

51. Locally the exterior derivative acts as

$$\begin{aligned} d\omega &= d(\omega_I dx^I) \\ &= \partial_\mu \omega_I dx^\mu \wedge dx^I \end{aligned}$$

which we can write as

$$= dx^\mu \wedge \partial_\mu \omega_I dx^I$$

In other words, we can think of  $d$  as the operator

$$d = dx^\mu \wedge \partial_\mu (\dots)$$

$$= dt \wedge \partial_t (\dots) + dx^\mu \wedge \partial_\mu (\dots)$$

$$\text{so } d\omega = dt \wedge \partial_t \omega + d_S \omega.$$