Exercise 2

The definition of an open set given in the text can be made more precise as follows:

$V \subseteq \mathbb{R}$ is defined to be open if for all $x \in V$ there exists $\delta > 0$ such that $B(x, \delta) \subseteq V$, where $B(x, \delta)$ is defined as $\{y \in \mathbb{R}^n \mid |x - y| < \delta\}$.

It can be shown that $B(x, \delta)$ is open according to this definition for any $x \in \mathbb{R}^n$, $\delta > 0$.

To prove the given statement, let $f : \mathbb{R}^n \to \mathbb{R}^n$ and suppose that $f^{-1}(V)$ is open for any $V \subseteq \mathbb{R}^n$ open (i.e. f is continuous according to the general topology definition). Let $x \in \mathbb{R}^n$ and $\epsilon > 0$. Let $V = B(f(x), \epsilon)$. Then V is open, and hence $f^{-1}(V)$ is open. Thus by the above definition of 'open', there exists $\delta > 0$ such that $x \in B(x, \delta) \subset f^{-1}(V)$. In other words, if $|y - x| < \delta$ then $|f(y) - f(x)| < \epsilon$, which proves the one direction.

For the converse, assume that f is continuous according to the epsilon-delta definition. Let $x \in f^{-1}(V)$. Since $f(x) \in V$, which is open, there exists $\epsilon > 0$ such that $f(x) \in B(f(x), \epsilon) \subset V$. Then since f is continuous, there exists $\delta > 0$ such that $f(B(x, \delta)) \subset B(f(x), \epsilon) \subset V$. Which implies that $B(x, \delta) \subset f^{-1}(V)$, hence $f^{-1}(V)$ is open. So f is continuous according to the definition from topology.

Exercise 3

Let $z = (0, 0, \ldots, 0, 1) \in S^n$. Define $U = S^n \setminus z$. Then $U = (\mathbb{R}^{n+1} \setminus z) \bigcap S^n$ is open in the induced topology since $(\mathbb{R}^{n+1} \setminus z) \subset \mathbb{R}^{n+1}$ is open. Define the stereographic projection $f : U \to \mathbb{R}^n$:

$f(x_1, x_2, \ldots, x_{n+1}) = (\frac{x_1}{1 - x_{n+1}}, \frac{x_2}{1 - x_{n+1}}, \ldots, \frac{x_n}{1 - x_{n+1}})$

It can be shown that f is a homeomorphism between U and $\mathbb{R}^n$ (the inverse can be given explicitly, and both are continuous). Similarly, we can define an open set $V = S^n \setminus \{-z\}$, with a similar homeomorphism to $\mathbb{R}^n$. The transition function can be shown to be smooth. Since U and V cover S^n, this proves that S^n with the given charts is a smooth manifold.

Exercise 4

Suppose that $\{(A_i, \phi_i)\}$ is an atlas for M. Then $\{(A_i \cap U, \tilde{\phi_i})\}$ is an atlas for U, where $\tilde{\phi_i}$ is the restriction of ϕ_i to U. The transition functions for the new atlas are exactly the same, only restricted to an open subset of their original domain.

Exercise 5

Let $\{(A_i, \alpha_i) \mid i \in I\}$ be an atlas for M, and $\{(B_j, \beta_j) \mid j \in J\}$ an atlas for N. Then define the open sets $C_{ij} = A_i \times B_j \subseteq M \times N$. The family C_{ij} forms an open cover for $M \times N$. Define $\delta_{ij} : A_i \times B_j \to \mathbb{R}^{m+n}$ by mapping (a, b) to $(\alpha_i(a), \beta_j(b))$. From the properties of α_i and β_j follows that δ_{ij} is a homeomorphism between $A_i \times B_j$ and $\alpha_i(A_i) \times \beta_j(B_j)$. The family of charts

$\{(C_{ij}, \delta_{ij}) \mid i, j \in I \times J\}$ is an atlas for $M \times N$. The new transition functions can also be expressed in terms of the original transition functions and shown to be smooth.

Exercise 6

In this case we first need to find two disjoint open subsets of $\mathbb{R}^n$ which are both diffeomorphic to $\mathbb{R}^n$. Define $X^+ = \{x \in \mathbb{R}^n \mid x_n > 0\}$. Then a diffeomorphism f^+ between $\mathbb{R}^n$ and X^+ is given by mapping $(x_1, x_2, \ldots, x_n)$ to $(x_1, x_2, \ldots, \exp(x_n))$. Similarly, a diffeomorphism f^-can be defined mapping $\mathbb{R}^n$ to $X^- = \{x \in \mathbb{R}^n \mid x_n < 0\}$ by mapping x_n to $-\exp(x_n)$.

Now let atlases for M and N be given as in the previous exercise. Then $\{(A_i \cup B_j)\}$ is a family of open sets covering $M \cup N$. Define $\delta_{ij} : A_i \cup B_j \to \mathbb{R}^n$ by mapping x to $f^+(\alpha_i(x))$ if $x \in A_i$, and $f^-(\beta_j(x))$ if $x \in B_j$. Since X^+ and X^- are disjoint, these maps are still bijective. The resulting atlas satisfies the requirements for making $M \cup N$ a manifold.

Exercise 7

We need to show that both $v + w$ and gw satisfy the three conditions at the top of page 26.

First for $v + w$:

$$\begin{aligned}(v+w)(f+g) &= v(f+g) + w(f+g) \\ &= v(f) + v(g) + w(f) + w(g) \\ &= (v+w)(f) + (v+w)(g)\end{aligned}$$

$$\begin{aligned}(v+w)(\alpha f) &= v(\alpha f) + w(\alpha f) \\ &= \alpha v(f) + \alpha w(f) \\ &= \alpha(v(f) + w(f))\end{aligned}$$

$$\begin{aligned}(v+w)(fg) &= v(fg) + w(fg) \\ &= v(f)g + fv(g) + w(f)g + fw(g) \\ &= (v(f) + w(f))g + f(v(g) + w(g)) = (v+w)(f)g + f(v+w)(g)\end{aligned}$$

Then for gw:

$$\begin{aligned}(gw)(f_1 + f_2) &= gw(f_1 + f_2) \\ &= g(wf_1 + wf_2) \\ &= gwf_1 + gwf_2 \\ &= (gw)f_1 + (gw)f_2\end{aligned}$$

$$
\begin{aligned}
(gw)(\alpha f) &= gw(\alpha f) \\
&= g\alpha w(f) \\
&= \alpha g w(f)
\end{aligned}
$$

$$
\begin{aligned}
(gw)(f_1 f_2) &= gw(f_1 f_2) \\
&= g(w(f_1)f_2 + f_1 w(f_2)) \\
&= gw(f_1)f_2 + gf_1 w(f_2) \\
&= g(w(f_1)f_2 + f_1 w(f_2)) \\
&= gw(f_1 f_2)
\end{aligned}
$$

Exercise 8

To show that two vector fields are equal, we need to show that they are equal for all $h \in C^\infty(M)$:

$$
\begin{aligned}
(f(v+w))(h) &= f(v+w)(h) \\
&= f(v(h) + w(h)) \\
&= fv(h) + fw(h) \\
&= (fv)(h) + (fw)(h) \\
&= (fv + fw)(h)
\end{aligned}
$$

$$
\begin{aligned}
((f+g)v)(h) &= (f+g)v(h) \\
&= fv(h) + gv(h) \\
&= (fv)(h) + (gv)(h) \\
&= (fv + gv)(h)
\end{aligned}
$$

$$
\begin{aligned}
((fg)v)(h) &= (fg)v(h) \\
&= f(gv(h)) \\
&= f((gv)(h)) \\
&= (f(gv))(h)
\end{aligned}
$$

$$
\begin{aligned}
(1v)(h) &= 1v(h) \\
&= v(h)
\end{aligned}
$$

Exercise 9

Apply the vector field to the coordinate functions:

$$
\begin{aligned}
v^\mu \partial_\mu x_i &= v^i \frac{\partial x_i}{\partial x_i} \\
&= v^i
\end{aligned}
$$

Thus $v^i = 0$ for all i.