Exercise 2

First, assume f is continuous according to the topological definition. Let $\vec{x} \in \mathbb{R}^n$, and let $\varepsilon > 0$ be a real number. Consider the ε-neighbourhood $N_\varepsilon(f(\vec{x}))$ around $f(\vec{x})$. By the continuity of f, $f^{-1}\left(N_\varepsilon(f(\vec{x}))\right)$ is open in $\mathbb{R}^n$. This open set contains a neighbourhood about each of its points. Thus there exists a $\delta > 0$ such that $N_\delta(\vec{x}) \subseteq f^{-1}\left(N_\varepsilon(f(\vec{x}))\right)$. That is, $f\left(N_\delta(\vec{x})\right) \subseteq N_\varepsilon\left(f(\vec{x})\right)$. This δ thus satisfies the condition, and f is continuous at $\vec{x}$ in the "ordinary" sense. Since $\vec{x}$ was arbitrarily chosen, f is continuous on $\mathbb{R}^n$ via the ε-δ definition.

Conversely, suppose that f is continuous via the epsilon-delta definition on $\mathbb{R}^n$. Let $V \subseteq \mathbb{R}^m$ be open. Let $U = f^{-1}(V)$. By hypothesis f is continuous at every $\vec{x} \in U$. Thus there is a $\delta_{\vec{x}} > 0$ such that

$f\left(N_{\delta_{\vec{x}}}(\vec{x})\right) \subseteq V$. Thus $N_{\delta_{\vec{x}}}(\vec{x}) \subseteq U$, so that

$$U = \bigcup_{\vec{x} \in U} N_{\delta_{\vec{x}}}(\vec{x}).$$

Thus U is open.