Baez and Munian

Exercises 22 – 24

22. Use the best coordinates we can! We have

$$v = \partial_r \quad , \quad w = \frac{1}{r}\partial_\theta$$

since 1 unit of arclength around circle $= \frac{1}{r}$ times angular distance in radians.

To calculate components of $[v,w]$, we set

$$[v,w] = a_r \partial_r + a_\theta \partial_\theta$$

and then use exercise 9 which tells us how to calculate a_r and a_θ: evaluate the vector field on π_r and π_θ (the projection functions which send $(r,\theta) \xmapsto{\pi_r} r$, $(r,\theta) \xmapsto{\pi_\theta} \theta$).

We have:

$$[v,w]\pi_r = \partial_r \frac{1}{r} \underbrace{\partial_\theta \pi_r}_{=0} - \underbrace{\frac{1}{r}\partial_\theta \underbrace{\partial_r \pi_r}_{=1}}_{=0}$$

$$= 0.$$

$$[v,w]\pi_\theta = \partial_r \frac{1}{r} \underbrace{\partial_\theta \pi_\theta}_{=1} - \frac{1}{r}\partial_\theta \underbrace{\partial_r \pi_\theta}_{=0}$$

$$= -\frac{1}{r^2}.$$

So,

$$[v,w] = -\frac{1}{r^2}\partial_\theta = -\frac{1}{r} w .$$

<u>NB</u> The thing we're really using here is that polar coordinates are a <u>valid</u> system; i.e. that there's an explicit change of coordinates $(x,y) \mapsto (r,\theta)$ accomplishing transformation in exercise.

23. $$V(W(f))(p) = \frac{\partial}{\partial t} W(f)(\phi_t(p)) \Big|_{t=0}$$
$$= \frac{\partial}{\partial t}\frac{\partial}{\partial s} f(\psi_s \phi_t(p)) \Big|_{t=0}$$

and similarly for WV; this gives
$$[V,W](f)(p) = \frac{\partial^2}{\partial t \partial s}\Big[f(\psi_s\phi_t(p)) - f(\phi_t \psi_s(p)) \Big]\Big|_{t=0}$$

since we can swap the order of $\frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}$.

24. 1. $[V,W](f) = (VW - WV)(f) = -(WV - VW)(f) = -[W,V](f)$.

2. Clear: α and β are constants so they can come in front of the vector fields.

3. $$[U,[V,W]] = [U, VW - WV]$$
$$= UVW - UWV - VWU + WVU$$

The other terms are cyclic permutations;

so we get

$$[U,[V,W]] + \underset{\text{perm.}}{\text{cyclic}} + \underset{\text{perm.}}{\text{cyclic}} = \cancel{UVW} - \cancel{UWV} - \cancel{VWU} + \cancel{WVU}$$
$$+ \cancel{VWU} - \cancel{WVU} - \cancel{WUV} + \cancel{VUW}$$
$$+ \cancel{WUV} - \cancel{VUW} - \cancel{UVW} + \cancel{UWV}$$
$$= 0.$$