Exercise 46

We claim that it is enough to prove this for

$$\omega = \alpha\, v_1 \wedge \cdots \wedge v_p$$

and

$$\mu = \beta\, v_{p+1} \wedge \cdots \wedge v_n$$

where $n = p+q$, and $v_1, \ldots, v_n$ form part of a basis of V. In general ω would be a linear combination of elements of the form $\alpha v_{i_1} \wedge \cdots v_{i_p}$. Suppose $\omega = \omega_1 + \omega_2$, and the property that we have to prove holds for the pairs (ω_1, μ) and (ω_2, μ). Then it also holds for (ω, μ) as the following shows

$$\begin{aligned}
\omega \wedge \mu &= \omega_1 \wedge \mu + \omega_2 \wedge \mu \\
&= (-1)^{pq} \mu \wedge \omega_1 + (-1)^{pq} \mu \wedge \omega_2 \\
&= (-1)^{pq} \mu \wedge (\omega_1 + \omega_2) \\
&= (-1)^{pq} \mu \wedge \omega
\end{aligned}$$

Therefore it is enough to assume that ω is of the form $\alpha v_{i_1} \wedge \cdots v_{i_p}$, and similarly for μ. We can also assume that they don't have any basis vectors in common, since then both sides of the desired equation would be 0, so it would be true. This proves the claim.

Finally,

$$\begin{aligned}
\omega \wedge \mu &= \alpha\beta v_1 \wedge \cdots \wedge v_p \wedge v_{p+1} \wedge \cdots \wedge v_n \\
&= \alpha\beta(-1)^p v_{p+1} \wedge v_1 \wedge \cdots \wedge v_p \wedge v_{p+2} \wedge \cdots \wedge v_n \\
&= \alpha\beta(-1)^{2p} v_{p+1} \wedge v_{p+2} \wedge v_1 \wedge \cdots \wedge v_p \wedge v_{p+3} \wedge \cdots \wedge v_n \\
&= \cdots \\
&= \alpha\beta(-1)^{qp} v_{p+1} \wedge v_{p+2} \wedge \cdots \wedge v_n \wedge v_1 \wedge v_2 \wedge \cdots \wedge v_p \\
&= (-1)^{pq} \mu \wedge \omega
\end{aligned}$$

as required. In the first step v_{p+1} is moved p steps to the left, and so on. The total number of transpositions is pq, hence the expression for the sign.

Exercise 47

The pullback map ϕ^* can be defined to operate on functions and 1-forms in the usual way. Since the given three operations (multiplying by a function, addition and taking the wedge product) generates all the differential forms starting with functions and 1-forms, the given three requirements are enough to define $\phi^*$for all differential forms. This shows that if such a map exists, it is unique.

The image of a differential form under the pullback map is well-defined in all cases because the three operations commute in all the necessary ways. (So you could define the pullback of a given differential form in many different ways, but they are actually all the same.) Therefore such a pullback map exists.

Exercise 48

If (v_1, v_2, v_3) is a right-handed basis, then $v_1 \times v_2$ is parallel to v_3 and points in the same direction. Applying ϕ to get (u_1, u_2, u_3), we have $u_1 \times u_2 = -v_1 \times -v_2 = v_1 \times v_2$ and $u_3 = -v_3$. So $u_1 \times u_2$ is still parallel to u_3, but points in the opposite direction, giving a left-handed basis. The converse is similar.

Here are the other two computations:

$$\phi^*(\omega_\mu dx^\mu) = -(\omega_\mu \circ \phi)dx^\mu$$

$$\phi^*(\frac{1}{2}\omega_{\mu\nu} dx^\mu \wedge dx^\nu) = \frac{1}{2}(\omega_{\mu\nu} \circ \phi)dx^\mu \wedge dx^\nu$$

Exercise 49

This follows from the Leibniz law and the definition of the differential of a function.

$$\begin{aligned} d(\omega_\mu dx^\mu) &= d\omega_\mu \wedge dx^\mu \\ &= \partial_\nu \omega_\mu dx^\nu \wedge dx^\mu \end{aligned}$$

Exercise 50

Let $\{(U_i, \phi_i)\}$ be an atlas for S. Then $\{(\mathbb{R} \times U_i, id_{\mathbb{R}} \times \phi_i)\}$ is an atlas for $\mathbb{R} \times S$. Choose any chart (U_i, ϕ_i) and let x^i be local coordinates. Then together with t they form local coordinates for the corrsponding chart on $\mathbb{R} \times S$. It follows that the set of 2-forms $dx^i \wedge dt$ and $dx^i \wedge dx^j$ with $i < j$ form a basis for $\Omega^2(U_i)$. Hence there exist unique α_i and β_{ij} such that the given 2-form F can be expressed locally as

$$\alpha_i dx^i \wedge dt + \beta_{ij} dx^i \wedge dx^j.$$

(Again, we assume $i < j$.) So locally, F can be expressed as a sum of 2-forms in the desired form. Now it remains to be shown that these 2-forms can be patched together to give the global decomposition of F. This would follow if we can show that the transition functions respect the local decomposition.

Suppose (U_i', ϕ_i') is another chart on S with local coordinates $x^{i\prime}$ and t'. Then the transition function between the corresponding charts on $\mathbb{R} \times S$ maps t to t'. It follows from this the induced pullback map on differential forms will map $\alpha_i dx^i \wedge dt$ to something of the form $\tilde{\alpha}_i dx^{i\prime} \wedge dt'$ and similarly for the other local 2-form. But due to the uniqueness of the local decomposition of F, we must have $\tilde{\alpha}_i = \alpha_i'$, where $\alpha_i' dx^{i\prime} \wedge dt'$ is the first component of F on U_i'. A similar statement holds for the second component. Thus the pullback map respects the local decomposition, and we can patch things together to obtain a local decomposition as required:

$$F = B + E \wedge dt$$

The global decomposition is unique due to the uniqueness of all the local ones.

(An aside: the coefficients B_{ij} are only unique if we add a condition such as $B_{ij} = -B_{ji}$.)

Exercise 51

This is similar to the previous exercise. It follows from the definition of the exterior derivative that this decomposition is possible locally. It is also locally unique since there are no choices involved. As in the previous exercise, the fact that the transition functions maps t to t' implies that the induced pullback map preserves the decomposition. Thus the local differential forms can be patched together to form global ones giving the required decomposition.

Exercise 52

Denote the map from V to V^* by ϕ. Let $v \in \ker \phi$. This means that $g(v, w) = 0 \, \forall w \in V$. Since g is nondegenerate, this implies that $v = 0$, so ϕ is one-to-one. Since the dimensions of V and V^* are equal, ϕ must be an isomorphism.

Note that V should be finite dimensional for this result to hold (from peeking at comments to other people's answers...).

Exercise 53

To show that the two 1-forms are the same, it is enough to apply each of them to a basis vector field, e_λ. First, applying the 1-form corresponding to the vector v under the metric g gives:

$$\begin{aligned} g(v, e_\lambda) &= g(v^\mu e_\mu, e_\lambda) \\ &= v^\mu g(e_\mu, e_\lambda) \\ &= v^\mu g_{\mu\lambda} \end{aligned}$$

Then, applying the given 1-form $v_\nu f^\nu$ gives:

$$\begin{aligned} (v_\nu f^\nu)(e_\lambda) &= (g_{\mu\nu} v^\mu f^\nu)(e_\lambda) \\ &= g_{\mu\lambda} v^\mu \\ &= v^\mu g_{\mu\lambda} \end{aligned}$$

which is the same.

Exercise 54

The question is to show that under the metric g, the 1-form $\omega = \omega_\mu f^\mu$ corresponds to the given vector field. Since the corrspondence goes both ways, we can start with the vector field and show that it corresponds to the given 1-form. Using the result from the previous exercise, the vector field $\omega^\nu e_\nu$corresponds to the 1-form $g_{\nu\lambda}\omega^\nu f^\lambda$. But

$$\begin{aligned} g_{\nu\lambda}\omega^\nu f^\lambda &= g_{\nu\lambda}g^{\mu\nu}\omega_\mu f^\lambda \\ &= \delta^\mu_\lambda \omega_\mu f^\lambda \\ &= \omega_\mu f^\mu \end{aligned}$$

where the first step uses the given definition of the vector field $\omega^\nu e_\nu$, and the second step uses the fact that the two matrices are inverses of each other.

Exercise 55

From the definition of the Minkowski metric

$$\eta(v, w) = -v^0 w^0 + v^1 w^1 + v^2 w^2 + v^3 w^3$$

it follows that $\eta(e_i, e_j)$ is 0 if $i \neq j$, -1 if $i = j = 0$, and 1 if $i = j$, and $i \neq 0$. This is the content of the given matrix $\eta_{\mu\nu}$.

Exercise 56

Lowering the index μ, gives

$$g_{\nu\mu} = g_{\mu\lambda} g^\lambda_\nu$$

which implies that

$$\begin{aligned} g^\lambda_\nu &= g^{\mu\lambda} g_{\nu\mu} \\ &= \delta^\lambda_\nu \end{aligned}$$

as desired.

Exercise 57

Firstly, we should consider only p-fold wedge products

$$e^{i_1} \wedge \cdots \wedge e^{i_p}$$

with strictly increasing indices ($i_1 < \cdots < i_p$). It was shown in an earlier exercise that these wedge products form a basis for the p-forms. To show that the basis is orthonormal, first consider the inner product of two distinct elements:

$$\langle e^{i_1} \wedge \cdots \wedge e^{i_p}, e^{i'_i} \wedge \cdots \wedge e^{i'_p} \rangle \quad = \quad \det[g(e^{i_j}, e^{i'_k})]$$

But since the elements are distinct, there must be some j such that $i_j \neq i'_k$ for all k. This means that the j'th row of the matrix will consist only of 0's (since $(e^1, \ldots, e^n)$ is an orthonormal basis), and hence the determinant will be 0.

Now consider the inner product of an element with itself:

$$\begin{aligned}
\langle e^{i_1} \wedge \cdots \wedge e^{i_p}, e^{i_i} \wedge \cdots \wedge e^{i_p} \rangle &= \det[g(e^{i_j}, e^{i_k})] \\
&= \prod_j g(e^{i_j}, e^{i_j}) \\
&= \prod_j \epsilon(i_j) \\
&= \epsilon(i_1) \cdots \epsilon(i_p)
\end{aligned}$$

as claimed, and which has absolute value 1. Hence the p-fold wedge products form an orthonormal basis of p-forms.

To see that this implies that the inner product of p-forms is nondegenerate, consider an aribrary non-zero p-form

$$\omega = \alpha e^{i_1} \wedge \cdots \wedge e^{i_p} + \cdots$$

with $\alpha \neq 0$. Then

$$\begin{aligned}
\langle \omega, e^{i_1} \wedge \cdots \wedge e^{i_p} \rangle &= \pm\alpha \\
&\neq 0
\end{aligned}$$

Exercise 58

For the Euclidean metric, the component matrix of the metric is the identity matrix. So it's inverse is also the identity, and hence the inner product on 1-forms is still just the dot product. This gives the first part

$$\langle E, E \rangle = E_x^2 + E_y^2 + E_z^2$$

For the second part, we can write $B_y dz \wedge dx$ as $-B_y dx \wedge dz$ and use the results of the previous exercise to derive the required result:

$$\langle B, B \rangle = B_x^2 + B_y^2 + B_z^2$$

Exercise 59

$$\begin{aligned}
F &= B + E \wedge dt \\
&= B - E \wedge dt
\end{aligned}$$

$$\begin{aligned}\langle F,F\rangle &= \langle B,B\rangle + \langle -dt\wedge E, -dt\wedge E\rangle \\ &= \langle B,B\rangle + \langle dt\wedge E, dt\wedge E\rangle \\ &= \langle B,B\rangle + \langle dt\wedge dt\rangle\langle E,E\rangle \\ &= \langle B,B\rangle - \langle E,E\rangle\end{aligned}$$

$$\text{-}\tfrac{1}{2}\langle \mathrm{F},\mathrm{F}\rangle = \frac{1}{2}(\langle E,E\rangle - \langle B,B\rangle)$$

which is the Lagrangian.