80. $d\frac{x\,dy}{x^2+y^2} = \frac{1}{(x^2+y^2)^2}[(y^2-x^2)\ dx - 2y\ dy] \wedge dy = \frac{y^2-x^2}{(x^2+y^2)^2} dx \wedge dy$. Similarly, $\frac{y\,dx}{x^2+y^2} = \frac{x^2-y^2}{(x^2+y^2)^2} dy \wedge dx = \frac{y^2-x^2}{(x^2+y^2)^2} dx \wedge dy$, and hence $dE = d\frac{x\,dy - y\,dx}{x^2+y^2} = 0$.
$\gamma_0 : [0,1] \to \mathbb{R}^2 : t \mapsto (\cos \pi(1-t), \sin \pi(1-t))$ has

$$\int_{\gamma_0} dE = \int_0^1 \frac{\cos \pi(1-t) \cdot -\pi \cos \pi(1-t)\ dt - \sin \pi(1-t) \cdot -\pi \cdot - \sin \pi(1-t)\ dt}{\cos^2 \pi(1-t) + \sin^2 \pi(1-t)} = -\pi \int_0^1 \frac{1}{1}\ dt$$

Similarly, $\gamma_1(t) = (\cos(-\pi(1-t)), \sin(-\pi(1-t)))$, which amounts to replacing π by $-\pi$.

81. Given two paths γ_0, γ_1 from p to q in $\mathbb{R}^n$, define, for each $\lambda \in (0,1)$, a path γ_λ by

$$\gamma_\lambda(t) := (1-\lambda)\gamma_0(t) + \lambda\gamma_1(t)$$

82. If ω is exact, i.e. $\omega = d\phi$, then for any loop γ based at p we have $\int_\gamma \omega = \phi(p) - \phi(p) = 0$.

Conversely, suppose that ω is not exact. We have seen that if $\int_\gamma E = \int_{\gamma'} E$ for *any path* from a point $p \in M$ to a point $q \in M$, then the map

$$\phi(q) := \int_\gamma E \qquad \gamma \text{ an arbitrary path } p \text{ to } q$$

is well–defined, and has $E = d\phi$. Hence if E is not exact, there must be p, q and two paths γ, γ' from p to q such that $\int_\gamma E \neq \int_{\gamma'} E$. Glueing γ' in reverse direction to γ yields a loop Γ based at p. (To be precise, define $\Gamma(t) := \gamma(t)$ for $t \leq T$, and $\Gamma(t) := \gamma'(T' + T - t)$ for $T \leq t \leq T + T'$) Then $\int_\Gamma E = \int_\gamma E - \int_{\gamma'} E \neq 0$.

83. Clearly if $\omega = d\theta$ on the coordinate patch $S^1 - \{1\} = \{(e^{i\theta} : 0 < \theta < 2\pi\}$, it can be extended uniquely to S^1, and then $\int_{S^1} \omega = 2\pi$. Hence ω cannot be exact. Now consider $\pi_0^*(\omega)$, where $\pi_0 : S^1 \times M \to S^1$ is the projection onto S^1.

84. For $i \leq n$, let $U_{\pm i} = \{(x_1, \ldots, x_n) : ||\mathbf{x}||^2 \leq 1, \pm x_i > 0\}$, and define $p_i(\mathbf{x}) = (x_1, \ldots, x_{i-1}, \ldots, x_{i+1}, \ldots x_n)$. Define $\varphi_: U_{\pm i} \to \mathbb{H}^n : \mathbf{x} \mapsto (p_i(\mathbf{x}), \sqrt{1 - ||\mathbf{x}||^2})$. The point $\mathbf{0}$ needs a chart also.

85. I'm going to give a very rough argument, as many concepts are inadequately defined in BM. If I recall, we didn't even prove that the tangent spaces of an ordinary n–dimensional manifold are n–dimensional. Assume this is known. Any chart containing a boundary point also contains a non–boundary point. For non–boundary points, the coordinate basis vectors ∂_i are linearly independent. The basis vector ∂_n is the only one which might give trouble at a boundary point. However, if $f : M \to \mathbb{R}$ is smooth, then it can be extended to coordinates with $x_n > -\varepsilon$, so tha $\partial_n f$ makes sense also at boundary points.

86. Suppose that $(U_\alpha, \varphi_\alpha)$ is a family of charts with associated partition of unity f_α, and that the same is true for $U'\beta, \varphi_\beta)$ and f'_β. Note that $g_\alpha dx^1 \wedge \cdots \wedge dx^n = \mathrm{Det}(\partial'_j x^i) g_\alpha\ dx'^1 \wedge$

$\cdots \wedge dx'^n$, so that $g'_\beta = \mathrm{Det}(\partial'_j x^i) g_\alpha$ on $U_\alpha \cap U'_\beta$. Hence

$$\begin{aligned}\sum_\alpha \int f_\alpha \omega &= \sum_\alpha \sum_\beta \int f_{\_}\beta f_\alpha g_\alpha \, dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_\beta \sum_\alpha \int f_\alpha f'_\beta g_\alpha \mathrm{Det}(\partial'_j x^i) \, dx'^1 \wedge \cdots \wedge dx'^n \\ &= \sum_\beta \int f'_\beta g'_\beta \, dx'^1 \wedge \cdots \wedge dx'^n = \sum_\beta \int f'_\beta \omega\end{aligned}$$

using the change of variables formula and the fact that the $\varphi_\alpha \circ {\varphi'_\beta}^{-1}$ are orientation–preserving.

87. Using the charts $(U_{\pm i}, \varphi_{\pm i})$ of exercise 84, we have $V_{\pm i} := U_{\pm i} \cap \partial D^n = \{(x_1, \ldots, x_n) : x_1^2 + \cdots + x_n^2 = 1, x_i = 0\}$. By definition, $\mathbf{x} \in \partial D^n$ iff $\varphi_{\pm i}\mathbf{x})$ has n^{th} coordinate $= 0$ for some $\pm i$. Thus we must have $\sqrt{1 - ||\mathbf{x}||^2} = 0$ i.e. $||\mathbf{x}||^2 = 1$.

    This is not entirely satisfactory — one would also like to know that a point x in a manifold M cannot simultaneously have a chart that is like $\mathbb{R}^n$, *and* one that is like $\mathbb{H}^n$. If that were the case, there would be a diffeomorphism from an open set in $U \subseteq \mathbb{R}^n$ to an open set in $V \subseteq \mathbb{H}^n$, where $V \cap \partial\mathbb{H}^n \neq \emptyset$. This is impossible, by the inverse function theorem.

88. Stokes: $\int_{[0,1]} df = \int_{\partial[0,1]} f$. By definition, $\int_{[0,1]} df = \int_0^1 f'(x)\, dx = f(1) - f(0)$, using the Fundamental Theorem of Calculus. On the other hand, we do not yet seem to have a definition for $\int_{\partial[0,1]} f$, the integral of a 0–form. $\partial[0,1]$ inherits an orientation from $[0,1]$: Pointing in the negative x–direction at $x = 0$, and in the positive x–direction at $x = 1$. So we must define $\int_{\partial[0,1]} = f(1) - f(0)$.

89. Obviously, $\partial[0,\infty) = \{0\}$. With the induced orientation, $\int_{\partial[0,\infty)} f = -f(0)$. Now $\int_{0,\infty} f'(x) \;\; dx = \lim_{a\to\infty} f(a) - f(0)$, so for Stokes' Theorem to hold, we must have $\lim_{a\to\infty} f(a) = 0$.