10.

$$\begin{aligned} & v = w \\ \iff & \forall f \in C^\infty(M)\ (v(f) = w(f)) \\ \iff & \forall f \in C^\infty(M)\ \forall p \in M\ (v(f)(p) = w(f)(p)) \\ \iff & \forall p \in M\ \forall f \in C^\infty(M)\ (v_p(f) = w_p(f)) \\ \iff & \forall p \in M\ (v_p = w_p) \end{aligned}$$

11. For $p \in M$, define $\mathrm{ev}_p : \mathrm{Vec}(M) \to T_pM : v \mapsto v_p$. Then ev_p is clearly linear:

$$\mathrm{ev}_p(v + \alpha w)(f) = (v + \alpha w)(f)(p) = v(f)(p) + \alpha w(f)(p) = \mathrm{ev}_p(v)(f) + \alpha \mathrm{ev}_p(w)(f)$$

It is asserted on p. 28 of Baez and Muniain (without proof) that ev_p is surjective.

12. Note that

$$\gamma'(t)(f + \alpha g) = \frac{d}{dt} f(\gamma(t)) + \alpha \frac{d}{dt} g(\gamma(t)) = \gamma'(t)(f) + \alpha\gamma'(t)(g)$$

and that

$$\gamma'(t)(fg) = \frac{d}{dt} f(\gamma(t))g(\gamma(t)) = f(\gamma(t))\frac{dg(\gamma(t))}{dt} + \frac{df(\gamma(t))}{dt} g(\gamma(t)) = f(\gamma(t))\cdot\gamma'(t)(g) + \gamma'(t)(f)\cdot g(\gamma(t))$$

13. $\phi^*\mathrm{id}_{\mathbb{R}} = \mathrm{id} \circ \phi = \phi$.

14.

$$\phi^*\pi_1 \begin{pmatrix} x \\ y \end{pmatrix} = \pi_1 \circ \phi \begin{pmatrix} x \\ y \end{pmatrix} = (1\ \ 0) \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = x\cos\theta - y\sin\theta$$

15. (a) For $f : M \to \mathbb{R}$ we have two definitions of smoothness: Say that f is S_1 iff $f \in C^\infty(M)$, i.e. iff $f \circ \varphi_\alpha^{-1} : V_\alpha \subseteq \mathbb{R}^n \to \mathbb{R}$ is smooth for all charts $\varphi_\alpha : U_\alpha \to V_\alpha \subseteq \mathbb{R}^n$ of M. Further, say that f is S_2 iff $f^*g \in C^\infty(M)$ for all $g \in C^\infty(\mathbb{R})$. We are required to show that f is S_1 iff it is S_2.

    If $f : M \to \mathbb{R}$ is S_1 and $g \in C^\infty(\mathbb{R})$, then certainly the composition $g \circ (f \circ \varphi_\alpha^{-1}) : V_\alpha \subseteq \mathbb{R}^n \to \mathbb{R}$ is smooth for any chart $\varphi_\alpha : U_\alpha \to V_\alpha$, so $g \circ f$ is S_1, i.e. $g \circ f \in C^\infty(M)$, and so $f^*g \in C^\infty(M)$. hence f is S_2

    Conversely, if f is S_2, then $f^*\mathrm{id}_{\mathbb{R}} \in C^\infty(M)$, so that $f \in C^\infty(M)$, i.e. f is S_1.

    (b) The two definitions of "smooth curve" are actually the same already. To make this explicit, say that $\gamma : \mathbb{R} \to M$ is S_3 if $f \circ \gamma : \mathbb{R} \to \mathbb{R}$ is smooth for any $f \in C^\infty(M)$, and define S_2–smoothness as in (a).

    Now note that γ is S_3 iff $f \circ \gamma \in C^\infty(\mathbb{R})$ for all $f \in C^\infty(M)$, iff $\gamma^*f \in C^\infty(\mathbb{R})$ for all $f \in C^\infty(M)$ iff γ is S_2.