Exercise 16

First consider the left side. Note that $\phi \circ \gamma$ is a composition of smooth functions, and hence it is smooth. Also,

$$(\phi \circ \gamma)(t) = \phi(\gamma(t)) = \phi(p) \in N,$$

and thus

$$(\phi \circ \gamma)'(t) \in T_{\phi(p)}N.$$

Then

$$\begin{aligned}
(\phi \circ \gamma)'(t)(f) &= \frac{d}{dt} f(\phi \circ \gamma)(t)) \\
&= \frac{d}{dt}(f \circ \phi \circ \gamma)(t)) \\
&= \frac{d}{dt}((f \circ \phi)(\gamma(t))) \\
&= \gamma'(t)(f \circ \phi) \\
&= \gamma'(t)(\phi^* f) \\
&= \phi_*(\gamma'(t))(f)
\end{aligned}$$

where $f \in C^\infty(N)$, and hence

$$(\phi \circ \gamma)'(t) = \phi_*(\gamma'(t)).$$

Exercise 17

Let $v, w \in T_pM$ and $f \in C^\infty(N)$. Then

$$\begin{aligned}
\phi_*(v+w)(f) &= (v+w)(\phi^* f) && \text{(definition of pushforward)} \\
&= v(\phi^* f) + w(\phi^* f) && \text{(definition of addition in tangent space)} \\
&= (\phi_* v)(f) + (\phi_* w)(f) && \text{(definition of pushforward)} \\
&= (\phi_* v + \phi_* w)(f) && \text{(definition of addition in tangent space)}
\end{aligned}$$

which implies that

$$\phi_*(v+w) = \phi_* v + \phi_* w.$$

Now let $a \in \mathbb{R}$. Then

$$\begin{aligned}
(\phi_*(av))(f) &= (av)(\phi^* f) && \text{(definition of pushforward)} \\
&= a[v(\phi^* f)] && \text{(scalar multiplication in tangent space)} \\
&= a[(\phi_* v)(f)] && \text{(pushforward)} \\
&= (a(\phi_* v))(f) && \text{(scalar multiplication)}
\end{aligned}$$

and hence

$$\phi_*(av) = a(\phi_* v).$$

Exercise 18

Let v be a vector field on M. Then define

$$\phi_* v \,:\, C^\infty(N) \to C^\infty(N)$$

by mapping f to $v(f \circ \phi) \circ \phi^{-1}$. To show that $\phi_* v$ is a vector field on N, we should show that it is linear and satisfies the product rule. We only show that it satisfies the first of the three properties (linearity is two properties) and assume the other two:

$$\begin{aligned}
(\phi_* v)(f+g) &= v((f+g)\circ\phi)\circ\phi^{-1} \\
&= v(f\circ\phi + g\circ\phi)\circ\phi^{-1} \\
&= [v(f\circ\phi) + v(g\circ\phi))\circ\phi^{-1} \\
&= v(f\circ\phi)\circ\phi^{-1} + v(g\circ\phi)\circ\phi^{-1} \\
&= (\phi_* v)(f) + (\phi_* v)(g)
\end{aligned}$$

where $f, g \in C^\infty(N)$.

Now let $p \in M$, and $q = \phi(p) \in N$. And let $f \in C^\infty(N)$. Then

$$\begin{aligned}
(\phi_* v)_q(f) &= [(\phi_* v)(f)](q) \\
&= (v(f\circ\phi)\circ\phi^{-1})(q) \\
&= [v(f\circ\phi)](p) \\
&= (v(\phi^* f))(p) \\
&= v_p(\phi^* f) \\
&= (\phi_*(v_p))(f)
\end{aligned}$$

and hence

$$(\phi_* v)_q = \phi_*(v_p).$$

Exercise 19

Denote the components of ϕ by u and v. In other words,

$$\phi(x,y) = (u(x,y), v(x,y)).$$

Then, as used in a previous exercise, we have:

$$\begin{aligned}
u(x,y) &= (\cos\theta)x - (\sin\theta)y \\
v(x,y) &= (\sin\theta)x + (\cos\theta)y
\end{aligned}$$

Let $f \in C^\infty(\mathbb{R}^2)$, seen as a function of (u, v). Then:

$$\begin{aligned}
(\phi_* \partial_x)(f) &= \partial_x(\phi^* f) \\
&= \frac{\partial}{\partial x}(f \circ \phi) \\
&= \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \\
&= (\partial_u f)\cos\theta + (\partial_v f)\sin\theta \\
&= (\partial_x f)\cos\theta + (\partial_y f)\sin\theta \\
&= (\cos\theta\partial_x + \sin\theta\partial_y)(f)
\end{aligned}$$

where in the second-last line we consider f as a function of (x, y) instead. It follows that

$$\phi_*\partial_x = \cos\theta\partial_x + \sin\theta\partial_y.$$

Similarly, we can show:

$$\phi_*\partial_y = -\sin\theta\partial_x + \cos\theta\partial_y.$$

Exercise 20

Suppose $\gamma(t)$ is an integral curve through $\gamma(0) = p = (a, b) \in \mathbb{R}^2$ of v. Then for t in a small interval around 0, we must have

$$\gamma'(t) = v_{\gamma(t)}.$$

Setting $\gamma(t) = (x(t), y(t))$, this can be rewritten as

$$x'(t)\partial_x + y'(t)\partial_y = (x(t))^2\partial_x + (y(t))\partial_y,$$

and given than $\partial_x$and ∂_y are independent, this is equivalent to:

$$\begin{aligned}
x'(t) &= (x(t))^2 \\
y'(t) &= y(t)
\end{aligned}$$

The solution depends on $p = (a, b)$. We can distinguish between the following cases:

1. $(a, b) = (0, 0)$. Then

$$\begin{aligned}
x(t) &= 0 \\
y(t) &= 0
\end{aligned}$$

is a solution defined for all t.

2. $a \neq 0$, $b = 0$. Then

$$\begin{aligned}
x(t) &= \frac{a}{1 - at} \\
y(t) &= 0
\end{aligned}$$

is a solution defined for $t > \frac{1}{a}$ if $a < 0$, and $t < \frac{1}{a}$ if $a > 0$. In other words, the solution is not defined for all t, and the image of γ is either the positive or negative x-axis. Note that the solution for $x(t)$ is simply $-\frac{1}{t}$, which has been reparametrised to force $x(0) = a$.

3. $a = 0$, $b \neq 0$. Then

$$\begin{aligned} x(t) &= 0 \\ y(t) &= be^t \end{aligned}$$

is a solution defined either for $t > 0$ or $t < 0$, again depending on the sign of b. The image of γ is either the positive or negative y-axis.

4. $a \neq 0$, $b \neq 0$. Then

$$\begin{aligned} x(t) &= \frac{a}{1 - at} \\ y(t) &= be^t \end{aligned}$$

is again only defined for either positive or negative t. It's image is a curve lying either in the first two quadrants, or in the last two quadrants.

These solutions exhaust the possibilities for $p = (a, b)$. The only integral curve defined for all t, is the first one, thus this vector field is not integrable.

Exercise 21

The first part follows from the definition of an integral curve through a point.

For the second part, fix $s = s_0 \in \mathbb{R}$. Define

$$\psi : \mathbb{R} \to \mathbb{R}$$

by $\psi(t) = s_0 + t$. For simplity, let $q = \phi_{s_0}(p)$. Now consider two different curves $\mathbb{R} \to X$.

The first is $\phi_t(q)$. It has the property that $\phi_0(q) = q$, and

$$\frac{d}{dt}|_{t=0}\phi_t(q) = v_q.$$

The other curve is the composition

$$\phi_s(p) \circ \psi,$$

seen as a function of t. Note that

$$\begin{aligned} (\phi_s(p) \circ \psi)(t) &= \phi_{\psi(t)}(p) \\ (\phi_s(p) \circ \psi)(0) &= \phi_{s_0}(p) \\ &= q \end{aligned}$$

and

$$\begin{aligned}
\frac{d}{dt}|_{t=0}(\phi_s(p)\circ\psi)(t) &= \frac{d}{ds}|_{s=s_0}\phi(p).\frac{d}{dt}|_{t=0}\psi(t)\\
&= v_{\phi_{s_0}(p)}.1\\
&= v_q
\end{aligned}$$

Thus both these functions have the same value and derivative at $t=0$. The first function is an integral curve of X, and the second is simply a reparametrisation of an integral curve of X, and hence also an integral curve. By the appropriate uniqueness property of differential equations, they must actually be equal for all $t\in\mathbb{R}$. In other words, for all t:

$$\begin{aligned}
\phi_t(q) &= (\phi_s(p)\circ\psi)(t)\\
\phi_t(\phi_{s_0}(p)) &= \phi_{\psi(t)}(p)\\
&= \phi_{s_0+t}(p)
\end{aligned}$$

This holds for all $s_0\in\mathbb{R}$, and hence:

$$\phi_t(\phi_s(p)) = \phi_{s+t}(p).$$

Exercise 22

Let $f\in C^\infty(\mathbb{R}^2)$. Denote $\dfrac{\partial f}{\partial x}$ by f_x, and $\dfrac{\partial^2 f}{\partial x^2}$ by f_{xx}, and similarly for y. Then:

$$\begin{aligned}
[v,w](f) &= (vw-wv)f\\
&= v(wf)-w(vf)
\end{aligned}$$

$$wf = \frac{xf_y-yf_x}{\sqrt{x^2+y^2}}$$

$$\begin{aligned}
v(wf) &= \frac{x}{\sqrt{x^2+y^2}}\left(\frac{\partial}{\partial x}\frac{xf_y-yf_x}{\sqrt{x^2+y^2}}\right)+\frac{y}{\sqrt{x^2+y^2}}\left(\frac{\partial}{\partial y}\frac{xf_y-yf_x}{\sqrt{x^2+y^2}}\right)\\
&= \dots\\
&= \frac{xf_y-xyf_{xx}+xyf_{yy}-yf_x}{x^2+y^2}+\frac{(yf_x-xf_y)}{x^2+y^2}
\end{aligned}$$

$$vf = \frac{xf_x+yf_y}{\sqrt{x^2+y^2}}$$

$$\begin{aligned}
w(vf) &= \frac{x}{\sqrt{x^2+y^2}}\left(\frac{\partial}{\partial y}\frac{xf_x+yf_y}{\sqrt{x^2+y^2}}\right)-\frac{y}{\sqrt{x^2+y^2}}\left(\frac{\partial}{\partial x}\frac{xf_x+yf_y}{\sqrt{x^2+y^2}}\right)\\
&= \dots\\
&= \frac{xf_y-xyf_{xx}+xyf_{yy}-yf_x}{x^2+y^2}
\end{aligned}$$

(A few steps have been left out to keep it short.)

$$\begin{aligned}
[v,w](f) &= v(wf) - w(vf) \\
&= \frac{yf_x - xf_y}{x^2+y^2} \\
&= (\frac{y\partial_x - x\partial_y}{x^2+y^2})(f)
\end{aligned}$$

Therefore:

$$[v,w] = (\frac{y\partial_x - x\partial_y}{x^2+y^2})$$

Exercise 23

Applying the given expression for $(vf)(p)$ with wf instead of f, yields:

$$\begin{aligned}
(vwf)(p) &= \frac{d}{dt}(wf)(\phi_t(p))|_{t=0} \\
&= \frac{d}{dt}(\frac{\partial}{\partial s}f(\psi_s(\phi_t(p)))|_{s=0})|_{t=0} \\
&= \frac{\partial^2}{\partial t \partial s}f(\psi_s(\phi_t(p)))|_{s=t=0}
\end{aligned}$$

A similar derivation yields a similar expression for $(wvf)(p)$, and subtracting the two gives the desired expression.

Exercise 24

1.

$$\begin{aligned}
[v,w](f) &= v(wf) - w(vf) \\
&= -[w(vf) - v(wf)] \\
&= -([w,v](f)) \\
&= (-[w,v])(f)
\end{aligned}$$

which implies that

$$[v,w] = -[w,v].$$

2.

$$\begin{aligned}
[u, \alpha v + \beta w](f) &= u((\alpha v + \beta w)(f)) - (\alpha v + \beta w)(uf) \\
&= u(\alpha vf + \beta wf) - (\alpha v(uf) + \beta w(uf)) \\
&= \alpha u(vf) + \beta u(wf) - \alpha v(uf) - \beta w(uf) \\
&= \alpha[u,v](f) + \beta[u,w](f) \\
&= (\alpha[u,v] + \beta[u,w])(f)
\end{aligned}$$

implying that

$$[u, \alpha v + \beta w] = \alpha[u, v] + \beta[u, w].$$

3.

$$\begin{aligned}
[u, [v, w]](f) &= u([v, w](f)) - [v, w](uf) \\
&= u(vwf - wvf) - (vw - wv)(uf) \\
&= uvwf - uwvf - vwuf + wvuf
\end{aligned}$$

which gives an expression for the first term:

$$[u, [v, w]] = uvw - uwv - vwu + wvu.$$

The expressions for the other two terms can be found by permuting the variables, and adding them yields:

$$\begin{aligned}
[u, [v, w]] + [v, [w, u]] + [w, [u, v]] &= uvw - uwv - vwu + wvu + \\
&\quad vwu - vuw - wuv + uwv + \\
&\quad wuv - wvu - uvw + vuw \\
&= 0
\end{aligned}$$