47. Given that ϕ^* has already been defined on 0– and 1–forms, and that each p–form on M is a linear combination (over $C^\infty(M)$) of p-fold wedge products of 1–forms, it is clear we must define

$$\phi^*(f_{i_1\dots i_p} dx^{i_1}\wedge\cdots\wedge dx^{i_p}) := \phi^*(f_{i_1\dots i_p}\phi^*(dx^{i_1})\wedge\cdots\wedge\phi^*(dx^{i_p}) = f_{i_1\dots i_p}\circ\phi\frac{\partial x^{i_1}}{\partial x'^{j^1}}\cdots\frac{\partial x^{i_p}}{\partial x'^{j_p}}dx'^{j_1}\dots dx'^{j_p}$$

and extend by linearity. There is no choice about this, so ϕ^* is unique.

48. $P^*(\omega_\mu(\mathbf{x})\ dx^\mu) = \omega_\mu(-\mathbf{x})\frac{\partial x^\mu\circ P}{\partial x^\nu}\ dx^\nu = -\omega_\mu(-\mathbf{x})\ dx^\mu$.
Similarly, $P^*(\omega_{\mu\nu}(\mathbf{x}) dx^\mu\ dx^\nu) = \omega_{\mu\nu}(-\mathbf{x})\ dx^\mu; dx^\nu$

49. $d(\omega_\mu\ dx^\mu = d\omega_\mu \wedge dx^\mu = \partial_\nu\omega_\mu\ dx^\nu\wedge dx^\mu$

50. Any 2–form on $\mathbb{R}\times S$ is locally $\frac{1}{2}\omega_{\mu\nu}dx^\mu\wedge dx^\nu$, where w.lo.g ω is antisymmetric, $x^0 := t$ is the coordinate on $\mathbb{R}$, and $x^i (i>0)$ the coordinates on S. For $i>0$, define $E_i := \omega_{i,0}$, and for $i,j>0$ define $B_{ij} = \omega_{ij}$.

51. If $\omega = \omega_I dx^I$, then $d\omega = \partial_0\omega_I\ dx^0\wedge dx^I + \partial_i\omega_I dx^i\wedge dx^I$, where in the last term summation is over coordinates of S, i.e. over $i>0$.

52. The bilinearity of g ensures the linearity of $g(v,\cdot): V\to\mathbb{R}$, i.e. if $v\in V$, then $Lv := g(v,\cdot)\in V^*$. Now non–degeneracy g of immediately implies that $\ker L = \{0\}$, so that L is injective. Since $\dim V = \dim V^*$, L is also surjective.

    Alternatively, suppose that $v^*\in V^*$, and that e_i is an orthonormal basis for V w.r.t g. Define $v = \sum_{i,j} g(e_i,e_j)v^*(e_i)e_j$ Note that the product (without summation) $g(e_i,e_j)g(e_j,e_k)$ is $=1$ if $i=j=k$, and is $=0$ otherwise. Now observe that $g(v,e_k) = g(\sum_{i,j} g(e_i,e_j)v^*(e_i)e_j, e_k) = \sum_{i,j} g(e_i,e_j)v^*(e_i)g(e_j,e_k) = v^*(e_k)$. Hence $L^{-1}: V^*\to V: v^*\mapsto v := \sum_{i,j} g(e_i,e_j)v^*(e_i)e_j$.

53. If $v = v^\mu e_\mu$, and $\omega := g(v,\cdot)$, then we can write $\omega = v_\nu f^\nu$ where the dual basis has $f^\nu(e_\mu) = \delta_{\mu^\nu}$. Now

$$v_\nu = v_\gamma f^\gamma(e_\nu) = \omega(e_\nu) = g(v,e_\nu) = v^\mu g(e_\mu,e_\nu) = g_{\mu\nu}v^\mu$$

54. Because of the isomorphism in exercise 52, we need merely show that $g(\omega^\nu e_\nu,\cdot) = \omega_\nu f^\nu$, where $\omega^\nu := g^{\mu\nu}\omega_\mu$. But $g(\omega^\nu e_\nu, e_\gamma) = g^{\mu\nu}\omega_\mu g(e_\nu,e_\gamma) = \omega_\mu g^{\mu\nu}g_{\nu\gamma} = \omega_\gamma = \omega_\nu f^\nu(e_\gamma)$.

55. Obvious (unless I'm missing something).

56. $g^\mu_\nu = g^{\mu\gamma}g_{\gamma\nu} = \delta^\mu_\nu$.

57. By definition,

$$\langle e^{\mu_1}\wedge\cdots\wedge e^{\mu_p}, e^{\nu_1}\wedge\cdots\wedge e^{\nu_p}\rangle = \det(g^{\mu_i,\nu_j}) = \sum_{\sigma\in S_p}(-1)^\sigma g^{\mu_1,\nu_{\sigma(1)}}\cdot\dots\cdot g^{\mu_p,\nu_{\sigma(p)}}$$

    Since $g^{\mu\nu}=0$ if $\mu\neq\nu$, we see that $\langle e^{\mu_1}\wedge\cdots\wedge e^{\mu_p}, e^{\nu_1}\wedge\cdots\wedge e^{\nu_p}\rangle\neq 0$ only when $\nu_1,\dots,\nu_p$ is a permutation of $\mu_1,\dots,\mu_p$, in which case $e^{\nu_1}\wedge\cdots\wedge e^{\nu_p} = \pm e^{\mu_1}\wedge\cdots\wedge e^{\mu_p}$, where the sign is $+(-)$ if that permutation is even (odd).

    Now clearly

$$\langle e^{\mu_1}\wedge\cdots\wedge e^{\mu_p}, e^{\mu_1}\wedge\cdots\wedge e^{\mu_p}\rangle = g^{\mu_1,\mu_1}\cdot\dots\cdot g^{\mu_p,\mu_p} = \epsilon(\mu_1)\cdot\dots\cdot\epsilon(\mu_p)$$

58. We have $\langle dx^i, dx^j \rangle = g^{ij} = \delta^{ij}$ so that $\langle E, E\rangle = E_i E_j \langle dx^i, dx^j\rangle = E_i E^i = \sum_{i=1}^3 E_i^2$. Similarly,

$$\langle dx^i \wedge dx^j, dx^k \wedge dx^l \rangle = \det \begin{pmatrix} g^{ik} & g^{il} \\ g^{jk} & g^{jl} \end{pmatrix} = g^{ik}g^{jl} - g^{il}g^{jk} = \begin{cases} 1 & \text{if } i=k, j=l \\ -1 & \text{if } i=l, j=k \\ 0 & \text{else} \end{cases}$$

Hence $\langle B_x\, dy \wedge dz, B_x\, dy \wedge dz\rangle = B_x^2$, from which it follows easily that $\langle B, B\rangle = B_x^2 + B_y^2 + B_z^2$.

59. Note that $\langle dx^i \wedge dt, dx^j \wedge dt\rangle = -\delta^{ij}$ for $i, j > 0$. Thus $\langle E_{x^i}\, dx^i \wedge dt, E_{x^i}\, dx^i \wedge dt\rangle = -E_{x^i}^2$, from which it follows that $\langle E \wedge dt, E \wedge dt\rangle = -(E_x^2 + E_y^2 + E_z^2)$. Clearly $\langle B, E \wedge dt\rangle = 0$, because $\rangle dx^i \wedge dx^j, dx^k \wedge dt\rangle = 0$ for all $i, j, k > 0$. Thus $\langle F, F\rangle = \langle B, B\rangle + \langle E \wedge dt, E \wedge dt\rangle + \langle B, E \wedge dt\rangle + \langle E \wedge dt, B\rangle = (B_x^2 + B_y^2 + B_z^2) - (E_x^2 + E_y^2 + E_z^2)$, so that $-\frac{1}{2}\langle F, F\rangle =$ Lagrangian.

60. Let T be transformation which takes e_i to $e_{\sigma(i)}$, where σ is a permutation of $1, 2, \dots, n$ (where n is the dimension of the space). Then $T_{ij} = 1$ if $j = \sigma(i)$, and $T_{ij} = 0$ else. Thus $\det(T) = \sum_{\tau \in S_n} (-1)^\tau T_{1,\tau(1)} \dots T_{n,\tau(n)} = (-1)^\sigma$, as only the term corresponding to $\tau = \sigma$ is non–zero.

61. I'm not sure if I have interpreted this question correctly. Let $V = dx^1 \wedge \cdots \wedge dx^n$ be the standard volume form on $\mathbb{R}^n$, and let ω be a volume form on M. If for some chart $(U, \alpha, \varphi_\alpha)$ we have that $\varphi_\alpha^*(V)$ belongs to the equivalence class of $-\omega$, then we can replace φ_α by a chart that interchanges to of the coordinates. To be specific, define $\psi_\alpha = (\pi_2, \pi_1, \pi_3, \dots, \pi_n) \circ \varphi_\alpha$. Then $\psi_\alpha^*(dx^1 \wedge \cdots \wedge dx^n) = \varphi_\alpha^*(dx^2 \wedge dx^1 \wedge dx^3 \wedge \cdots \wedge dx^n) = -\varphi_\alpha^*(dx^1 \wedge \cdots \wedge dx^n)$ belongs to the equivalence class of ω. Hence we can cover M with charts $(U_\alpha, \varphi_\alpha)$ such that when $U_\alpha \cap U_\beta \neq \emptyset$, then $\varphi_\alpha^*(V), \varphi_\beta^*(V)$ have the same orientation, namely that of ω.

62. We need to show that if there are "orientation–preserving charts" $(U_\alpha, \varphi_\alpha)$ on M, i.e. charts such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ are orientation–preserving, then M has a volume form.

Note that if $f : (U, y^1, \dots, y^n) \to (V x^1, \dots, x^n)$, then $f^*(dx^1 \wedge \cdots \wedge dx^n) = f^*(dx^1) \wedge \cdots \wedge f^*(dx^n) = (\frac{\partial f^1}{\partial y^{j_1}}\, dy^{j_1}) \wedge \cdots \wedge (\frac{\partial f^n}{\partial y^{j_n}}\, dy^{j_n}) = \det(\frac{\partial f^i}{\partial x^j})_{ij}\, dy^1 \wedge \cdots \wedge dy^n$. Thus f is orientation–preserving iff $\det(\frac{\partial f^i}{\partial x^j})_{ij} > 0$.

To construct a volume form ω on M, start with the volume form $V = dx^1 \wedge \cdots \wedge dx^n$ on $\mathbb{R}^n$, and pull it back to M via the charts. This defines ω locally by $\omega | U_\alpha = \varphi_\alpha^*(V)$. If $y^1, \dots, y^n$ are the local coordinates of $(U_\alpha, \varphi_\alpha)$ (i.e. if $y^i = x^i \circ \varphi_\alpha$), then $\omega = \varphi_\alpha^*(dx^1) \wedge \cdots \wedge \varphi_\alpha^*(dx^n) = dy^1 \wedge \cdots \wedge dy^n$.

The trouble that may arise is that when U_α, U_β overlap, the orientations of $\varphi_\alpha^*(V), \varphi_\beta^*(V)$ are opposite, for then the orientation of ω is not well–defined. Now if (U_β, φ_β) has coordinates $z^1, \dots, z_n$ then $\varphi_\alpha^*(V)$ and $\varphi_\beta^*(V)$ have the same orientation iff $dz^1 \wedge \cdots \wedge dz^n$ is a positive function times $dy^1 \wedge \cdots \wedge dy^n$. But $dy^1 \wedge \cdots \wedge dy^n = \det(\frac{\partial y^i}{\partial z^j})_{ij} dz^1 \wedge \cdots \wedge dz^n$, and $\det(\frac{\partial y^i}{\partial z^j}) > 0$, since the transformation $z \mapsto y(z)$ is none $\varphi_\alpha \circ \varphi_\beta^{-1}$, which is orientation–preserving by assumption.

63. At p we have $e^i = T^i_j\ dx^j$ for some invertible matrix T. Hence $e^1\wedge\cdots\wedge e^n = \det T\ dx^1\wedge\cdots\wedge dx^n$. However, $g(e^i,e^j) = \pm\delta^{ij}$, and hence $T^i_s T^j_t g(dx^s\ dx^t) = \pm\delta^{ij}$, i.e. $T^i_s g^{st} T^j_t = \pm\delta^{ij}$. Taking determinants, we obtain $(\det T)(\det g^{-1})(\det T) = \pm 1$, i.e. $\det g = \pm(\det T)^2$. But $\det T > 0$, because it preserves orientation. Hence $\det T = \sqrt{|\det g|}$, and so $e^1\wedge\cdots\wedge e^n = \sqrt{|\det g|}\ dx^1\wedge\cdots\wedge dx^n = \text{vol}$.

64. We have, using exercises 57 and 63,

$$(e^{i_1}\wedge\cdots\wedge e^{i_p})\wedge\star(e^{i_1}\wedge\cdots\wedge e^{i_p}) = \langle e^{i_1}\wedge\cdots\wedge e^{i_p}, e^{i_1}\wedge\cdots\wedge e^{i_p}\rangle\text{vol} = \epsilon(i_1)\cdot\ldots\epsilon(i_p)e^1\wedge\cdots\wedge e^n$$

It follows immediately that $\star(e^{i_1}\wedge\cdots\wedge e^{i_p}) = \pm e^{i_{p+1}}\wedge\cdots\wedge e^{i_n}$. To determine which sign ($+$ or $-$), just note that

$$e^{i_1}\wedge\cdots\wedge e^{i_n} = \text{sgn}(i_1,\ldots,i_n)e^1\wedge\cdots\wedge e^{\text{sgn}}(i_1,\ldots,i_n)\text{vol}$$

and hence that the sign is $\text{sgn}(i_1,\ldots,i_n)\epsilon(i_1)\ldots\epsilon(i_p)$, as asserted.

65. If $\omega := \omega_x\ dx + \omega_y\ dy + \omega_z\ dz$, then $d\omega = (\partial_z\omega_x - \partial_x\omega_z)\ dz\wedge dx + (\partial_x\omega_y - \partial_y\omega_x)\ dx\wedge dy + (\partial_y\omega_z - \partial_z\omega_y)\ dy\wedge dz$, so that

$$\star d\omega = (\partial_y\omega_z - \partial_z\omega_y)\ dx - (\partial_x\omega_z - \partial_z\omega_x)\ dy + (\partial_x\omega_y - \partial_y\omega_x)\ dz = \begin{vmatrix} dx & dy & dz \\ \partial_x & \partial_y & \partial_z \\ \omega_x & \omega_y & \omega_z \end{vmatrix} = \text{“curl”}\ \omega$$

66. Looking at just one term: $\star d\star(\omega_x\ dx) = \star d(\omega_x\ dy\wedge dz) = \star(\partial_x\omega_x\ dx\wedge dy\wedge dz) = \partial_x\omega_x$. Hence

$$\star d\star\omega = \text{“div”}\ \omega$$

67. I'll do a few: $\star dt = \text{sgn}(0,1,2,3)\epsilon(0)\ dx\wedge dy\wedge dz = -\ dx\wedge dy\wedge dz$.
$\star dx = \text{sgn}(1,0,2,3)\epsilon(1)\ dt\wedge\ dy\wedge dz = -\ dt\wedge\ dy\wedge dz$.
$\star(dt\wedge dy) = \text{sgn}(0,2,1,3)\epsilon(0)\epsilon(2)\ dx\wedge dz = dx\wedge dz$
$\star(dx\wedge dz) = \text{sgn}(1,3,0,2)\epsilon(1)\epsilon(3)\ dt\wedge dy = dt\wedge dy$
$\star(dt\wedge dx\wedge dz) = \text{sgn}(0,1,3,2)\epsilon(0)\ldots\epsilon(1)\epsilon(3) = dy$.
The second part of this exercise is generalized in the next.

68. Clearly $\star^2$ takes a p–form to a $p--form$, and $\star^2\omega = \pm\omega$ for all ω. To determine the sign, note that

$$\star^2(dx^{i_1}\wedge\cdots\wedge dx^{i_p}) = \text{sgn}(i_1,\ldots,i_n)\text{sgn}(i_{p+1},\ldots,i_n,i_1,\ldots,i_p)\epsilon(1)\ldots\epsilon(n)dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

Now

$$\begin{aligned}\text{sgn}(i_1,\ldots,i_n) &= (-1)^p\text{sgn}(i_{p+1},i_1,\ldots,i_p,i_{p+2}\ldots i_n)\\ &= (-1)^{2p}\text{sgn}(i_{p+1},i_{p+2},i_1\ldots,i_p,i_{p+3},\ldots,i_n)\\ &= \ldots\\ &= \text{sgn}(-1)^{p(n-p)}\text{sgn}(i_{p+1},\ldots,i_n,i_1,\ldots,i_p)\end{aligned}$$

which yields

$$\text{sgn}(i_1,\ldots,i_n)\text{sgn}(i_{p+1},\ldots,i_n,i_1,\ldots,i_p)\epsilon(1)\ldots\epsilon(n) = (-1)^{p(n-p)+s}$$

69. Note that $\epsilon^{i_1\dots i^p}_{j_1\dots j_{n-p}} = g^{i_1k_1}\dots g^{i_pk_p}\epsilon_{k_1\dots k_pj_1\dots j_{n-p}} = \epsilon(i_1)\dots\epsilon(i_p)\mathrm{sgn}(i_1\dots i_pj_1\dots j_{n-p})$.

70. $\star_S\, d_S \star_S E_x\, dx = \star_S\, d_S \star_S E_x\, dy\wedge dz = \star_S \partial_x E_x\, dx\wedge dy\wedge dz = \partial_x E_x$.
Similarly $\star_S\ d_S \star_S B_x\ dy\wedge dz = \star_S\ d_S B_x\ dx = \star_S(\partial_z B_x\ dz\wedge dx - \partial_y B_x\ dx\wedge dy) = \partial_z B_x\ dy - \partial_y B_x\ dz$

71. $\star F = \star(B_x\ dy\wedge dz + \dots + E_x\ dx\wedge dt + \dots) = (B_x\ dt\wedge dx + \dots) - (E_x\ dy\wedge dz + \dots)$