Exercise 72

For F_+:

$$\begin{aligned} F_+ &= \frac{1}{2}(F + \star F) \\ \star F_+ &= \frac{1}{2}(\star F + \star^2 F) \\ &= \frac{1}{2}(\star F + F) \\ &= F_+ \end{aligned}$$

and F_- is similar:

$$\begin{aligned} F_- &= \frac{1}{2}(F - \star F) \\ \star F_- &= \frac{1}{2}(\star F - \star^2 F) \\ &= \frac{1}{2}(\star F - F) \\ &= -F_- \end{aligned}$$

Exercise 73

Trial and error shows that if we define F_+ as

$$F_+ = \frac{1}{2}(F - \imath \star F)$$

then

$$\begin{aligned} \star F_+ &= \frac{1}{2}(\star F + \imath F) \\ &= \imath F_+ \end{aligned}$$

and defining F_- as

$$F_- = \frac{1}{2}(F + \imath \star F)$$

gives

$$\begin{aligned} \star F_- &= \frac{1}{2}(\star F - \imath F) \\ &= -\imath F_- \end{aligned}$$

and

$$F = F_+ + F_-.$$

Exercise 74

In the first equation, $\star_S E = \imath B$:

$$\begin{aligned} \star_S E &= E_1 dx^2 \wedge dx^3 + E_2 dx^3 \wedge dx^1 + E_3 dx^1 \wedge dx^2 \\ \imath B &= \imath(B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2) \end{aligned}$$

So equating coefficients gives

$$\imath B_i = E_i$$

or

$$B_i = -\imath E_i$$

which is equivalent to the given equations.

In the second equation, $\star_S B = -\imath E$:

$$\begin{aligned} \star_S B &= B_1 dx^1 + B_2 dx^2 + B_3 dx^3 \\ -\imath E &= -\imath(E_1 dx^1 + E_2 dx^2 + E_3 dx^3) \end{aligned}$$

So equating coefficients again gives

$$B_i = \imath E_i$$

showing that the second equation is equivalent to the first and to the given equations.

Exercise 75

Using

$$B = \mathbf{B} e^{\imath k_\mu x^\mu}$$

and

$$E = \mathbf{E} e^{\imath k_\mu x^\mu}$$

gives

$$\begin{aligned} \partial_t B + d_S E &= \mathbf{B} \imath k_0 e^{\imath k_\mu x^\mu} + d_S e^{\imath k_\mu x^\mu} \wedge \mathbf{E} \\ &= \imath e^{\imath k_\mu x^\mu} k_0 \mathbf{B} + \imath e^{\imath k_\mu x^\mu}\, {}^3k \wedge \mathbf{E} \\ &= (\imath e^{\imath k_\mu x^\mu})(k_0 \mathbf{B} + {}^3k \wedge \mathbf{E}) \end{aligned}$$

So

$$\partial_t B + d_S E = 0$$

is equivalent to

$$k_0 \mathbf{B} + {}^3k \wedge \mathbf{E} = 0$$

or

$${}^3k \wedge \mathbf{E} = -k_0 \mathbf{B}$$

Which is what we're supposed to get, except for the sign.

Exercise 76

Expanding both sides gives

$$\begin{aligned}
{}^3k \wedge E &= k_\mu dx^\mu \wedge E_\nu dx^\nu \\
&= (k_2E_3 - k_3E_2)dx^2 \wedge dx^3 + \\
&\quad (k_3E_1 - k_1E_3)dx^3 \wedge dx^1 + \\
&\quad (k_1E_2 - k_2E_1)dx^1 \wedge dx^2
\end{aligned}$$

and

$$-\imath k_0 \star_S E = -\imath k_0(E_1 dx^2 \wedge dx^3 + E_2 dx^3 \wedge dx^1 + E_3 dx^1 \wedge dx^2)$$

Setting coefficients equal, and writing it all as a matrix, gives

$$A \cdot \begin{bmatrix} E_1 \\ E_2 \\ E_3 \end{bmatrix} = 0$$

where

$$A = \begin{bmatrix} \imath k_0 & -k_3 & k_2 \\ k_3 & \imath k_0 & -k_1 \\ -k_2 & k_1 & \imath k_0 \end{bmatrix}$$

Since A is skew-symmetric and has odd dimension, its determinant is 0. Calculating the determinant, and setting it to 0 gives

$$-k_0^2 + k_1^2 + k_2^2 + k_3^2 = 0$$

which is the same as

$$k_\mu k^\mu = 0.$$

Exercise 77

Firstly,

$$k = t - x = k_\mu x^\mu$$

Then, $\mathbf{E} = dy - \imath dz$ written in the old notation is

$$(0, 1, -\imath)$$

and

$$\begin{aligned}
\mathbf{B} &= -\imath \star_S E \\
&= -\imath(dz \wedge dx - \imath dx \wedge dy) \\
&= -\imath dz \wedge dx - dx \wedge dy
\end{aligned}$$

which is

$$(0, -\imath, -1)$$

in the old notation. Thus

$$\begin{aligned} E &= e^{\imath k_\mu x^\mu}\mathbf{E} \\ &= e^{\imath(t-x)}(0,1,-\imath) \\ &= (0, e^{\imath(t-x)}, -\imath e^{\imath(t-x)}) \end{aligned}$$

and similarly for B.

Exercise 78

I get stuck with the following approach: consider the real part of the 1-form E at a single point as a function of time. This can be seen as a vector rotating in a plane containing the point. Using just E, find the normal to this plane, and find a direction of the normal describing the rotation. The cross product of the vector with its derivative should give such a normal vector. But its hard to find the general expression for this vector.

If this could be done, then the rest should follow somehow using the equation just above exercise 76.

Exercise 79

It was shown previously that $F = B + E \wedge dt$ is self-dual if either (and hence both) of the following hold:

$$\begin{aligned} \star_S E &= \imath B \\ \star_S B &= -\imath E \end{aligned}$$

Similarly, from $\star F = -\imath F$ it follows that F is anti-self-dual if

$$\begin{aligned} \star_S E &= -\imath B \\ \star_S B &= \imath E \end{aligned}$$

Now

$$P^* F = P^* B + (P^* E) \wedge dt$$

so we must show that if (B, E) satisfies the first pair of equations, then $(P^* B, P^* E)$ satisfies the second pair. The converse follows from $P \circ P = id$. In other words, assume F is self-dual. Then

$$\begin{aligned} \star_S P^* B &= \star_S B \\ &= -\imath E \\ &= \imath P^* E \end{aligned}$$

and

$$\begin{aligned} \star_S P^* E &= -\star_S E \\ &= -\imath B \\ &= -\imath P^* B \end{aligned}$$

shows that $P^* F$ is anti-self-dual.