Exercise 11

We just have to show that $(\rho \oplus \rho')$ is associative using that ρ and ρ' are associative.

Let $g_1, g_2 \in G$. Then for all $v \in V$, $v' \in V'$,

$$\begin{aligned}
(\rho \oplus \rho')(g_1 g_2)(v, v') &= (\rho(g_1 g_2)v, \rho'(g_1 g_2)v') \\
&= (\rho(g_1)v\rho(g_2)v, \rho'(g_1)v'\rho'(g_2)v') \\
&= (\rho(g_1)v, \rho'(g_1)v')(\rho(g_2)v, \rho'(g_2)v') \\
&= (\rho \oplus \rho')(g_1)(v, v')(\rho \oplus \rho')(g_2)(v, v')
\end{aligned}$$

so

$$(\rho \oplus \rho')(g_1 g_2) = (\rho \oplus \rho')(g_1)(\rho \oplus \rho')(g_2)$$

Exercise 12

Pick a basis $\{e_i\}$ for V and $\{e'_j\}$ for V' as in the text. Then define F from $V \otimes V'$ to W by requiring that

$$F(e_i \otimes e'_j) = f(e_i, e'_j)$$

and extending to the entire $V \otimes V'$ through linearity. This is possible and does not require any assumptions on $f(e_i, e'_j)$ since $\{e_i \otimes e'_j\}$ forms a basis. But then the following holds:

$$\begin{aligned}
F(\nu \otimes \nu') &= F(\nu^i \nu'^j e_i \otimes e'_j) \\
&= \nu^i \nu'^j F(e_i \otimes e'_j) \\
&= \nu^i \nu'^j f(e_i, e'_j) \\
&= f(\nu^i e_i, \nu'^j e'_j) \\
&= f(\nu, \nu')
\end{aligned}$$

Since the function f is fixed, it shows that F is unique.

Exercise 13

Well-definedness follows from the following

$$\begin{aligned}
\rho(g)\nu \otimes \rho'(g)\nu' &= \rho(g)\nu^i e_i \otimes \rho'(g)\nu'^j e'_j \\
&= \nu^i \nu'^j \rho(g) e_i \otimes \rho'(g) e'_j
\end{aligned}$$

since every element of $V \otimes V'$can be uniquely expressed as $\nu^i \nu'^j e_i \otimes e'_j$. The image of g is also clearly linear. And the fact that $\rho \otimes \rho'$ is a group homomorphism can also be shown using that ρ and $\rho'$are homomorphisms.

Exercise 14

Identify V with the subspace $\{\nu \oplus 0 : \nu \in V\}$ of $V \oplus V'$. Then V is an invariant subspace of the representation $\rho \oplus \rho'$ of G on $V \oplus V'$:

$$\begin{aligned}(\rho \oplus \rho')g(\nu \oplus 0) &= \rho(g)\nu \oplus \rho'(g)0 \\ &= \rho(g)\nu \oplus 0\end{aligned}$$

Thus we can define a subrepresentation of $\rho \oplus \rho'$ on V by letting $g \in G$ map to the transformation of V taking $\nu \oplus 0$ to $\rho(g)\nu \oplus 0$. But under the identification of V, this is just the original representation ρ.

The claim for ρ' follows by symmetry.

Exercise 15

For any $\theta \in \mathbb{R}$, $e^{\imath n\theta}$ is just a complex number, and multiplying by a fixed complex number is a linearly transformation of $\mathbb{C}$. The following shows that the map from $U(1)$ to $GL(\mathbb{C})$ is a group homomorphism:

$$\begin{aligned}\rho_n(e^{\imath\theta_1}e^{\imath\theta_2})\nu &= \rho_n(e^{\imath(\theta_1+\theta_2)})\nu \\ &= e^{\imath n(\theta_1+\theta_2)}\nu \\ &= \rho_n(e^{\imath\theta_1})\rho_n(e^{\imath\theta_2})\nu\end{aligned}$$

Exercise 16

Let $\rho : U(1) \to \mathbb{C}^*$ be a one-dimensional representation of $U(1)$. Consider the set $\rho^{-1}(1)$. We claim that this set is either the entire $U(1)$, or it is discrete, and hence finite (since $U(1)$ is compact). If it is not discrete, then we can find $\theta \in \mathbb{R}$ with $|\theta|$ arbitrarily small, but not 0, such that $\rho(e^{\imath\theta}) = 1$. But since ρ is a homomorphism this implies that the set of points in $U(1)$ mapping to the identity is dense in $U(1)$, and since ρ is continuous, it is the entire $U(1)$.

So assume that ρ is not the trivial representation, and choose x such that $|x|$ is as small as possible but not 0, such that $e^{\imath 2\pi x} \in \rho^{-1}(1)$. There should be two possible choices for x (x or $-x$).

We claim that x is rational. Otherwise the set $\{e^{\imath 2\pi xn} | n \in \mathbb{Z}\}$ has infinite cardinality and lies in $\rho^{-1}(1)$, which contradicts the fact that the latter set is finite.

Next, let $x = \frac{p}{q}$. We can assume the fraction is irreducible, and proper. Thus the gcd of p and q is 1, so we can find m and n such that

$$pm + qn = 1$$

or equivalently

$$\frac{pm}{q} + n = \frac{1}{q}$$

But then

$$\begin{aligned}\rho(e^{\imath 2\pi/q}) &= \rho(e^{\imath 2\pi p/q})^m \rho(e^{\imath 2\pi})^n \\ &= 1\end{aligned}$$

And since $\frac{1}{q} \leq \frac{p}{q}$, x must be of the form $\frac{1}{q}$.

Now we want to show that ρ maps $e^{\imath\theta}$ to multiplication by $\pm e^{\imath q\theta}$. We know that

$$\rho(e^{\imath 2\pi/q}) = 1$$

This implies that

$$\rho(e^{\imath 2\pi/2q}) = -1$$

(it has to be a square root of 1, but cannot be 1 since $\frac{1}{2q} < x$). Next

$$\rho(e^{\imath 2\pi/4q}) = \pm\imath$$

Both signs are compatible with what we're trying to prove. So suppose it's $\imath$ (the proof would just go slightly different otherwise). But now

$$\rho(e^{\imath 2\pi/8q})$$

must be a primitive 8-th root of unity, and using the continuity of ρ we can determine which one. Namely, if it does not lie between 1 and $\imath$, then there must be some $0 < k < 1$ such that

$$\rho(e^{\imath 2\pi k/4q}) = \imath$$

and thus

$$\rho(e^{\imath 2\pi k/q}) = 1$$

but $\frac{k}{q} < x$ gives the contradiction. Actually, the second-last equation could also equal -1 instead of $\imath$, it depends if we go clockwise or anti-clockwise from 1 to $\imath$. But we would get a similar contradiction.

So

$$\rho(e^{\imath 2\pi/8q}) = e^{\imath 2\pi/8}$$

We could continue in the same manner to show that

$$\rho(e^{\imath 2\pi/(2^k)q}) = e^{\imath 2\pi/(2^k)}$$

for all $k > 0$.

This shows that ρ agrees with the representation that maps $e^{\imath\theta}$ to multiplication by $e^{\imath q\theta}$ on a sequence of points converging to $1 \in SU(1)$. This is enough to show that they agree on a dense set, and by continuity, on the whole set. Thus ρ is equivalent to one of the desired representations.

Exercise 20

Applying the definitions gives the described 2x2 matrix as

$$\begin{pmatrix} (a - \imath d) & (-c - \imath b) \\ (c - \imath b) & (a + \imath d) \end{pmatrix}$$

Call it U. Calculating the determinant gives 1. And multiplying by the conjugate transpose, assuming a, b, c and d are real, gives the identity.