Exercise 65-66

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Let $\omega = \omega_1 dx^1 + \omega_2 dx^2 + \omega_3 dx^3$ be a 1-form on $\mathbb{R}^3$. Then,

$$\begin{aligned}
d\omega &= \partial_2\omega_1 dx^2 \wedge dx^1 + \partial_3\omega_1 dx^3 \wedge dx^1 + \partial_1\omega_2 dx^1 \wedge dx^2 \\
&= \partial_3\omega_2 dx^3 \wedge dx^2 + \partial_1\omega_3 dx^1 \wedge dx^3 + \partial_2\omega_3 dx^2 \wedge dx^3 \\
&= (\partial_2\omega_3 - \partial_3\omega_2)dx^2 \wedge dx^3 + (\partial_1\omega_3 - \partial_3\omega_1)dx^1 \wedge dx^3 + (\partial_1\omega_2 - \partial_2\omega_1)dx^1 \wedge dx^2 \\
\star d\omega &= (\partial_2\omega_3 - \partial_3\omega_2)dx^1 - (\partial_1\omega_3 - \partial_3\omega_1)dx^2 + (\partial_1\omega_2 - \partial_2\omega_1)dx^3
\end{aligned}$$

if the components are Cartesian, then the components are that of the curl a vector ω. That is

$$\star d\omega = (\boldsymbol{\nabla} \times \omega) \cdot d\mathbf{x}$$

Notice that

$$\begin{aligned}
\star\omega &= \omega_1 dx^2 \wedge dx^3 + \omega_2 dx^3 \wedge dx^1 + \omega_3 dx^1 \wedge dx^2 \\
d \star \omega &= (\partial_1\omega_1 + \partial_2\omega_2 + \partial_3\omega_3)dx^1 \wedge dx^2 \wedge dx^3 \\
\star d \star \omega &= \partial_1\omega_1 + \partial_2\omega_2 + \partial_3\omega_3 = \boldsymbol{\nabla} \cdot \omega \quad \text{in Cartesian coordinate}
\end{aligned}$$