Exercise 25

Given that ω and μ are 1-forms, we first show that $\omega + \mu$ is a 1-form:

$$\begin{aligned}
(\omega+\mu)(v+w) &= \omega(v+w)+\mu(v+w) \\
&= \omega(v)+\omega(w)+\mu(v)+\omega(v) \\
&= (\omega+\mu)(v)+(\omega+\mu)(w)
\end{aligned}$$

$$\begin{aligned}
(\omega+\mu)(gv) &= \omega(gv)+\mu(gv) \\
&= g\omega(v)+g\mu(v) \\
&= g(\omega(v)+\mu(v)) \\
&= g(\omega+\mu)(v)
\end{aligned}$$

and then that $f\omega$ is a 1-form, for $f \in C^\infty(M)$:

$$\begin{aligned}
(f\omega)(v+w) &= f\omega(v+w) \\
&= f(\omega(v)+\omega(w)) \\
&= f\omega(v)+f\omega(w) \\
&= (f\omega)(v)+(f\omega)(w)
\end{aligned}$$

$$\begin{aligned}
(f\omega)(gv) &= f\omega(gv) \\
&= fg\omega(v) \\
&= gf\omega(v) \\
&= g(f\omega(v)) \\
&= g((f\omega)(v))
\end{aligned}$$

Exercise 26

In the following, let $v \in \mathrm{Vect}(M)$. Then just check the 4 conditions on p26:

$$\begin{aligned}
(f(\omega+\mu))(v) &= f(\omega+\mu)(v) \\
&= f(\omega(v)+\mu(v)) \\
&= f\omega(v)+f\mu(v) \\
&= (f\omega)(v)+(f\mu)(v) \\
&= (f\omega+f\mu)(v)
\end{aligned}$$

which implies that $f(\omega+\mu) = f\omega + f\mu$.

$$
\begin{aligned}
((f+g)\omega)(v) &= (f+g)\omega(v) \\
&= f\omega(v) + g\omega(v) \\
&= (f\omega)(v) + (g\omega)(v) \\
&= (f\omega + g\omega)(v)
\end{aligned}
$$

which implies that $(f+g)\omega = f\omega + g\omega$.

$$
\begin{aligned}
((fg)\omega)(v) &= (fg)\omega(v) \\
&= f(g\omega(v)) \\
&= f((g\omega)(v)) \\
&= (f(g\omega))(v)
\end{aligned}
$$

which implies that $(fg)(\omega) = f(g(\omega))$, and finally:

$$
\begin{aligned}
(1\omega)(v) &= 1\omega(v) \\
&= \omega(v)
\end{aligned}
$$

which implies that $1\omega = \omega$, and hence $\Omega^1(M)$ is a module over $C^\infty(M)$.

Exercise 27

As before, let $v \in \mathrm{Vect}(M)$. Then we check each of the 4 conditions:

$$
\begin{aligned}
(d(f+g))(v) &= v(f+g) \qquad \text{(definition of d)} \\
&= v(f) + v(g) \qquad \text{(linearity of v)} \\
&= (df)(v) + (dg)(v) \qquad \text{(definition of d)} \\
&= (df + dg)(v) \qquad (\mathrm{Vect}(M)\text{is a module over } C^\infty(\mathbb{R}))
\end{aligned}
$$

which implies that $d(f+g) = df + dg$.

$$
\begin{aligned}
(d(\alpha f))(v) &= v(\alpha f) \qquad \text{(definition of d)} \\
&= \alpha v(f) \qquad \text{(linearity of v)} \\
&= \alpha(df)(v) \qquad \text{(definition of d)} \\
&= (\alpha df)(v) \qquad (\Omega^1(M)\text{is a module over } C^\infty(\mathbb{R}))
\end{aligned}
$$

which implies that $d(\alpha f) = \alpha df$.

$$
\begin{aligned}
((f+g)dh)(v) &= (f+g)(dh)(v) \\
&= (f+g)(vh) \\
&= fvh + gvh \\
&= f(dh)(v) + g(dh)(v) \\
&= (fdh)(v) + (gdh)(v) \\
&= (fdh + gdh)(v)
\end{aligned}
$$

which implies thhat $(f+g)df = fdh + gdh$.

$$
\begin{aligned}
(d(fg))(v) &= v(fg) \\
&= fvg + gvf \\
&= f(dg)(v) + g(df)(v) \\
&= (fdg)(v) + (gdf)(v) \\
&= (fdg + gdf)(v)
\end{aligned}
$$

which implies that $d(fg) = fdg + gdf$.

Exercise 28

Assume that $\{\partial_\mu\}$ forms a basis for the vector fields on $\mathbb{R}^n$, and let $v = v^\mu \partial_\mu$ be any vector field on $\mathbb{R}^n$. Then starting with the right-hand side of the given equation:

$$
\begin{aligned}
(\partial_\mu f dx^\mu)(v) &= (\partial_\mu f) dx^\mu(v) \\
&= \partial_\mu f v x^\mu \\
&= \partial_\mu f (v^\nu \partial_\nu x^\mu) \\
&= \partial_\mu f (v^\mu) \\
&= (v^\mu \partial_\mu)(f) \\
&= vf \\
&= (df)(v)
\end{aligned}
$$

which implies that $\partial_\mu f dx^\mu = df$.

Exercise 29

$\omega = 0$ means that $\omega v = 0$ for any vector field v. In particular, let $v = \partial_\nu$. Then

$$
\begin{aligned}
\omega \partial_\nu &= \omega_\mu dx^\mu \partial_\nu \\
&= \omega_\nu
\end{aligned}
$$

so $\omega_\nu = 0$. This holds for all $1 \leq \nu \leq n$, which implies that the basis functions are independent.

Exercise 30

For the first part of the question, we should show that if v and w are two vector fields on M such that $v_p = w_p$, then $\omega(v)(p) = \omega(w)(p)$. This would imply that ω_p is well-defined.

If we define $z = v - w$, then $z_p = v_p - w_p = 0$, and $\omega(z)(p) = 0$ would imply that $\omega(v)(p) = \omega(w)(p)$. This uses the linearity of both vector fields and 1-forms. Therefore it is enough to prove that given a vector field v on M with $v_p = 0$, it follows that $\omega(v)(p) = 0$. (Just using v instead of z.)

Let x^μ be local coordinates in a chart around p. In this chart, we can write $v = v^\nu \partial_\nu$, where the v^ν are simply functions defined in a neighbourhood of p. So then

$$v_p = (v^\nu \partial_\nu)_p$$

and applying the tangent vector to a coordinate function yields

$$\begin{aligned} v_p(x^\mu) &= (vx^\mu)(p) \\ &= (v^\mu \partial_\nu x^\mu)(p) \\ &= v^\mu(p) \end{aligned}$$

But since $v_p = 0$, the functions v^μ all vanish at p.

Using the chart, we can also expand the 1-form as

$$\omega = \omega_\mu dx^\mu$$

which leads to

$$\begin{aligned} \omega(v)(p) &= (\omega_\mu dx^\mu)(v^\nu \partial_\nu)(p) \\ &= (\omega_\mu v^\nu)(p) \\ &= \omega_\mu(p).v^\mu(p) \\ &= \omega_\mu(p).0 \\ &= 0 \end{aligned}$$

which is what we wanted to prove.

For the second part, let v be any vector field on M, and let $p \in M$. Then

$$\begin{aligned} \omega(v)(p) &= \omega_p(v) \\ &= \nu_p(v) \\ &= \nu(v)(p) \end{aligned}$$

implies that $\omega(v) = \nu(v)$, and since v was arbitrary, this implies that $\omega = \nu$.

Exercise 31

Denote the identity map on V by f, and let $\nu \in V^*$. Then for any $v \in V$, we have

$$\begin{aligned} f^*(\nu)(v) &= \nu(fv) \\ &= \nu(v) \end{aligned}$$

which implies that $f^*(\nu) = \nu$. It follows that f^* is the identity map on V^*.

For the second part of the question, let $\xi \in X^*$. Then for any $v \in V$, we have

$$\begin{aligned} (g \circ f)^*(\xi)(v) &= \xi((g \circ f)v) \\ &= \xi(g(f(v))) \end{aligned}$$

and

$$\begin{aligned} (f^* \circ g^*)(\xi)(v) &= (f^*(g^*(\xi)))(v) \\ &= g^*(\xi)(fv) \\ &= \xi(g(fv)) \end{aligned}$$

which implies that $(g \circ f)^*(\xi) = (f^* \circ g^*)(\xi)$ and hence $(g \circ f)^* = (f^* \circ g^*)$.

Exercise 32

The idea is to define a 1-form ν on M and show that for any $p \in M$, the cotangent vector ν_p is the same as $\phi^*(\omega_q)$.

Define ν as a map from $\mathrm{Vect}(M)$ to $C^\infty(M)$ which takes the vector v to the function $\phi^*(\omega(\phi_* v))$. I think it can be shown that this map is linear over $C^\infty(M)$, and hence it is a 1-form on M.

The following shows that the two tangent vectors are the same:

$$\begin{aligned} \nu_p(v_p) &= \nu(v)(p) \\ &= \phi^*(\omega(\phi_* v))(p) \\ &= (\omega(\phi_* v))(q) \\ &= \omega_q((\phi_* v)_q) \\ &= \omega_q(\phi_* v_q) \\ &= (\phi^* \omega_q)(v_q) \end{aligned}$$

Finally, uniqueness follows from exercise 30.

Exercise 33

$$\begin{aligned}\phi^* dx &= d(\phi^* x)\\ &= d(\sin t)\\ &= (\partial_t \sin t)\, dt\\ &= \cos t\, dt\end{aligned}$$

Exercise 34

$(x, y) \in \mathbb{R}^2$ is mapped by ϕ to $(\cos\theta\, x - \sin\theta\, y, \sin\theta\, x + \cos\theta\, y) \in \mathbb{R}^2$. Therefore,

$$\begin{aligned}\phi^* dx &= d(\phi^* x)\\ &= d(\cos\theta\, x - \sin\theta\, y)\\ &= \cos\theta\, dx - \sin\theta\, dy\end{aligned}$$

where the last step uses the first two properties from exercise 27.

Exercise 35

If we use precise notation, then the question is to show that

$$\phi^*(dx^\mu) = d(\phi^* x^\mu).$$

But if we consider ϕ as a map between manifolds U and $\phi(U) \subset \mathbb{R}^n$, then this is the content of the equation at the top of p48, namely that d is a natural transformation.

Exercise 36

Both $\{dx^\mu\}$ and $\{dx'^\nu\}$ form a basis for 1-forms on $\mathbb{R}^n$. Hence we can write

$$dx^\mu = C^\mu_\nu dx'^\nu$$

where C^μ_ν is a matrix of functions on $\mathbb{R}^n$. Now apply each side to ∂'_λ. The left-hand side becomes:

$$\begin{aligned}dx^\mu \partial'_\lambda &= \partial'_\lambda x^\mu\\ &= \frac{\partial x^\mu}{\partial x'^\lambda}\end{aligned}$$

and the right-hand side becomes:

$$C^\mu_\nu dx'^\nu \partial'_\lambda = C^\mu_\lambda$$

so in other words:

$$C^\mu_\lambda = \frac{\partial x^\mu}{\partial x'^\lambda}$$

and

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu$$

If we reverse the two coordinate systems at the beginning, we obtain the first equation of the exercise, but this form is more useful for the rest.

Given the 1-form ω expressed in each of the bases as shown, we can use the above equation to write:

$$\begin{aligned} \omega'_\nu dx'^\nu &= \omega_\mu dx^\mu \\ &= \omega_\mu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu \end{aligned}$$

Equating coefficients gives the desired:

$$\omega'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} \omega_\mu$$

Exercise 37

Apply both sides of the given equation to one of the coordinate vector fields ∂_λ on $\mathbb{R}^m$. The left hand side is

$$\begin{aligned} \phi^*(dx'^\nu)\partial_\lambda &= dx'^\nu \phi^* \partial_\lambda \\ &= dx'^\nu \frac{\partial x'^\nu}{\partial x^\lambda} \partial'_\nu \\ &= \frac{\partial x'^\nu}{\partial x^\lambda} dx'^\nu \partial'_\nu \\ &= \frac{\partial x'^\nu}{\partial x^\lambda} \end{aligned}$$

And the right hand side is

$$\frac{\partial x'^\nu}{\partial x^\mu} dx^\mu \partial_\lambda = \frac{\partial x'^\nu}{\partial x^\lambda}$$

which is the same.

Exercise 38

For the one direction, suppose $T^\nu_\mu(p)$ is invertible for all $p \in U$. For each p, let $S^\mu_\nu(p)$ be its inverse. This gives a matrix of functions on U, namely S^μ_ν. Assume that they are smooth (I don't know how to show this).

Then

$$\begin{aligned} S_\nu^\mu e_\mu &= S_\nu^\mu T_\mu^\lambda \partial_\lambda \\ &= \delta_\nu^\lambda \partial_\lambda \\ &= \partial_\nu \end{aligned}$$

Since $\{\partial_\nu\}$ forms a basis for the vector fields on U, and any ∂_ν can be expressed as a linear combination of e_μ's, it follows that $\{e_\mu\}$ forms a basis as well.

For the other direction, suppose $\{e_\mu\}$ is a basis. Then we can express

$$\partial_\nu = S_\nu^\mu e_\mu$$

for smooth functions S_ν^μ on U. But that means that

$$\partial_\nu = S_\nu^\mu T_\mu^\lambda \partial_\lambda$$

which implies that $S_\nu^\mu T_\mu^\lambda = \delta_\nu^\lambda$. So at any $p \in U$, the product of the two induced matrices is the identity matrix, and so the matrix $T_\mu^\lambda(p)$ is invertible.

Exercise 39

Use the same matrix of function S_λ^μ on U (with different indices) to define

$$f^\mu = S_\lambda^\mu dx^\lambda$$

Apply these 1-forms to the vector field basis function from the previous exercise:

$$\begin{aligned} f^\mu(e_\nu) &= S_\lambda^\mu dx^\lambda T_\nu^\omega \partial_\omega \\ &= S_\lambda^\mu T_\nu^\omega dx^\lambda \partial_\omega \\ &= S_\lambda^\mu T_\nu^\omega \delta_\omega^\lambda \\ &= S_\lambda^\mu T_\nu^\lambda \\ &= \delta_\nu^\mu \end{aligned}$$

where the last step follows from the previous exercise (for any p, T is invertible and square, so both ST and TS is the identity). Hence the dual basis exists. It's unique because if we had started with another set of basis function satisfying the same property, they would again have had to be inverses to T at each point, and inverses are unique (note that we could have started by defining the S_λ^μ as the expansion for f^μ).

Exercise 40

By the definition of dual basis, we know that

$$f'^\mu(e'_\lambda) = \delta_\lambda^\mu,$$

and it is uniquely defined by this property. So we just need to show that the expression on the right hand side gives the same delta function when applied to a basis vector field.

$$\begin{aligned}
(T^{-1})^{\mu}_{\nu} f^{\nu}(e'_{\lambda}) &= (T^{-1})^{\mu}_{\nu} f^{\nu}(T^{\omega}_{\lambda} e_{\omega}) \\
&= (T^{-1})^{\mu}_{\nu} T^{\omega}_{\lambda} f^{\nu}(e_{\omega}) \\
&= (T^{-1})^{\mu}_{\nu} T^{\omega}_{\lambda} \delta^{\nu}_{\omega} \\
&= (T^{-1})^{\mu}_{\nu} T^{\nu}_{\lambda} \\
&= \delta^{\mu}_{\lambda}
\end{aligned}$$

For transforming between the components of the vector field, we just use the tranformation matrices between the different bases:

$$\begin{aligned}
v'^{\mu} e'_{\mu} &= v^{\mu} e_{\mu} \\
&= v^{\mu} (T^{-1})^{\lambda}_{\mu} e'_{\lambda} \\
&= v^{\nu} (T^{-1})^{\mu}_{\nu} e'_{\mu}
\end{aligned}$$

so by equating coefficients we get the desired result

$$v'^{\mu} = (T^{-1})^{\mu}_{\nu} v^{\nu}$$

For transforming between the 1-form components we do the same

$$\begin{aligned}
\omega'_{\mu} f'^{\mu} &= \omega_{\mu} f^{\mu} \\
&= \omega_{\nu} f^{\nu} \\
&= \omega_{\nu} T^{\nu}_{\mu} f'^{\mu} \\
&= T^{\nu}_{\mu} \omega_{\nu} f'^{\mu}
\end{aligned}$$

and hence

$$\omega'_{\mu} = T^{\nu}_{\mu} \omega_{\nu}.$$

Exercise 41

Let

$$u = u_x dx + u_y dy + u_z dz$$

and

$$\begin{aligned}
v \wedge w &= (v_x w_y - v_y w_x) dx \wedge dy \\
&\quad + (v_y w_z - v_z w_y) dy \wedge dz \\
&\quad + (v_z w_x - v_x w_z) dz \wedge dx
\end{aligned}$$

Then

$$\begin{aligned} u \wedge (v \wedge w) &= u_x(v_y w_z - v_z w_y) dx \wedge dy \wedge dz \\ &\quad + u_y(v_z w_x - v_x w_z) dy \wedge dz \wedge dx \\ &\quad + u_z(v_x w_y - v_y w_x) dz \wedge dx \wedge dy \\ &= \det(M) dx \wedge dy \wedge dz \end{aligned}$$

where we use $dx \wedge dx = 0$ (and same for y and z) for the first step, and $dx \wedge dy = -dy \wedge dx$ for the second step. M is the matrix given in the question.

Using definitions of dot product and cross product, we get

$$\overrightarrow{u} \cdot (\overrightarrow{v} \times \overrightarrow{w}) = \det(M).$$

Exercise 42

By the previous exercise, we get

$$a \wedge b \wedge c \wedge d = (a_x dx + a_y dy + a_z dz) \wedge (k\, dx \wedge dy \wedge dz)$$

where $k \in \mathbb{R}$. Expanding this, and using $dx \wedge dx = 0$ (and same for y and z) gives the desired result.

Exercise 43

- If V is 1-dimensional, then $\wedge V$ consists of all linear combinations of 1 and dx.
- If V is 2-dimensional, then $\wedge V$ consists of all linear combinations of 1, dx, dy and $dx \wedge dy$.
- If V is 4-dimensional, then $\wedge V$ consists of linear combinations of 16 different elements.

So it appears that $\dim_k(\wedge V) = 2^{\dim V}$, where V is a vector space over the field k. (This is for $\wedge V$ seen as a vector space.)

Exercise 44

Let $\{v_1, \ldots, v_n\}$ be a basis for V. Then $\wedge^p V$ is generated by elements of the form

$$v_{i_1} \wedge \cdots \wedge v_{i_p}$$

where each i_j is an index between 1 and n. This follows from the definition of $\wedge^p V$ by writing each vector in the wedge product as a linear combination of basis vectors.

We can restrict ourselves to elements of this form for which $i_j \leq i_{j+1}$. This is because rearranging the vectors such that the indices are in non-decreasing order will at most change the sign of the vector, so it's still a linear multiple of the original vector.

Note that if there is an index j for which $i_j = i_{j+1}$, then $v_{i_j} \wedge v_{i_{j+1}} = 0$ and hence the entire element is 0. If $p > n$ then this must happen every time, and hence $\wedge^p V = \{0\}$ in this case (or is it empty as the text claims?).

So we assume that $p \leq n$ and the indices are strictly increasing. It's clear (i.e. I'm not sure how prove it) that the set of such elements is independent. Thus this set forms a basis for $\wedge^p V$.

The number of such elements (the dimension of $\wedge^p V$) is the number of ways of choosing p objects from a set of n objects, which is the given expression.

Exercise 45

Well, the given subspaces generate the entire space, and the intersection of any two distinct such subspaces is $\{0\}$. This implies the first part.

Using the first part, the dimension of $\wedge^p V$ can be calculated as the sum of the dimensions of the subspaces. The expression for each of these was given in the previous exercise, and the fact that their sum is 2^n is a well-known result in combinatorics.