rule is formulated as a propositional equality), then the syntactic category associated to type theory is still a path category.

One motivation: type theory without definitional equality, and all computation rules as propositional equalities.

Slicing for path categories

Factorisation (Brown)

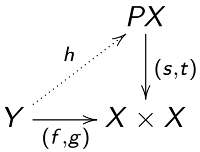
In a path category every map factors as an equivalence followed by a fibration.

Slicing for path categories (Brown)

Let $\mathcal{C}$ be a path category and X be an object in $\mathcal{C}$. Write $\mathcal{C}(X)$ for the full subcategory of $\mathcal{C}/X$ whose objects are fibrations. With the equivalences and fibrations as in $\mathcal{C}$, this is again a path category.

Homotopy in a path category

If $f, g : Y \to X$ are two parallel maps, then we say that f and g are *homotopic* and write $f \simeq g$ if there is a map $h : Y \to PX$ making



commute.

Theorem

The homotopy relation $\simeq$ is a congruence on $\mathcal{C}$.

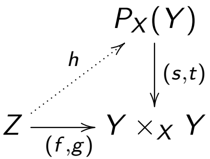
The quotient is the *homotopy category* of $\mathcal{C}$. A map which becomes an isomorphism in the homotopy category is called a *homotopy equivalence*.

Theorem

The equivalences and homotopy equivalences coincide in a path category.

Fibrewise homotopy in a path category

We also need a notion of fibrewise homotopy. Suppose $p : Y \to X$ is a fibration and $f, g : Z \to Y$ are two maps with $pf = pg$. Note that p is an object in $\mathcal{C}(X)$ and therefore there is an object $P_X(Y)$. We will say that f and g are *fibrewise homotopic* and write $h : f \simeq_X g$ if there is a map $h : Z \to P_X(Y)$ making



commute.

Fact

Every object X in a path category carries an ∞-groupoid structure up to homotopy.

Section 2

Homotopy type theory in path categories

Transport

Transport

Suppose $p : Y \to X$ is a fibration and consider the following pullback:

$$\begin{array}{ccc} Y \times_X PX & \xrightarrow{p_2} & PX \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle s} \\ Y & \xrightarrow[p]{} & X. \end{array}$$

A *transport structure* on p is a map $\Gamma : Y \times_X PX \to Y$ such that

1. $p\Gamma = tp_2$
2. $\Gamma(1, rp) \simeq_X 1_Y : Y \to Y$.

Theorem

Every fibration $p : Y \to X$ carries a transport structure; these transport structures are unique up to fibrewise homotopy over X.

Univalence

Proposition

Suppose $p : Y \to X$ is a fibration and $f, g : Z \to X$ are homotopic maps via a homotopy H. Then the homotopy H and the transport structure on p induce a map $f^*p \to g^*p$ over Z and this map is an equivalence.

Definition

A fibration $p : Y \to X$ is *univalent* if for any pair of maps $f, g : Z \to X$ and any equivalence $w : f^*p \to g^*p$ there is a homotopy H between f and g such that w is, up to fibrewise homotopy over Z, the equivalence induced by H, as in the previous proposition.

Homotopy n-types

Definition

The *fibrations of n-types* ($n \geq -2$) are defined inductively as follows:

- A fibration $f : Y \to X$ is a *fibration of* (-2)*-types* if f is trivial.
- A fibration $f : Y \to X$ is a *fibration of* $(n+1)$*-types* if $P_X(Y) \to Y \times_X Y$ is a fibration of n-types.

(-2)-types	contractible
(-1)-types	propositions
0-types	sets
1-types	groupoids

Homotopy exponentials

Definition

If X and Y are objects in a path category, then we will X^Y the *homotopy exponential* of X and Y if it comes equipped with a map $\mathrm{ev} : X^Y \times Y \to X$ such that for any map $h : A \times Y \to X$ there is a map $H : A \to X^Y$ such that $\mathrm{ev}(H \times 1_Y) \simeq h$, with H being unique up to homotopy with this property.

- Computation rule only in propositional form
- Function extensionality

There is a similar definition of "homotopy Π-types".

Theorem

If $\mathcal{C}$ is a path category with homotopy Π-types, then $\Pi_f : \mathcal{C}(Y) \to \mathcal{C}(X)$ preserves fibrations of n-types for *any* fibration $f : Y \to X$.

Proof.

This is Theorem 7.1.9 in the HoTT book. □

Section 3

The path category $\mathbb{EFF}$

The category $\mathbb{EFF}$

Objects of the category $\mathbb{EFF}$ consist of:

1. A set A.
2. A function $\alpha : A \to \mathbb{N}$ (sending an element $a \in A$ to its *realizer*).
3. For each pair of elements $a, a' \in A$ a subset $\mathcal{A}(a, a')$ of $\mathbb{N}$.
4. A function which computes for a realizer of $a \in A$ an element in $\mathcal{A}(a, a)$.
5. A function which given realizers for a, a' and $\pi \in \mathcal{A}(a, a')$ computes an element $\pi^{-1} \in \mathcal{A}(a', a)$.
6. A function which given realizers for a, a', a'' and $\pi \in \mathcal{A}(a, a'), \pi' \in \mathcal{A}(a', a'')$ computes an element $\pi' \circ \pi \in \mathcal{A}(a, a'')$.

A morphism $f : (B, \beta, \mathcal{B}) \to (A, \alpha, \mathcal{A})$ in $\mathbb{EFF}$ consists of:

1. A function $f : B \to A$ such that a realizer of $f(b)$ can be computed from a realizer of b.
2. For each $b, b' \in B$ a function $f_{(b,b')} : \mathcal{B}(b, b') \to \mathcal{A}(fb, fb')$ such that $f_{(b,b')}(\pi)$ can be computed from realizers for b, b' and π.

Homotopy in $\mathbb{EFF}$

Homotopy in $\mathbb{EFF}$

Two parallel maps $f, g : (B, \beta, \mathcal{B}) \to (A, \alpha, \mathcal{A})$ are *homotopic* if there is a function computing for every realizer for $b \in B$ an element in $\mathcal{A}(fb, gb)$ (this is called a *homotopy*). A map $f : (B, \beta, \mathcal{B}) \to (A, \alpha, \mathcal{A})$ is a *(homotopy) equivalence* if there is a morphism g in the other direction (the homotopy inverse) such that both composites fg and gf are homotopic to the identity.

Fibrations in $\mathbb{EFF}$

A map $f : (B, \beta, \mathcal{B}) \to (A, \alpha, \mathcal{A})$ in $\mathbb{EFF}$ is a *fibration* if:

1. for any b, a and $\pi : f(b) \to a$ one can effectively find $b', \rho : b \to b'$ such that $f(b') = a$ and $f(\rho) = \pi$ (meaning that there is a function picking such which is also tracked).
2. for any $b, b' \in B$, $\rho \in \mathcal{B}(b, b')$ and $\pi \in \mathcal{A}(fb, fb')$ one can compute $\rho' \in \mathcal{B}(b, b')$ with $f(\rho') = \pi$.

$\mathbb{EFF}$ as a path category

Theorem

The category $\mathbb{EFF}$ with the fibrations and equivalences as defined on the previous page is a path category with homotopy Π-types.

Proposition (AC)

The homotopy category of $\mathbb{EFF}$ is Hyland's effective topos.

This should be compared with: Rosolini's paper "The category of equilogical spaces and the effective topos as homotopical quotients".

Section 4

Discrete fibrations

Small fibrations

Let us suppose $\mathcal{S}$ is a subclass of the class of fibrations which is stable under homotopy pullbacks, and let us refer to the elements of $\mathcal{S}$ as the "small fibrations".

- Let us call such a class of small fibrations *impredicative* or *polymorphic* if it is closed under Π_f for *any* fibration f.
- A *representation* for such a class is an element $\pi : E \to U$ such that any other small fibration $f : B \to A$ can be obtained as a homotopy pullback of that one via some map $A \to U$.

Impredicative and representable classes of small fibrations are needed to obtain models of the Calculus of Constructions.

We will be especially interested in class of small maps with a univalent representation. In that case the classifying map $A \to U$ is unique up to homotopy.

Discrete fibrations

Definition

A map $f : (B, \beta, \mathcal{B}) \to (A, \alpha, \mathcal{A})$ is a *canonically discrete fibration* if distinct elements $b, b' \in B_a$ have distinct realizers; a *discrete fibration* is a fibration which is homotopy equivalent to one which is a canonically discrete.

Note that we demand: elements with identical realizers are *identical* (not homotopic!).

Proposition

Discrete fibrations in $\mathbb{EFF}$ are closed under homotopy pullback and stable under Π_f for *arbitrary* fibrations f.

Compare: "Discrete objects in the effective topos" by Hyland, Robinson, Rosolini.

Univalent polymorphism

Proposition

The class of discrete fibrations of sets is impredicative class of small fibrations, which, however, does not have a univalent representation.

In fact, it is unclear whether the discrete fibrations have any representation at all.

Proposition

The class of discrete propositional fibrations is impredicative and has a univalent representation.

In fact, the univalent representation is essentially the subobject classifier of the effective topos.

This means: $\mathbb{EFF}$ is roughly a model of CoC with a univalent Prop (hence: univalent polymorphism).

Propositions in $\mathbb{EFF}$

In fact, $\mathbb{EFF}$ also satisfies propositional truncation, mirroring the epi-mono factorisation of the effective topos.

Proposition (AC)

Every propositional fibration is discrete ("propositional resizing").

Compare: Taichi Uemura, *Cubical assemblies and the independence of propositional resizing*. arXiv:1803.06649.

Proposition

In $\mathbb{EFF}$ Church's Thesis (formulated with $\exists$) holds.

Section 5

Open questions and directions for future research

All the way up

Not so good:

Proposition

In $\mathbb{EFF}$ every fibration is a fibration of sets.

My paper also contains a more complicated version $\mathbb{EFF}_1$ in which every fibration is a fibration of groupoids and in which the discrete fibrations of sets are an impredicative class of fibrations with a univalent representation.

Clearly, we want a version without any restriction on hlevels, but that's not so easy!

Open questions

1. Cubical partitioned assemblies?
2. Eliminating AC from the proofs
3. Write down a syntax for propositional version of CoC
4. Discrete reflection a modality?
5. Is it possible to have a model of HoTT (cubical type theory) with lccc Π in which Church's Thesis (with $\exists$) holds?

THANK YOU!