Baez and Muniain Exercises

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Symmetry

Exercise 1

Let Q be the bilinear form of the Minkowski metric and L be the Lorentz transform mixing up the t and x coordinates.

Let $X = (x, y, z, t)\,,\ X' = (x', y', z', t') \in \mathbb{R}^4$.

$$\begin{aligned}
Q(LX, LX') &= -(t\cosh\phi - x\sinh\phi)(t'\cosh\phi - x'\sinh\phi) \\
&\quad + (-t\sinh\phi + x\cosh\phi)(-t'\sinh\phi + x'\cosh\phi) + yy' + zz' \\
&= -tt'\cosh^2\phi - xx'\sinh^2\phi + xt'\cosh\phi\ \sinh\phi + tx'\cosh\phi\ \sinh\phi \\
&\quad + tt'\cosh^2\phi + xx'\sinh^2\phi - xt'\cosh\phi\ \sinh\phi - tx'\cosh\phi\ \sinh\phi + yy' + zz' \\
&= -tt'\left(\cosh^2\phi - \sinh^2\phi\right) + xx'\left(\cosh^2\phi - \sinh^2\phi\right) + yy' + zz' \\
&= -tt' + xx' + yy' + zz' \\
&= Q(X, X')\,.
\end{aligned}$$

Then $L \in O(3,1)$. Since $\mathsf{det}(L) = \cosh^2\phi - \sinh^2\phi = 1$, $L \in SO(3,1)$.

With the same way wee can prove that Lorentz transforms mixing up t and y coordinates or t and z coordinates belong to $SO(3,1)$.

Exercise 2

The linear maps $P : (t, x, y, z) \longmapsto (t, -x, -y, -z)$ and $T : (t, x, y, z) \longmapsto (-t, x, y, z)$ can be represented by the matrices $\mathsf{diag}(1, -1, -1, -1)$ and $\mathsf{diag}(-1, 1, 1, 1)$ respectively, so $\mathsf{det}(P) = \mathsf{det}(T) = -1$.
Therefore $T, P \notin SO(3,1)$

Let $X = (x, y, z, t)\,,\ X' = (x', y', z', t') \in \mathbb{R}^4$.

$$\begin{aligned}
Q(PX, PX') &= -tt' + (-x)(-x') + (-y)(-y') + (-z)(-z') \\
&= -tt' + xx' + yy' + zz' \\
&= Q(X, X')\,,
\end{aligned}$$

and

$$\begin{aligned}
Q(TX, TX') &= -(-t)(-t') + xx' + yy' + zz' \\
&= -tt' + xx' + yy' + zz' \\
&= Q(X, X')\,.
\end{aligned}$$

Then $T, P \in O(3,1)$. It follows that $PT \in O(3,1)$. Since $\mathsf{det}(PT) = \mathsf{det}(P)\,\mathsf{det}(T) = 1$, $PT \in SO(3,1)$.

Exercise 3

1. The "restriction" of the determinant, $\mathsf{det} : \mathsf{GL}(n,\mathbb{R}) \longrightarrow \mathbb{R}^*$ is a group homomorphism with the multiplication. Therefore $\mathsf{SL}(n,\mathbb{R})$ is a subgroup since $\mathsf{SL}(n,\mathbb{R}) = \mathsf{det}^{-1}(1) = \ker(\mathsf{det})$.

2. We denote Q the bilinear form associated to $O(p,q)$. Let $A, B \in O(p,q)$ and $X, X' \in \mathbb{R}^4$.

$$Q(ABX, ABX') = Q(BX, BX') = Q(X, X'),$$

and

$$Q(X, X') = Q\left(BB^{-1}X, BB^{-1}X'\right) = Q\left(B^{-1}X, B^{-1}X'\right).$$

Therefore $AB,\ B^{-1} \in O(p,q)$. It follows that $O(p,q)$ is a subgroup of $\mathsf{GL}(n,\mathbb{R})$.
Since $SO(p,q) = O(p,q) \cap \mathsf{SL}(n,\mathbb{R})$, $SO(p,q)$ is also a subgroup of $\mathsf{GL}(n,\mathbb{R})$.

The other proofs are similar.

Exercise 4

1. The set of matrices $M(n,\mathbb{R})$ has a natural structure of manifold isomorphic to $\mathbb{R}^{n^2}$. The determinant $\mathsf{det} : M(n,\mathbb{R}) \longrightarrow \mathbb{R}$ is a smooth map and $\mathsf{GL}(n,\mathbb{R}) = \mathsf{det}^{-1}(\mathbb{R} \setminus \{0\})$, therefore $\mathsf{GL}(n,\mathbb{R})$ is an open subset of $M(n,\mathbb{R})$. It follows that $\mathsf{GL}(n,\mathbb{R})$ is a sub-manifold of $M(n,\mathbb{R})$ of dimension n^2. Multiplication and inverse of matrices in $\mathsf{GL}(n,\mathbb{R})$ are smooth since they are polynomial and fractional function of the components. Thus $\mathsf{GL}(n,\mathbb{R})$ is a Lie group.

   By analogous reasoning $\mathsf{GL}(n,\mathbb{C})$ is a complex (resp. real) Lie group of complex dimension n^2 (resp. real dimension $2n^2$ ($\mathbb{C}^{n^2} \cong \mathbb{R}^{2n^2}$)).

2. Here we will prove that the determinant is a submersion on $\mathsf{GL}(n,\mathbb{R})$. Let $X \in \mathsf{GL}(n,\mathbb{R})$, by expandin-galong the i^{th} column we have

$$\mathsf{det}X = \sum_k (-1)^{i+k} X_{ik} |X_i^k|$$

where $|X_i^k|$ is the determinant of the (i,k) minor of X. Then

$$\begin{aligned} \frac{\partial \mathsf{det}X}{\partial X_{ij}} &= (-1)^{i+j} |X_i^j| \\ &= \mathsf{det}X \left(\frac{(-1)^{i+j} |X_i^j|}{\mathsf{det}X} \right). \\ &= \mathsf{det}X \ \left(X^{-1}\right)_{ji} \end{aligned}$$

Thus

$$\begin{aligned} d(\mathsf{det})_X &= \sum_{i,j} \mathsf{det}X \ \left(X^{-1}\right)_{ji} \mathrm{d}X_{ij} \\ &= \mathsf{det}X \ \sum_{i,j} \ \left(X^{-1}\right)_{ji} \mathrm{d}X_{ij} \end{aligned}.$$

Therefore, for $B \in T_X \mathsf{GL}(n, \mathbb{R}) \cong M(n, \mathbb{R})$,

$$\begin{aligned} d(\mathsf{det})_X(B) &= \mathsf{det} X \left(\sum_{i,j} \left(X^{-1}\right)_{ji} \mathrm{d}X_{ij} \right)(B) \\ &= \mathsf{det} X \sum_{i,j} \left(X^{-1}\right)_{ji} B_{ij} \\ &= \mathsf{det} X \;\; tr\left(X^{-1}B\right) \end{aligned}$$

where $tr\left(X^{-1}B\right)$ is the trace of $X^{-1}B$. Now since $d(\mathsf{det})_X : M(n, \mathbb{R}) \longrightarrow \mathbb{R}$ is a linear map, $d(\mathsf{det})_X \neq 0$ if there exists $B \in M(n, \mathbb{R})$ suth that $d(\mathsf{det})_X(B) \neq 0$. Let chose $B = X$, we have

$$d(\mathsf{det})_X(B) = \mathsf{det} X \;\; tr\left(X^{-1}X\right) = \mathsf{det} X \;\; tr(I_n) = (\mathsf{det} X) \;\; n \neq 0.$$

Therefore $d(\mathsf{det})_X$ never vanishes for all $X \in \mathsf{GL}(n, \mathbb{R})$. Then the detrminant is a submersion on $\mathsf{GL}(n, \mathbb{R})$.

3. The determinant is a submersion on $\mathsf{GL}(n, \mathbb{R})$, so $\mathsf{SL}(n, \mathbb{R}) = \mathsf{det}^{-1}(1)$ is a submanifold of $\mathsf{GL}(n, \mathbb{R})$ of dimension $n^2 - 1$. Thus is a Lie subgroup of $\mathsf{GL}(n, \mathbb{R})$.

   By analogous reasoning $\mathsf{SL}(n, \mathbb{C})$ is a complex Lie group of complex dimension $n^2 - 1$ and then a real Lie group of real dimension $2\left(n^2 - 1\right)$.

4. Let $J = diag(I_p, -I_q)$ be the canonical representative of the quadratique form of $O(p, q)$

$$A \in O(p, q) \quad \text{if and only if} \quad {}^tAJA = J.$$

Let consider the map

$$\begin{aligned} M(n, \mathbb{R}) &\longrightarrow M(n, \mathbb{R}) \\ A &\longmapsto {}^t AJA \end{aligned}$$

Since ${}^t\left({}^tAJA\right) = {}^t AJA$, we can reduce this map to a map

$$\begin{aligned} G : M(n, \mathbb{R}) &\longrightarrow Sym(n, \mathbb{R}) \\ A &\longmapsto {}^t AJA \end{aligned}$$

Where $Sym(n, \mathbb{R})$ is the space of symetric $n \times n$ matrices over $\mathbb{R}$. $Sym(n, \mathbb{R})$ is a submanifold of $M(n, \mathbb{R})$ of dimension $n(n+1)/2$ since it's a subalgebra. The dimension is the number of entries for the "upper (or lower) triangle" of a $n \times n$ matrix. Since multiplication is smooth G is a smooth map between two manifolds.

$$dG = d\left({}^tAJA\right) = d\left({}^tA\right)JA + {}^t AJdA$$

$A \mapsto A$ and $A \mapsto {}^t A$ are linear maps so there differential are themselves. Thus for $B \in O(p, q)$, $dG_B : X \mapsto {}^t XJB + {}^t BJX$ and

$$\begin{aligned} dG_B(X) = 0 \quad \text{if and only if} \quad {}^tXJB &= -{}^tBJX \\ {}^t\left({}^tBJX\right) &= -{}^tBJX. \end{aligned}$$

Then $\ker dG_B = \{X \in M(n, \mathbb{R}) :{}^t BJX antisymetric\}$. Let $AS(n, \mathbb{R})$ be the space of antisymetric $n \times n$ matrices over $\mathbb{R}$. The map

$$\begin{aligned} \ker dG_B &\longrightarrow AS(n, \mathbb{R}) \\ X &\longmapsto {}^t BJX \end{aligned}$$

is an isomorphism of vector space (with invert $X \longmapsto \left({}^tBJ\right)^{-1} Y$). Since $\dim AS(n,\mathbb{R}) = n(n-1)/2$ (number of entries of the "strictly upper triangle" of a $n \times n$ matrix), $\dim(\ker dG_B) = n(n-1)/2$ and then

$$dim\left(\mathrm{Im}\left(dG_B\right)\right) = n^2 - n(n-1)/2 = n(n+1)/2 = \dim Sym(n,\mathbb{R}).$$

Therefore dG_B has maximal rank for all $B \in O(p,q)$. Thus $O(p,q) = G^{-1}(J)$ is a submanifold of $M(n,\mathbb{R})$ of dimension $n(n-1)/2$.

The fact that $O(p,q)$ is a Lie group is immediat.

5. Let $A \in O(p,q)$, the identity ${}^tAJA = J$ implies $\mathsf{det}(A)^2 = 1$. Therefore $\mathsf{det}(A) = \pm 1$. Then we have a continuous map

$$\begin{aligned} O(p,q) &\longrightarrow \{1,-1\} \\ A &\longmapsto \mathsf{det}(A) \end{aligned},$$

thus $SO(p,q)$ is an open set in $O(p,q)$ since it's the preimage of the open set $\{1\}$ under this map. Therefore $SO(p,q)$ is a submanifold of $O(p,q)$ of the same dimension. It follows that it's a Lie subgroup.

6. Using the fact that

$$U(n) = \{A \in M(n,\mathbb{C}) : A^*A = I\},$$

$$SU(n) = \{A \in U(n) : \mathsf{det} A = 1\}$$

and by the same process as for $O(p,q)$ and $SO(p,q)$, we can prove that $SU(n)$ and $U(n)$ are real Lie groups.

Exercise 5

Let G_0 be the identity component of the Lie group G and e its identity element. G_0 is a sub-manifold since connected components of a manifold are always sub-manifolds. The map

$$\begin{aligned} G_0 \times G_0 &\longrightarrow G \\ (g,h) &\longmapsto gh^{-1} \end{aligned}$$

is continuous since the group multiplication and the inverse are continuous. The image of a connected set by a continuous map is a connected set so $G_0G_0^{-1}$ is connected. $G_0 \subset G_0G_0^{-1}$ since $e \in G_0$, therefore $G_0 = G_0G_0^{-1}$ because G_0 is a connected component. Thus G_0 is a group. Multiplication and inverse in G_0 are still smooth, then G_0 is a Lie group.

Exercise 6

1. Let $A \in O(3)$. The characteristic polynomial of A has degree 3, so A has at least one real eigenvalue. Let λ be a real eigenvalue of A with eigenvector x, the fact that

$$< Ax, Ax >=< \lambda x, \lambda x >= \lambda^2 < x, x >=< x, x >$$

forces $\lambda^2 = 1$, therefore $\lambda = \pm 1$. Putting A into a normal form in an appropriate (real) basis (v_1, v_2, v_3) gives the identity or matrices of the form $\mathrm{Diag}(1,-1,-1)$, $\mathrm{Diag}(-1,1,1)$, or

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Re}[\mu] & -\mathrm{Im}[\mu] \\ 0 & \mathrm{Im}[\mu] & \mathrm{Re}[\mu] \end{pmatrix}, \qquad \begin{pmatrix} -1 & 0 & 0 \\ 0 & \mathrm{Re}[\mu] & -\mathrm{Im}[\mu] \\ 0 & \mathrm{Im}[\mu] & \mathrm{Re}[\mu] \end{pmatrix},$$

where μ is a complex eigenvalue of A. The identity ${}^tAA = I$ implies $\mathsf{det}A^2 = 1$, then $\mathsf{det}A = \pm 1$ and

$$\mathrm{Re}\,[\mu]^2 + \mathrm{Im}\,[\mu]^2 = 1.$$

We can then reduce all this case to

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, \qquad \theta \in \mathbb{R}.$$

The former is the rotation about the axis of the vector v_1 and the later is the composition of $\mathrm{Diag}\,(-1,1,1)$, the reflexion throught the plane generated by the vectors $v_2,\ v_3$ and a rotation like in the former case.

2. For $\theta \in \mathbb{R}$ let denote

$$A(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}.$$

From 1., the $A(\theta)$'s are the elements of $O(3)$ with determinant 1. Therefore $SO(3)$ is the collection of all these matrices. For θ fixed, let consider the map

$$\begin{aligned} \gamma : [0,1] &\longrightarrow SO(3) \\ t &\longmapsto A(t\theta) \end{aligned}$$

γ is continuous with $\gamma(0) = I_3$ and $\gamma(1) = A(\theta)$. Therefore $SO(3)$ is path connected then connected. Since we have the disjoint union

$$O(3) = SO(3) \amalg \mathsf{det}^{-1}(-1),$$

$SO(3)$ is the identity component of $O(3)$.

Exercise 7

The Lorentz group preserves the value of the expression $t^2 - x^2 - y^2 - z^2$, so we have a natural smooth action of $SO(3,1)$ on the hyperboloid H of equation

$$t^2 - x^2 - y^2 - z^2 = 1.$$

This action can be defined by:

$$\begin{aligned} F : SO(3,1) \times H &\longrightarrow H \\ (g, X) &\longmapsto gX \end{aligned}.$$

1. Let suppose that there exists a continuous path $\gamma(t)$ from the identity to the element PT in $SO(3,1)$. Let denote $x_0 = (1,0,0,0)$, the map $\gamma(t) \cdot x_0$ is then a continuous path in H from $(1,0,0,0)$ to $(-1,0,0,0)$, two points in the two disjoint components of H, which is impossible. Therefore there is no path from the identity to the element PT in $SO(3,1)$.

2. Let $U = \{(t,x,y,z) \in H : t > 0\}$, the connected component of H with $t > 0$.

   The action of the Lorentz group on H is continuous, $Id \cdot U = U$ and U is a connected component. Therefore we have an induced action

$$\begin{aligned} SO_0(3,1) \times U &\longrightarrow U \\ (g, X) &\longmapsto gX \end{aligned}.$$

of the identity component on U. We are going to show that this action is transitive.
Let $x_0 = (1, 0, 0, 0)$, the stabilizer of x_0 by the action of $SO(3, 1)$ is

$$Stab(x_0) = \{g \in SO(3, 1) : g \cdot x_0 = x_0\}$$

Let $g \in Stab(x_0)$, the identity $g \cdot x_0 = x_0$ implies $g_{11} = 1$, $g_{21} = g_{31} = g_{41} = 0$ and by using transpose we get $g_{12} = g_{13} = g_{14} = 0$. Then

$$g = \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix}.$$

Since $g \in SO_0(3, 1)$, A must preserve the Euclidean scalar product. Therefore

$$Stab(x_0) = \{\begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} : A \in SO(3)\}$$

and $Stab(x_0) \subset SO_0(3, 1)$.

Let $y \in U$. Since elements of $SO(3)$ are composition of rotations about some axis, we can find an element of $Stab(x_0)$ which map y to an element a of the hyperbol of equation

$$x_1^2 - x_2^2 = 1, \quad x_3 = x_4 = 0.$$

We have $a = (\cosh\phi, \sinh\phi, 0, 0)$ $(x_1 > 0)$ for some $\phi \in \mathbb{R}$. Then the transformation

$$L = \mathrm{Diag}\left(\begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}, 1, 1\right)$$

map a to $(1, 0, 0, 0)$. We can check that L belongs to $SO_0(3, 1)$. Therefore there exists $g \in SO_0(3, 1)$ such that $g \cdot y = (1, 0, 0, 0)$. Thus $SO_0(3, 1)$ act transitively on U.

Let denote $G := SO(3, 1)$ and $G_0 = SO_0(3, 1)$. Let $g \in G$,

- If $g \cdot x_0 \in U$, then there exists $g' \in G_0$ such that $g'g \cdot x_0 = x_0$ because G_0 act transitively on U. $Stab(x_0) \subset G_0$, therefore $g'g \in G_0$ and then $g \in G_0$.
- If $g \cdot x_0 \notin U$ then $PTgx_0 \in U$ and by the above result $PTg \in G_0$, thus $g \in (PT)^{-1} G_0 = PTG_0$.

Then we have either $g \in G_0$ or $g \in PTG_0$. It follows that $G = G_0 \cup PTG_0$. Since translations are continuous, PTG_0 is connected. Therefore $SO(3, 1)$ has two connected components, $SO_0(3, 1)$ and $PTSO_0(3, 1)$.