Exercise 1

Let T be the given transformation. Then applying it to

$$\nu = (\nu^0, \nu^1, \nu^2, \nu^3)$$

gives

$$T\nu = (\nu^0 \cosh\phi - \nu^1 \sinh\phi, -\nu^0 \sinh\phi + \nu^1 \cosh\phi, \nu^2, \nu^3)$$

and

$$\begin{aligned}
\eta(T\nu, T\omega) &= -(\nu^0 \cosh\phi - \nu^1 \sinh\phi)(\omega^0 \cosh\phi - \omega^1 \sinh\phi) \\
&\quad +(-\nu^0 \sinh\phi + \nu^1 \cosh\phi)(-\omega^0 \sinh\phi + \omega^1 \cosh\phi) \\
&\quad +\nu^2\omega^2 + \nu^3\omega^3 \\
&= -\nu^0\omega^0 + \nu^1\omega^1 + \nu^2\omega^2 + \nu^3\omega^3 \\
&= \eta(\nu, \omega)
\end{aligned}$$

where the identity

$$\cosh^2\phi - \sinh^2\phi = 1$$

is used twice. Using the same identity shows that the determinant of T is 1. Hence $T \in SO(3,1)$. The same follows for using y or z instead of x for reasons of symmetry.

Exercise 2

Proceeding as in the previous exercise, we can show that both P and T lie in $O(3,1)$.

$$\begin{aligned}
P\nu &= (\nu^0, -\nu^1, -\nu^2, -\nu^3) \\
T\nu &= (-\nu^0, \nu^1, \nu^2, \nu^3)
\end{aligned}$$

$$\begin{aligned}
\eta(P\nu, P\omega) &= -\nu^0\omega^0 + (-\nu^1)(-\omega^1) + (-\nu^2)(-\omega^2) + (-\nu^3)(-\omega^3) \\
&= \eta(\nu, \omega)
\end{aligned}$$

$$\begin{aligned}
\eta(T\nu, T\omega) &= -(-\nu^0)(-\omega^0) + \nu^1\omega^1 + \nu^2\omega^2 + \nu^3\omega^3 \\
&= \eta(\nu, \omega)
\end{aligned}$$

But

$$\det(P) = \det(T) = -1$$

which shows that neither is in $SO(3,1)$, but that their product must be, since it has det 1.

Exercise 3

The identity is in the special linear groups since its determinant is 1, and in the orthogonal and unitary groups since applying it to any vector gives the same vector again.

The special linear groups are closed under multiplication and taking inverses, because taking the determinant commutes with both these operations.

The othogonal and unitary groups are closed under multiplication:

$$\begin{aligned} \langle (AB)\nu, (AB)\omega \rangle &= \langle A(B\nu), A(B\omega) \rangle \\ &= \langle B\nu, B\omega \rangle \\ &= \langle \nu, \omega \rangle \end{aligned}$$

so essentially because applying matrices to vectors has the property $(AB)\nu = A(B\nu)$.

They are closed under taking inverses: let ν and ω be in $\mathbb{R}^n$ or $\mathbb{C}^n$, and let A be in the group being considered. Then it can be shown that $\det A \neq 0$. So A^{-1} is a matrix, and although we do not yet know if it is in the group, it can be used to define

$$\begin{aligned} \nu' &= A^{-1}(\nu) \\ \omega' &= A^{-1}(\omega) \end{aligned}$$

Then since A is in the group, we know that

$$\langle A\nu', A\omega' \rangle = \langle \nu', \omega' \rangle$$

which becomes

$$\langle \nu, \omega \rangle = \langle A\nu, A\omega \rangle$$

using that $AA^{-1} = I_n$, and showing that A^{-1} is in the group.

Exercise 4

Showing that the products and inverses are smooth is the same for all the groups and follows as described in the text. If $A = (a_{ij})$ and $B = (b_{ij})$ are two matrices, then the ij'th entry of the product is $\sum_k a_{ik}b_{kj}$ which is smooth as a function from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$. The inverse is shown to be smooth using the explicit formula with the adjugate matrix.

The general linear group is a submanifold since it is an open subset, the pre-image of $\mathbb{R} \setminus \{0\}$ under the determinant mapping.

For the special linear group we use a theorem which goes something like this: if ϕ is a smooth map from M to N with the dimension of M not less than that of N then the pre-image of any regular value is a smooth manifold. A regular value is a point in N such that all the pre-images are regular points in M. A regular point is a point for which the induced tangent map is surjective (which is why M should have dimension at least equal to that of N). (From 'Topology from the differentiable viewpoint' by Milnor.)

So if 1 is a regular value of 'det' then the special linear group is a submanifold of the general linear group. To see this, let A be in the special linear group. Let A_ϵ be the product of A with a matrix which is the same as the identity, except that the first entry in the diagonal is $1+\epsilon$ instead of 1. Then the vector at A pointing in the direction of A_ϵ will not map to 0 (in the tangent space at 1), since $\det(A_\epsilon)$ is $1+\epsilon \neq 1$, hence A is a regular point of the determinant map.

The same argument shows that if the orthogonal (unitary) group is a Lie group, then the special orthogonal (unitary) group must also be a Lie group.

The orthogonal group is characterized as matrices for which $AA^T = I$, where I is the identity if the signature of the metric is $(n, 0)$, otherwise, if it is (p, q), then the diagonal has p 1's and q 0's. In this case the map from $A \in \mathbb{R}^{2n}$ to $AA^T \in \mathbb{R}^{2n}$ could not be regular since the dimensions would not work out. Perhaps if we restrict the image to the lower triangular entries or something like that, it might work. I don't know how to do this part. The unitary group can be characterized in a similar way, so whatever works for the orthogonal group would probably work there as well.

Exercise 5

We need to show that G_0 is closed under multiplication and taking inverses. Firstly, it can be shown that manifolds are locally-path connected, and that for locally-path connected spaces, being connected is the same as being path-connected. So we rather work with this latter concept.

Let $x, y \in G_0$. Then there exists a continuous function (path)

$$\gamma : [0,1] \to G$$

such that $\gamma(0) = 1_G$, and $\gamma(1) = x$. Let $i : G \to G$ be the inverse function, which is continuous since G is a Lie group. Consider the path $i \circ \gamma$. This is a path in G from 1_G to $x^{-1} \in G$, which implies that $x^{-1} \in G_0$.

Next, let $m_y : G \to G$ be the continous function mapping $z \in G$ to $z \cdot y \in G$. Then the composition $m_y \circ \gamma$ is a path from $y \in G_0$ to $x \cdot y \in G$, which implies that $x \cdot y \in G_0$.

Thus G_0 is a subgroup of G. Furthermore, the above shows that when restricting the inverse and product functions to G_0 and $G_0 \times G_0$ respectively, their images lie in the same sets. Hence G_0 is a Lie group.

Exercise 6

Let $g \in O(3)$. If g is the identity, then it can be considered as a rotation of 0 radians about any axis, not followed by a reflection, and since it is in the identity component of $O(3)$, it does not contradict what we have to show.

We can also consider $O(3)$ as acting on vectors based at $0 \in \mathbb{R}^3$. It follows from the definition of $O(3)$ that elements of $O(3)$ must preserve the length of vectors, and that if two vectors are orthogonal, their images are also orthogonal (actually the angle between two vectors must stay the same, but this is probably not needed).

Now suppose g is not the identity. Then there must be some $x \in \mathbb{R}^3$ which is not fixed by g. Consider the plane P through the three points 0, x and $g(x)$, equivalently, spanned by the vectors x and $g(x)$. There are two vectors of unit length orthogonal to P. Choose one and call it y, then the other must be $-y$. Since y is orthogonal to x, $g(y)$ must be orthogonal to $g(x)$. By applying the inverse g^{-1}, it follows that since y is orthogonal to $g(x)$, $g(y)$ must be orthogonal to $g^{-1}(g(x)) = x$. In other words $g(y)$ is orthogonal to the same plane P, so it must equal either y or $-y$.

Now we actually need that g preserves the angle between vectors. Suppose $g(y) = y$. Then we claim that g is a rotation about the axis spanned by y. First consider a vector z in the plane spanned by x and y. Since the angles must be preserved, $g(z)$ must lie in the plane spanned by $g(x)$ and y, at the same position that rotation about y would leave it. Then consider a vector z in the plane spanned by x and $g(x)$. Again using preservation of angles shows that $g(z)$ lies where it should. And then combining these two facts would prove the claim.

If $g(y) = -y$, then we can compose with a reflection through the plane P, and it then follows from the above that the composition is a rotation. This completes the proof that g is a rotation possibly combined with a reflection.

If g is just a rotation, say through an angle θ, then we can construct a path γ in G, such that $\gamma(t)$ is a rotation through $t\theta$ radians about the same axis. This shows that g is in the identity component.

To show that rotations with reflections are not in the identity component, we could use the fact that the determinant function is continuous and has image $\{1, -1\}$, to divide G into two disjointed open sets. Using the fact that reflection through a plane has determinant -1, and that if $h \in G$ is a rotation, then $g \in G$ and gh are in the same connected component of G, it follows that the elements with determinant 1 are precisely the ones which are only rotations, and not reflections.

Exercise 7

I can do the first part (showing that there is no path from the identity to the element PT):

Consider the vector $u_t \in \mathbb{R}^4$, where

$$u_t = (1, 0, 0, 0)$$

Let $A = (a_{ij}) \in SO(3,1)$. Then $Au_t = (a_{11}, a_{21}, a_{31}, a_{41})$, the first column of the matrix A. Since

$$\langle Au_t, Au_t \rangle = \langle u_t, u_t \rangle = -1$$

it follows that

$$-a_{11}^2 + a_{21}^2 + a_{31}^2 + a_{41}^2 = -1$$

It follows that $a_{11}^2 \geq 1$. But for $A = PT$, $a_{11} = -1$, whereas for the identity, $a_{11} = 1$. Thus if we have a path from the identity to PT, then the function from

$[0,1]$ to a_{11} should be continous, but this is impossible given the restriction on a_{11}. So $SO(3,1)$ has at least two connected components.

For the second part, it seems from wikipedia that the connected component of the identity is generated by elements such as in exercise 1, together with transformations from $SO(3)$ which leave the time component unchanged. I don't understand the details though.

Exercise 8

Suppose 1_H is in the image of ρ. Then there exists $g \in G$ such that $\rho(g) = 1_H$. Then

$$\begin{aligned}
\rho(1_G) &= 1_H \rho(1_G) \\
&= \rho(g)\rho(1_G) \\
&= \rho(g 1_G) \\
&= \rho(g) \\
&= 1_H
\end{aligned}$$

I don't see how to show this in general.

For the second part

$$\begin{aligned}
\rho(g^{-1}) &= \rho(g^{-1}) \cdot 1_H \\
&= \rho(g^{-1})\rho(g)\rho(g)^{-1} \\
&= \rho(g^{-1}g)\rho(g)^{-1} \\
&= \rho(g)^{-1}
\end{aligned}$$

Exercise 9

Let $\alpha \in U(1)$. Then for all $x, y \in \mathbb{C}$,

$$\langle x, y \rangle = \langle \alpha x, \alpha y \rangle$$

so

$$\overline{x} y = \overline{\alpha x} \alpha y$$

so $\overline{\alpha}\alpha = 1$, i.e. $|\alpha| = 1$. So as a set $U(1)$ is as given. Also, $e^{\imath\theta_1} e^{\imath\theta_2} x = e^{\imath(\theta_1+\theta_2)} x$, so as a group, $U(1)$ is as given.

The hint gives the proof of the second part. Applying the given matrix to $\mathbb{R}^2$ gives an anti-clockwise rotation about the origin. All elements of $SO(2)$ is of this form. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ shows that multiplying by $e^{\imath\theta}$ has exactly the same effect.

Exercise 10

This is immediate by just writing out the definitions. For instance

$$\begin{aligned} 1(g,h) &= (1,1)(g,h) \\ &= (1g,1h) \\ &= (g,h) \end{aligned}$$

So $(1,1)$ is the identity on the left. Etc.