

### Exercise 53

If  $V = V^\mu e_\mu$  is a vector field on a chart, then the corresponding 1-form  $g(V \cdot)$  can be calculated as follows:

Let  $u = u^\beta e_\beta$  be an arbitrary vector field, then,

$$\begin{aligned}g(V \cdot)u &= g(V, u) = g(V^\mu e_\mu, u^\beta e_\beta) \\&= g(e_\mu, e_\beta) V^\mu u^\beta = g_{\mu\beta} V^\mu u^\beta \\&= g_{\mu\nu} \delta_\beta^\nu V^\mu u^\beta = g_{\mu\nu} V^\mu u^\beta f^\nu(e_\beta) \\&= (g_{\mu\nu} V^\mu f^\nu) u^\beta e_\beta\end{aligned}$$

$$\Rightarrow g(V \cdot) = V_\nu f^\nu, \quad \text{where, } V_\nu = g_{\mu\nu} V^\mu$$