

Exercise 54

If $\omega = \omega_\mu f^\mu$ is a 1-form on a chart, the corresponding vector field can be calculated as follows:

Let $\eta = \eta_\alpha f^\alpha$ be an arbitrary 1-form

then,

$$\begin{aligned} g^{-1}(\omega \cdot) \eta &= g^{-1}(\omega, \eta) = g^{-1}(\omega_\mu f^\mu, \eta_\alpha f^\alpha) \\ &= g^{-1}(f^\mu, f^\alpha) \omega_\mu \eta_\alpha \\ &= g^{\mu\alpha} \omega_\mu \eta_\alpha = g^{\mu\nu} \sum_\alpha \omega_\mu \eta_\alpha \\ &= g^{\mu\nu} \omega_\mu \eta_\alpha f^\alpha(e_\nu) \\ &= (g^{\mu\nu} \omega_\mu e_\nu) \eta_\alpha f^\alpha \end{aligned}$$

$$\Rightarrow g^{-1}(\omega \cdot) = \omega^\nu e_\nu, \text{ where, } \omega^\nu = g^{\mu\nu} \omega_\mu$$