Baez and Munian

Exercise 16–19

16. $$(\phi\circ\gamma)'(t)(f) = \frac{d}{dt} f\circ(\phi\circ\gamma) \qquad [\text{defn}]$$

$$= \frac{d}{dt}(f\circ\phi)\circ\gamma$$

$$= \gamma'(t)\,(f\circ\phi) \qquad [\text{defn of } \gamma'(t)]$$

$$= \gamma'(t)\,(\phi^* f) \qquad [\text{defn of } \phi^*]$$

$$= \phi_*\,(\gamma'(t)) \qquad [\text{defn of } \phi_*].$$

17. $$\phi_*(v+w)(f) = (v+w)(\phi^* f) \qquad [\text{defn of } \phi_*]$$

$$= v(\phi^* f) + w(\phi^* f) \qquad [\text{defn of } v+w]$$

$$= (\phi_* v)(f) + (\phi_* w)(f) \qquad [\text{defn of } \phi_*]$$

$$= (\phi_* v + \phi_* w)(f) \qquad [\text{defn of addition of vectors in } T_{\phi(p)}N]$$

$$\therefore\ \phi_*(v+w) = \phi_* v + \phi_* w$$

Similarly, $$\phi_*(\lambda v)(f) = (\lambda v)(\phi^* f)$$

$$= \lambda\, v(\phi^* f)$$

$$= \lambda\,(\phi_* v)(f)$$

$$= [\lambda(\phi_* v)](f).$$

18.' To be explicit, we are really defining the vector field

$$\phi_* V : C^\infty(N) \longrightarrow C^\infty(N).$$

as

$$(\phi_* V)(f)(q) := \underbrace{V(\phi^* f)}_{\in C^\infty(M)}(p), \quad \text{where } p = \phi^{-1}(q).$$

$$\underline{\text{or}} \quad (\phi_* V)(f) = V(\phi^* f) \circ \phi^{-1}.$$

Now we check it satisfies the rules of a vector field. Just really need to check Leibniz:

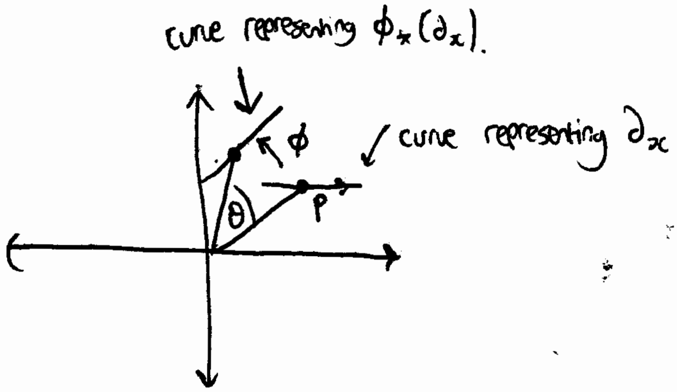
$$
\begin{aligned}
(\phi_* V)(f.g) &= V(\phi^*(f.g)) \circ \phi^{-1} && \text{(see above)} \\
&= V((f.g)\circ\phi) \circ \phi^{-1} && \text{(defn of } \phi^*) \\
&= V((f\circ\phi).(g\circ\phi)) \circ \phi^{-1} && \text{(defn of product of functions)} \\
&= \left[V(f\circ\phi).(g\circ\phi) + (f\circ\phi).V(g\circ\phi)\right] \circ \phi^{-1} && \text{(V satisfies Leibniz)} \\
&= \left[V(f\circ\phi)\circ\phi^{-1}\right].g + f.\left[V(g\circ\phi)\circ\phi^{-1}\right] \\
&= (\phi_* V)(f).g + f.(\phi_* V)(g).
\end{aligned}
$$

19. We compute:

$$\phi_*((\partial_x)_p)(f) = \frac{d}{dt}\bigg|_{t=0} f\big(\phi(p) + t(\cos\theta, \sin\theta)\big)$$

$$= \partial_x f\big|_{\phi(p)} \cdot \cos\theta + \partial_y f\big|_{\phi(p)} \cdot \sin\theta$$

$$= \left(\cos\theta\, \partial_x\big|_{\phi(p)} + \sin\theta\, \partial_y\big|_{\phi(p)}\right)(f).$$



so $\phi_*((\partial_x)_p) = \cos\theta\, \partial_x\big|_{\phi(p)} + \sin\theta\, \partial_y\big|_{\phi(p)}$.

Similarly,

$$\phi_*((\partial_y)_p)(f) = \frac{d}{dt}\bigg|_{t=0} f\big(\phi(p) + t(-\sin\theta, \cos\theta)\big)$$

$$= \partial_x f\big|_{\phi(p)} \cdot (-\sin\theta) + \partial_y f\big|_{\phi(p)} \cdot \cos\theta$$

$$= \left(-\sin\theta\, \partial_x\big|_{\phi(p)} + \cos\theta\, \partial_y\big|_{\phi(p)}\right)(f)$$

$\therefore\ \phi_*((\partial_y)_p) = -\sin\theta\, \partial_x\big|_{\phi(p)} + \cos\theta\, \partial_y\big|_{\phi(p)}$.