Baez and Munian

Exercise 2, 2 May 2010

2. Suppose $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous according to the "abstract" definition. Let $x \in \mathbb{R}^{\cancel{m}n}$, and let $B(f(x), \epsilon)$ be an open ball around $f(x)$ with radius ϵ. Since f is "abstractly continuous", we know that $f^{-1}(B(f(x),\epsilon))$ is open in $\mathbb{R}^n$, and hence it is a union of open balls around each of its points. Set δ to be the radius of the open ball around x; by definition this open ball maps into $B(f(x), \epsilon)$.

Conversely, suppose $f: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is continuous according to "$\epsilon - \delta$". Let V be open in $\mathbb{R}^m$. If $f^{-1}V = \phi$, we are done since ϕ is open in $\mathbb{R}^n$. Suppose $f^{-1}V = U$ for an open U in $\mathbb{R}^n$. Let $x \in U$. Then since V is open, there must be an open ball W around $f(x)$ which is contained in V. And since f is continuous according to "$\epsilon - \delta$", we can find an open ball around x mapping into W. In other words, we can find an open ball contained in U around each point $x \in U$. So U is open.