Baez and Munian

Exercises 41 - 45

41. $$u \wedge v \wedge w = (u_x dx + u_y dy + u_z dz) \wedge (v_x dx + v_y dy + v_z dz) \wedge (w_x dx + w_y dy + w_z dz)$$

If you stare at the right hand side, you see that the only term that survives is the $dx \wedge dy \wedge dz$ term, since eg. $dy \wedge dy = 0$.

Moreover, the coefficient of $dx \wedge dy \wedge dz$ will be the signed sum of all products like $u_x v_y w_z$. (We need to take the signed sum since eg. the term

$$u_y dy \wedge v_x dx \wedge w_z dz = -u_y v_x w_z \, dx \wedge dy \wedge dz$$

that is, we need a minus sign to interchange the dx's) Thus we have

$$u \wedge v \wedge w = \left(\sum_{\substack{\pi : \{x,y,z\} \to \{x,y,z\} \\ \text{permutations}}} \operatorname{sign} \pi \; u_{\pi(x)} v_{\pi(y)} w_{\pi(z)} \right) dx \wedge dy \wedge dz$$

$$= \det \begin{pmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{pmatrix} dx \wedge dy \wedge dz.$$

42. If we expand $a = a_x dx + a_y dy + a_z dz$ and do the same for b, c and d, then $a \wedge b \wedge c \wedge d$ will consist of terms which must repeat one of the dx's eg. $dx \wedge \underbrace{dy \wedge dy}_{=0} \wedge dz$. All such terms are zero.

43. <u>V is 1-dim</u>: vectors in ΛV take the form

$$a + b\epsilon \qquad (\Lambda V \text{ is 2-dimensional})$$

where $\epsilon \wedge \epsilon = 0$.

<u>V is 2-dim</u>: Vectors in ΛV take the form

ΛV is (4-dimensional)

$$a + b_1 e_1 + b_2 e_2 + c\, e_1 \wedge e_2 .$$

<u>V is 4-dim</u> vectors in ΛV take the form

$$\begin{aligned} a &+ b_1 e_1 + b_2 e_2 + b_3 e_3 + b_4 e_4 \\ &+ c_{12} e_1 \wedge e_2 + c_{13} e_1 \wedge e_3 + c_{14} e_1 \wedge e_4 \\ &\qquad + c_{23} e_2 \wedge e_3 + c_{24} e_2 \wedge e_4 + c_{34} e_3 \wedge e_4 \\ &+ d_{123} e_1 \wedge e_2 \wedge e_3 + d_{124} e_1 \wedge e_2 \wedge e_4 + d_{134} e_1 \wedge e_3 \wedge e_4 \\ &\qquad + d_{234} e_2 \wedge e_3 \wedge e_4 \\ &+ k \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4 \qquad (\Lambda V \text{ is 16-dimensional}). \end{aligned}$$

44. Clearly $\Lambda^p V$ is empty (equals the zero space) for $p > n$, since if we expand in a basis, any wedge product of greater than n vectors must have a "repeated wedge", which equals zero, eg:

$$e_1 \wedge e_3 \wedge e_5 \wedge e_2 \wedge e_1 \wedge e_4 = -\underbrace{e_1 \wedge e_1}_{=0} \wedge e_3 \wedge e_5 \wedge e_2 \wedge e_4 .$$

The vectors

$$e_{i_1} \wedge \cdots \wedge e_{i_p} \qquad 1 \leq i_1 < i_2 < \cdots < i_p \leq n$$

clearly span $\Lambda^p V$ by definition (any formal span of formal wedge products can be reduced into a sum of vectors of this form).

That they are <u>linearly independent</u> is a bit more tricky (one has to spell out the definition of ΛV a bit more precisely than we have here). Nevertheless, they <u>are</u> linearly independent. Clearly then we have

$$\dim \Lambda^p V = \binom{n}{p} = \frac{n!}{p!\,(n-p)!}$$

4.5. The spaces $\Lambda^p V$ are linearly independent from each other (there is no rule which can convert a vector in $\Lambda^p V$ to one in $\Lambda^q V$), hence $\Lambda V = \bigoplus_{p=0}^{n} \Lambda^p V$.

$$\therefore \dim \Lambda V = \sum_{p=0}^{n} \dim \Lambda^p V$$

$$= \sum_{p=0}^{n} \binom{n}{p} = 2^n.$$