16. By applying the definitions, we see that, for all $f \in C^\infty(N)$, we have

$$\phi_*(\gamma'(t))(f) = \gamma'(t)(f \circ \phi) = \frac{d}{dt} f(\phi(\gamma(t))) = \frac{d}{dt} f(\phi \circ \gamma(t)) = (\phi \circ \gamma)'(t)(f)$$

17. For $f, g \in C^\infty(N)$ and $\alpha \in \mathbb{R}$, we have, by linearity of $v \in T_pM$, that

$$(\phi_* v)(f + \alpha g) = v((f + \alpha g) \circ \phi) = v(f \circ \phi) + \alpha v(g \circ \phi) = (\phi_* v)f + \alpha(\phi_* v)(g)$$

18. For $v \in \text{Vect}(M)$ and $\phi : M \to N$ a diffeomorphism, we may define $\phi_* v \in \text{Vect}(N)$ as follows: For $q \in N$ and $f \in C^\infty(N)$, define

$$(\phi_* v)(f)(q) = v(f \circ \phi)(\phi^{-1}(q)) \qquad \text{i.e.} \qquad (\phi_* v)_{\phi(p)}(f) := v_p(\phi^* f)$$

(where $f \in C^\infty(N)$ and $v_p(g) := v(g)(p)$, etc.) To show that $\phi_* v \in \text{Vect}(N)$, note that

$$(\phi_* v)_{\phi(p)}(fg) = v_p((fg) \circ \phi) = f(\phi(p))(\phi_* v)(g) + (\phi_* v)(f) g(\phi(p))$$

to obtain the Leibniz property. Linearity follows similarly.

19. Note that $\phi(x, y) = (x \cos\theta - y \sin\theta, x \sin\theta + y \cos\theta)$, and so

$$\begin{aligned} ((\phi_* \partial_x)f, (\phi_* \partial_y)f) &= (\partial_x(f \circ \phi), \partial_y(f \circ \phi))|_{\phi(x,y)} \\ &= D(f \circ \phi)|_{\phi(x,y)} \\ &= Df|_{\phi(x,y)} D\phi|_{(x,y)} \\ &= (\partial_x f, \partial_y f)|_{\phi(x,y)} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \\ &= (\cos\theta \partial_x f + \sin\theta \partial_y f, -\sin\theta \partial_x f + \cos\theta \partial_y f) \end{aligned}$$

where D denotes the derivative as linear operator, given by the Jacobian matrix.

20. If $\gamma : \mathbb{R} \to \mathbb{R}^2 : t \mapsto (x(t), y(t))$ is an integral curve, we want that $\gamma'(t) = x(t)^2 \partial x + y(t) \partial_y$, i.e. that

$$\partial_x f \frac{dx}{dt} + \partial_y f \frac{dy}{dt} = \frac{d}{dt} f(x(t), y(t)) = \gamma'(t)(f) = x(t)^2 \partial_x f + y(t) \partial_y f$$

for all $f \in C^\infty(\mathbb{R}^2)$. So e

$$\frac{dx}{dt} = x(t)^2 \qquad \frac{dy}{dt} = y(t)$$

and hence either $x(t) = 0$ or

$$x(t) = \frac{1}{A - t} \qquad y(t) = Be^t$$

for constants A, B. These blow up at $t = A$.

21. The semigroup property is intuitively clear from the "water" analogy in BM: If a water molecule is at point p at time zero, it will be at $\phi_t(p)$ at time t. s units of time later, i.e. at time $t + s$ the point particle will be at $\phi_s(\phi_t(p))$ — this assumes that the vector field is constant over time. Doing this in one step, the particle will also be at $\phi_{t+s}(p)$ at time $t + s$.

    Actually, though intuitively clear, I didn't find the semigroup property all that easy to prove (semi)–formally, partly because BM aren't very precise, and partly because there's probably an easier way I am not seeing. Anyway,

(i) If v is a vector field on a manifold M, there is at a unique integral curve $\gamma_p(t)$ through each point $p \in M$ such that $\gamma_p(0) = p$ This is implied by a statements in BM ("*Let $\phi_t(p)$ be* **the** *integral curve of v through $p \in M$*"), and "*Show that ϕ_0 is the identity map*", but presumably needs some Lipschitz–like conditions, as for ODE's.

(ii) Certainly the integral curve as function is not unique, but nor is it sufficient to simply specify the range of an integral curve as a subset of M. The speed at which the curve is traversed is important, but the starting point is not. To be more precise, note that if $\gamma(t)$ is an integral curve through a point p, and $s \in \mathbb{R}$ is fixed, then $\bar{\gamma}(t) := \gamma(t+s)$ is also an integral curve through p (but starting at a different point). Indeed, if $f \in C^\infty(M)$, and $h_s(t) := t + s$, then by the chain rule

$$(f \circ \bar{\gamma})'(t) = (f \circ \gamma \circ h_s)'(t) = (f \circ \gamma)'(h_s(t))h_s'(t) = (f \circ \gamma)'(t+s)$$

Hence $\bar{\gamma}'(t) = \gamma'(t)$ as tangent vectors in $T_{\gamma(t+s)}M$, and hence $\bar{\gamma}$ is also an integral curve through p. Yet $\bar{\gamma} \neq \gamma$ as curves.

(iii) For $p \in M$, let $\psi_p(t) := \phi_t(p)$, where $\{\phi_t : M \to M\}$ is the flow generated by a given vector field v, with $\phi_0 = \mathrm{id}_M$. Then by the previous bullet point, for fixed s, we have that $\psi_p(t+s)$ and $\psi_p(t)$ are the "same" integral curve, namely the integral curve through p. However $\psi_p(t+s)$ starts at the point $\psi_p(s)$ at 0, as does the integral curve $\psi_{\psi_p(s)}(t)$. By uniqueness, it follows that $\psi_p(t+s) = \psi_{\psi_s(p)}(t)$. Going back from ψ's to ϕ's, we see that

$$\phi_{t+s}(p) = \psi_p(t+s) = \psi_{\psi_s(p)}(t) = \phi_t(\phi_s(p))$$

i.e. $\phi_{t+s} = \phi_t \circ \phi_s$.

22. $v = \partial_r$, $w = \frac{1}{r}\partial_\theta$, so

$$[v, w] = \partial_r(\tfrac{1}{r}\partial_\theta) - \tfrac{1}{r}\partial_\theta\partial_r = -\tfrac{1}{r^2}\partial_\theta = -\frac{w}{r} = -\frac{y\partial_x - x\partial_y}{x^2 + y^2}$$

23.
$$vw(f)(p) = \frac{d}{dt}w(f)(\phi_t(p))\Big|_{t=0} = \frac{d}{dt}\frac{d}{ds}f(\psi_s(\phi_t(p)))\Big|_{t,s=0}$$

24. 1), 2) are easy.
The Jacobi identity can be proved as follows: Multiplying out the lefthand side $[u,[v,w]] + [v,[w,u]] + [w,[u,v]]$ we will obtain 12 terms. Each such term can be regarded as a permutation of uvw — of which there are 3!=6 in total — and each term will occur exactly twice, but with opposite signs. Thus $+abc$ will occur in the expansion of $[a,[b,c]]$, and $-abc$ in that of $[c,[a,b]]$. But this "explanation" is more complicated than simply multiplying the whole thing out:

$$\begin{aligned}
[u,[v,w]] &= uvw - uwv - vwu + wvu \\
[v,[w,u]] &= vwu - vuw - wuv + uwv \\
[w,[u,v]] &= wuv - wvu - uvw + vuw
\end{aligned}$$

Adding, we see that all terms cancel.

25. For $\omega, \mu \in \Omega^1(M)$, $v, w \in \text{Vect}(M)$ and $f, g \in C^\infty(M)$, we have

$$(\omega+\mu)(v+w) = \omega(v+w)+\mu(v+w) = \omega(v)+\omega(w)+\mu(v)+\mu(w) = (\omega+\mu)(v)+(\omega+\mu)(w)$$

and

$$(\omega + \mu)(gv) = \omega(gv) + \mu(gv) = g\omega(v) + g\mu(v) = g(\omega + \mu)(v)$$

Also

$$(f\omega)(v + w) = f(\omega(v + w)) = f(\omega(v) + \omega(w)) = (f\omega)(v) + (f\omega)(w)$$

and

$$(f\omega)(gv) = f(\omega(gv)) = f(g\omega(v)) = g(f\omega)(v)$$

26. We have

$$\begin{aligned} f(\omega + \mu)(v) &= f(\omega(v) + \mu(v)) = f\omega(v) + f\mu(v) \\ (f + g)(\omega)(v) &= f\omega(v) + g\omega(v) \\ (fg)(\omega(v) &= f(g\omega(v)) \\ 1\omega(v) &= \omega(v) \end{aligned}$$

27.

$$\begin{aligned} d(f + g)(v) &= v(f + g) = v(f) + v(g) = df(v) + dg(v) \\ d(\alpha f)(v) &= v(\alpha f) = \alpha v(f) = \alpha df(v) \\ (f + g)dh(v) &= (f + g)v(h) = fv(h) + gv(h) = f\ dh(v) + g\ dh(v) \\ d(fg)(v) &= v(fg) = gv(f) + fv(g) = g\ df(v) + f\ dg(v) \end{aligned}$$

28. If $v \in \text{Vect}(\mathbb{R}^n)$, then $v = v^i(x)\partial_i$ for some $v^i \in C^\infty(\mathbb{R}^n)$. Note that we have $dx^j(v) = v^i(x)\partial_i x^j = v^j(x)$ so that

$$df(v) = v(f) = v^i(x)\partial_i f = \partial_i f dx^i(v)$$

29. Suppose that $\omega := \omega_\mu\ dx^\mu = 0$, and define $v := \omega^\nu\partial_\nu$, where $\omega^\nu := \omega_\nu$ (i.e. equal as functions in $C^\infty(\mathbb{R}^n)$). Then

$$0 = \omega(v) = \omega_\mu\ dx^\mu(\omega^\nu\partial_\nu) = \omega_\mu\omega_\nu\delta^\mu_\nu = \omega_\mu\omega^\mu = \sum_{\mu=1}^{n}(\omega_\mu)^2$$

30. I can't do this (yet) without coordinates, i.e. without assuming every 1-form is locally of the form $\omega = \omega_\mu\ dx^\mu$. Assuming that, it is easy.

31. $(\text{id}^*\omega)(v) = \omega(\text{id}(v)) = \omega(v)$.
$(gf)^*(\omega)(v) = \omega(g(f(v)) = (g^*\omega)(f(v)) = (f^*g^*\omega)(v)$.

32. For $\phi : M \to N$, we put $\phi^* : \Omega^1(N) \to \Omega^1(M)$ by $\phi^*(\omega)(v)(p) = \omega(\phi_*(v))(\phi(p))$. Let $f \in C^\infty(M), v \in \text{Vec}(M)$, and put $q := \phi(p)$. Note that $\phi_*(fv)_p(g) = (fv)_p(g \circ \phi) = f(p)v_p(g \circ \phi) = (f\phi_*(v))_p(g)$, so that $\phi_*(fv) = f\phi_*(v)$. Hence

$$\phi^*(\omega)_p(fv)_p = \omega_q(\phi_*(fv)_p) = \omega_q(f\phi_*(v)_p) = f(p)\omega_q(\phi_*(v)_p) = (f\phi^*(\omega))_p(v_p)$$

so that indeed $\phi^*(\omega)(fv) = f\phi^*(\omega)(v)$.

33. Note first that $dx = d\text{id}_{\mathbb{R}}$. Hence

$$\phi^*(dx)_t(v) = dx_{\text{id}(t)}(\phi_*(v)) = \phi_*(v)_t(\text{id}) = v_t(\text{id} \circ \phi) = v_t(\phi) = d\phi_t(v)$$

i.e. $\phi^*(dx) = \; d\phi$. But clearly $d\phi_t = \frac{\partial \phi}{\partial t}\, dt = \cos t\; dt$.

34. $\phi(x, y) := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta)$, so with π_1 denoting the projection onto the first component, we see that

$$\phi^*(dx) = \phi^*(d\pi_1) = d(\phi^*\pi_1) = d(\pi_1 \circ \phi) = d(x\cos\theta - y\sin\theta) = \cos\theta\; dx - \sin\theta\; dy$$

using linearity of d.

35. Let $\varphi : U \to \mathbb{R}^n$ be a chart. We are writing x^μ instead of $x^\mu \circ \varphi = \varphi^*(x^\mu)$, ∂_μ instead of $(\varphi^{-1})_*(\partial_\mu)$, and dx^μ instead of $\varphi^*(dx^\mu)$. We have to check that

$$d(\phi^* x^\mu) = \varphi^*(dx^\mu)$$

but we already know that.

36. By definition, $dx^\lambda(\partial_\mu) = \partial_\mu(x^\lambda) = \delta_\mu\lambda$. Thus if $dx'^\nu = S^\nu_\mu\; dx^\mu$, we see that $dx'^\nu(\partial_\lambda) = S^\nu_\lambda$. But $dx'^\nu(\partial_\lambda) = \partial_\lambda x'^\nu$ and hence $S^\nu_\mu = \frac{\partial x'^\nu}{\partial x^\mu}$. Then if $\omega'_\nu\; dx'^\nu = \omega_\mu\; dx^\mu$, we see that

$$\omega'_\nu \frac{\partial x'^\nu}{\partial x^\mu}\; dx^\mu = \omega_\mu\; dx^\mu$$

from which $\omega_\mu = \frac{\partial x'^\nu}{\partial x^\mu}\omega'_\nu$. Now to obtain $\omega'_\nu = \frac{\partial x^\mu}{\partial x'^\nu}\omega_\mu$ either interchange primed and non–primed coordinates *and* μ and ν, or else multiply both sides by $\frac{\partial x^\lambda}{\partial x'^\nu}$ (with summation convention), noting that $\frac{\partial x^\lambda}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^\mu} = \frac{\partial x^\lambda}{\partial x^\mu} = \delta^\lambda_\mu$.

37. On the left we have

$$\phi^*(dx'^\nu)(\partial_\lambda) = d(\phi^* x'^\nu)(\partial_\lambda) = \partial_\lambda(\phi^* x'^\nu) = \text{ sloppy } \frac{\partial x'^\nu}{\partial x^\lambda}$$

and on the right we obtain the same:

$$\frac{\partial x'^\nu}{\partial x^\mu} dx^\mu(\partial_\lambda) = \frac{\partial x'^\nu}{\partial x^\mu}\delta^\mu_\lambda = \frac{\partial x'^\nu}{\partial x^\lambda}$$