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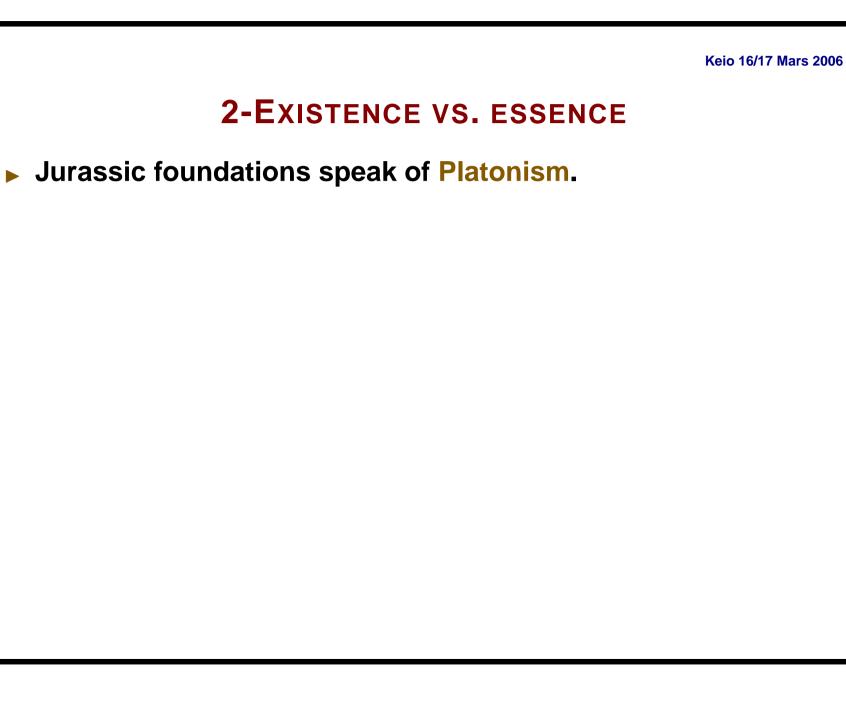
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 - For instance many isomorphic (standard !) versions of N.
 - Non internally isomorphic.
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- Got beyond the essential(ist) circularity of logic, the blind spot.

I-THE BLIND SPOT



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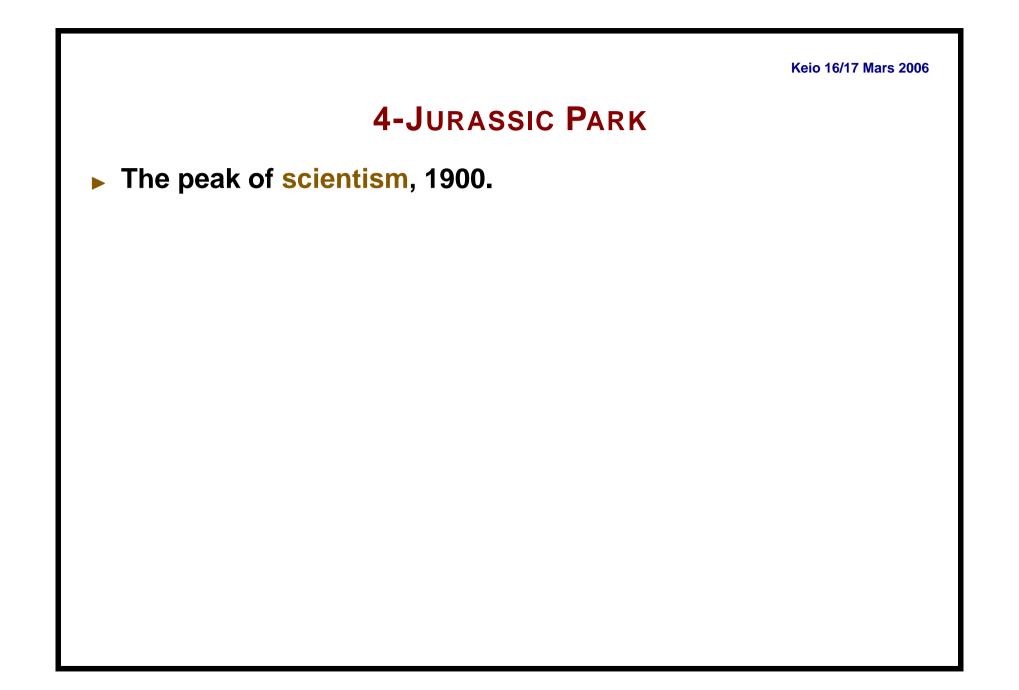
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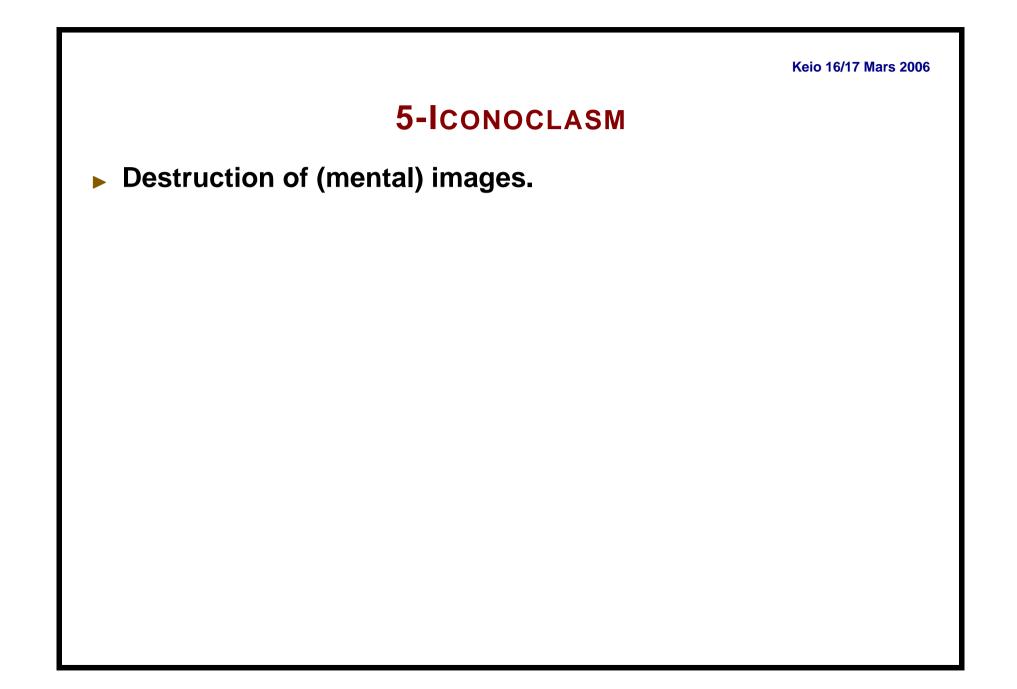
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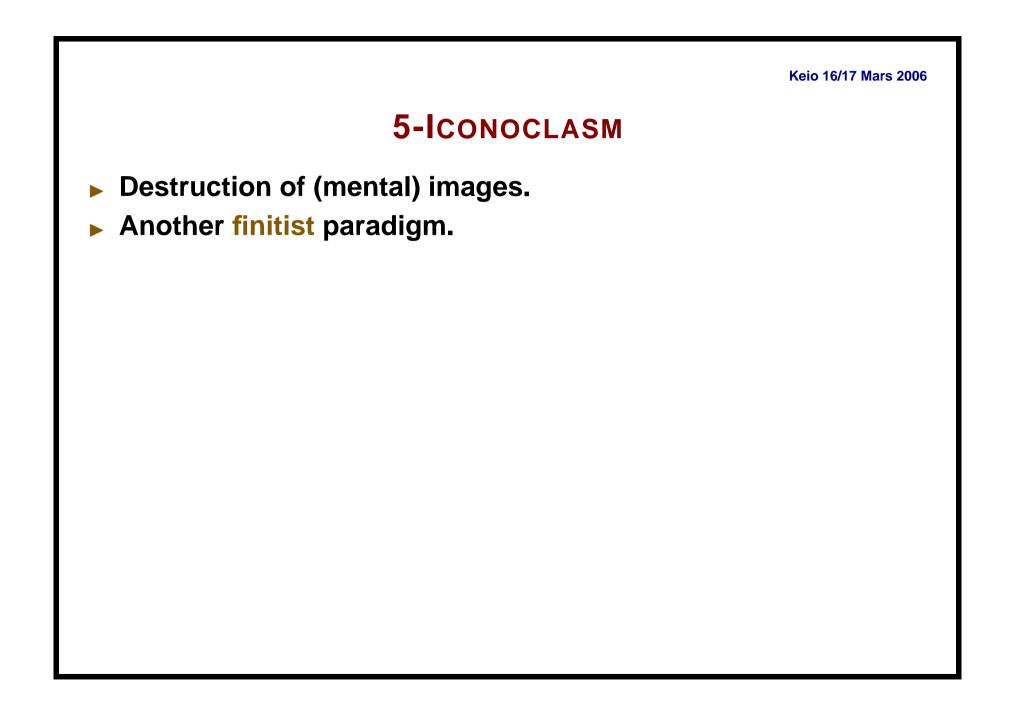
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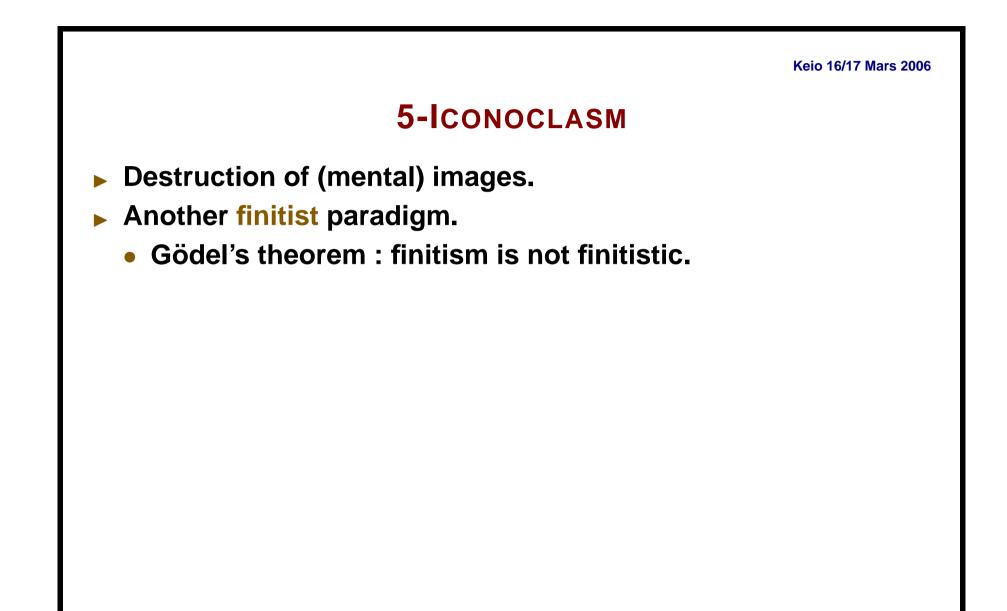
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- Set theory problematic in extreme situations (foundations).







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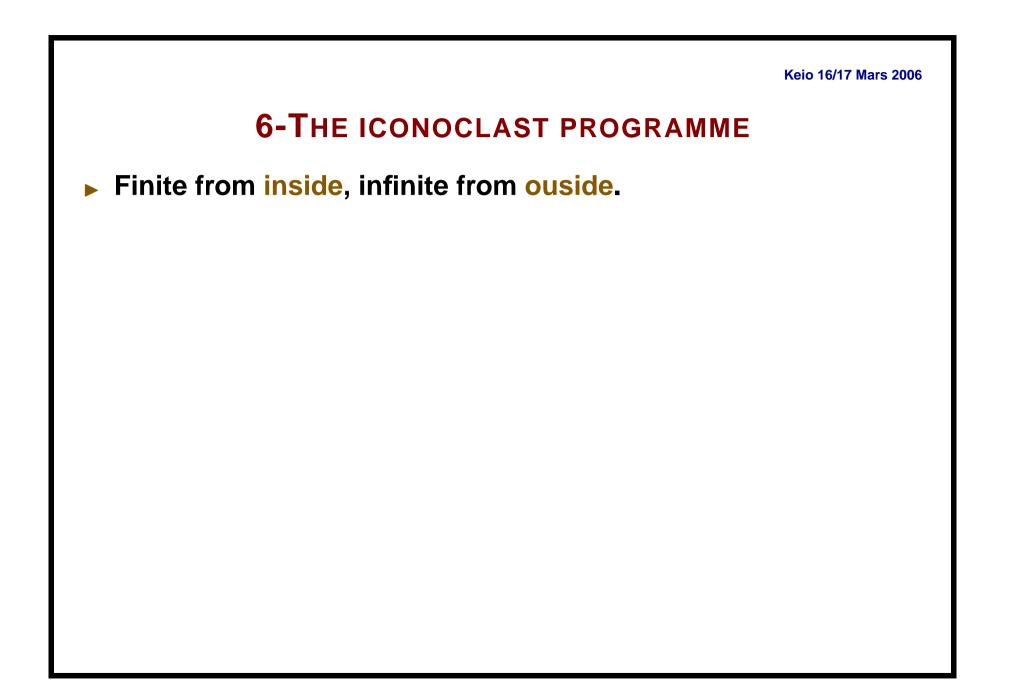
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- Accept foundations with most of operations external.



Keio 16/17 Mars 2006

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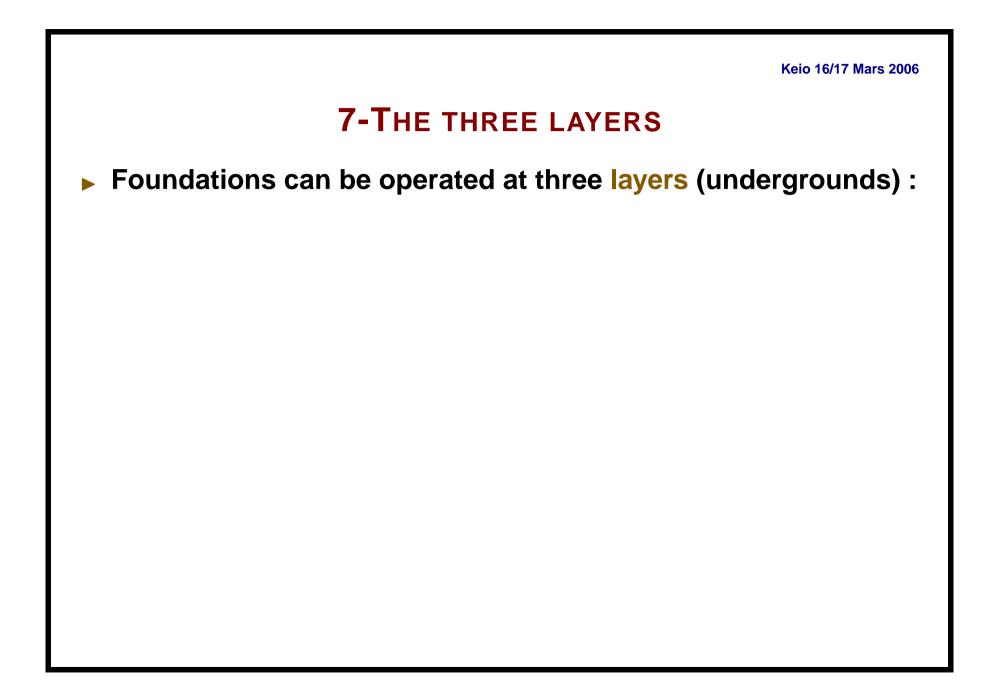
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- Forget the idea of creation in 7 days, from simple to complicated (sets, algebra, reals, function spaces) since it does not work anyway (Incompleteness theorem).

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II-THE CATEGORICAL LAYER





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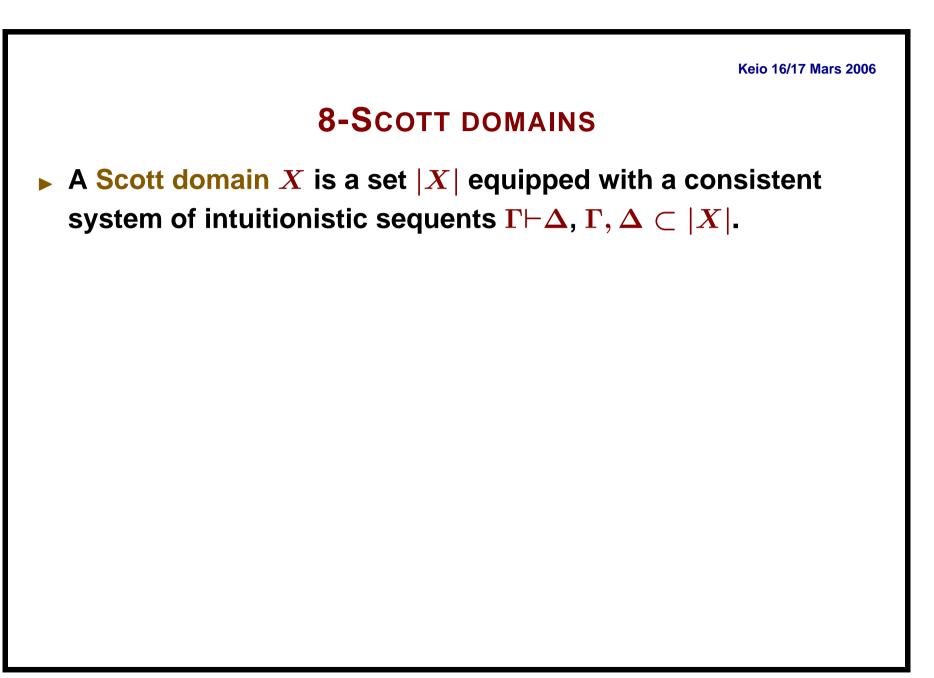
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- Level -2 not fit to go beyond the blind spot.



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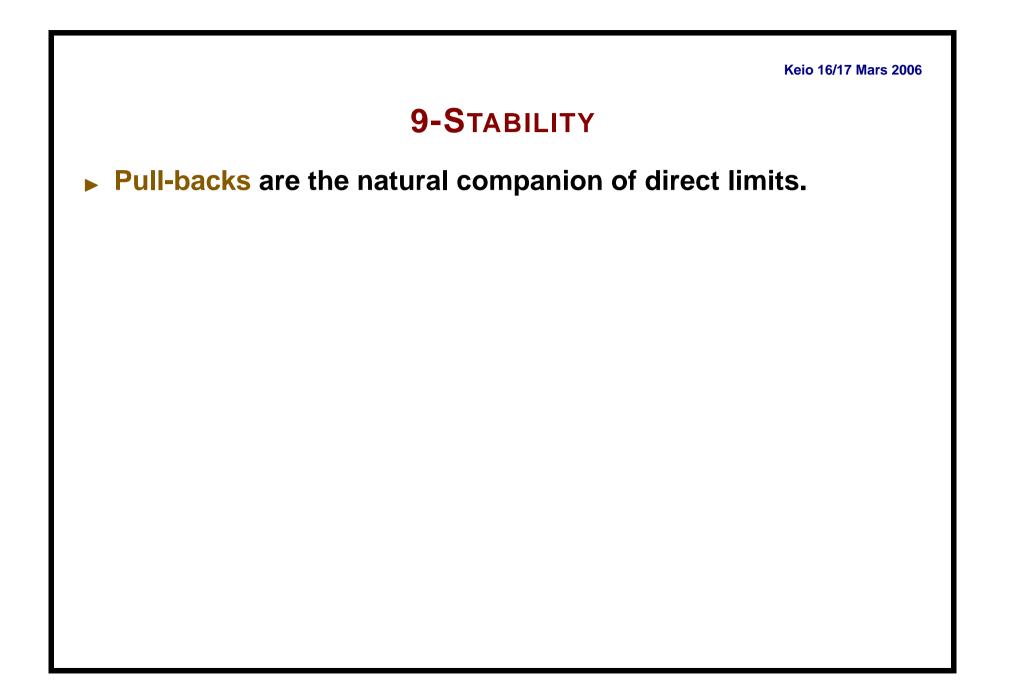
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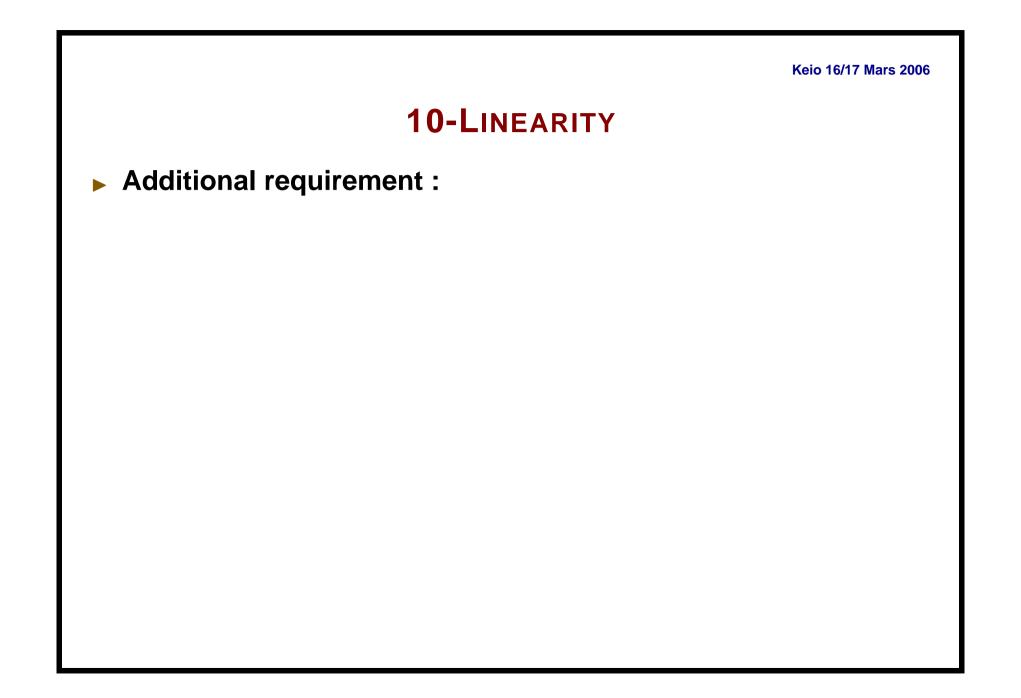
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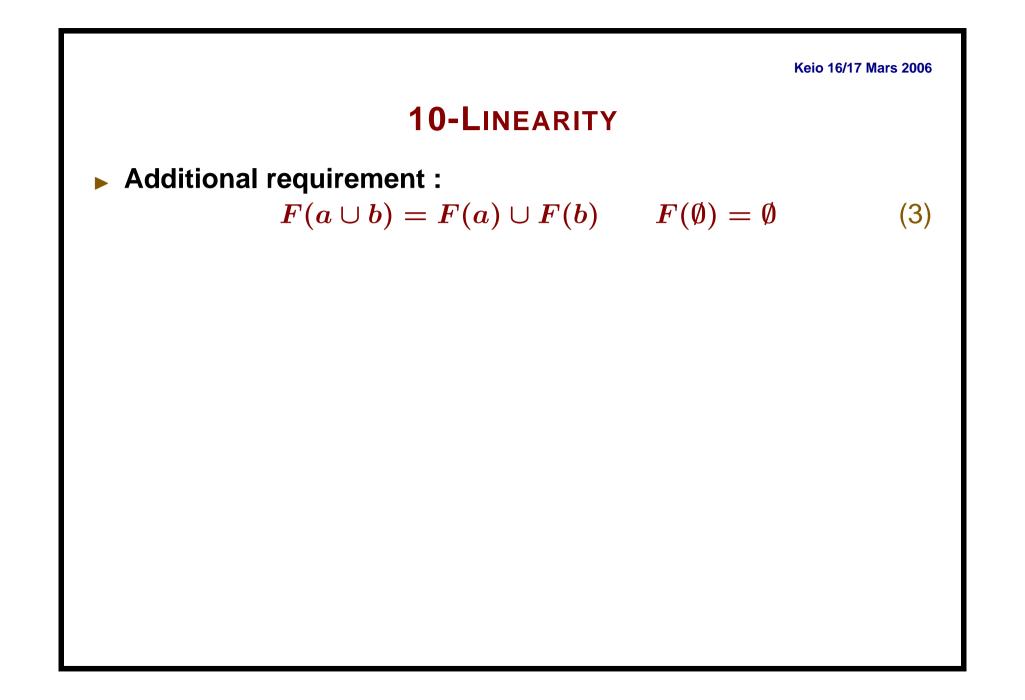
9-STABILITY

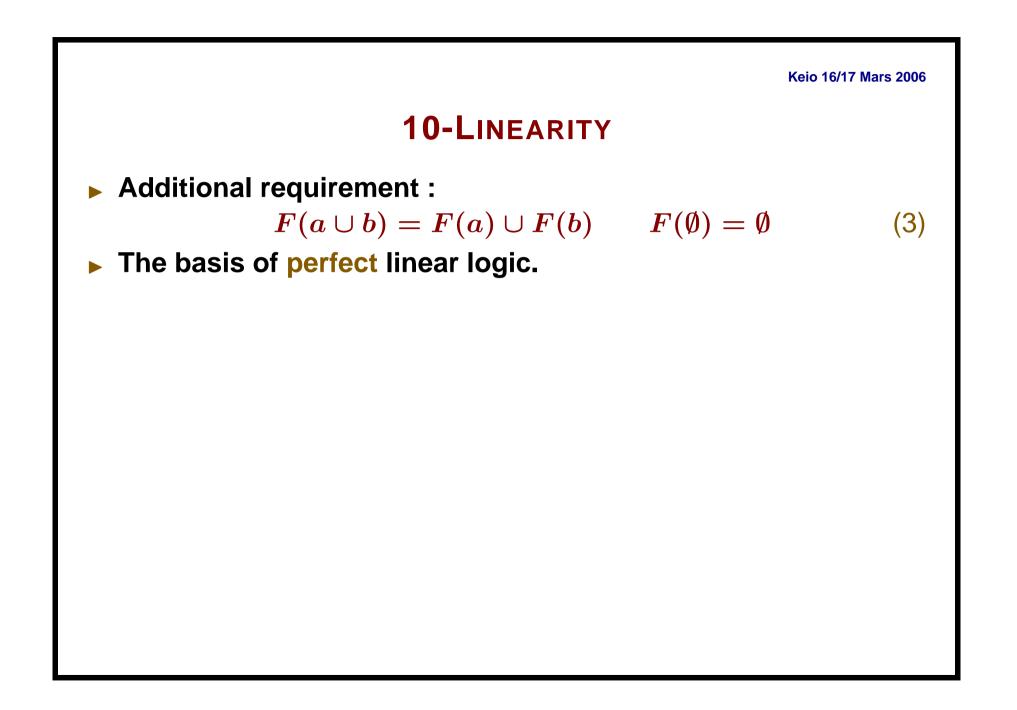
- Pull-backs are the natural companion of direct limits.
- Correspond to $a \cap b$ provided $a \cup b$ is consistent.
- Preservation of pull-backs a.k.a. stability (Berry) :

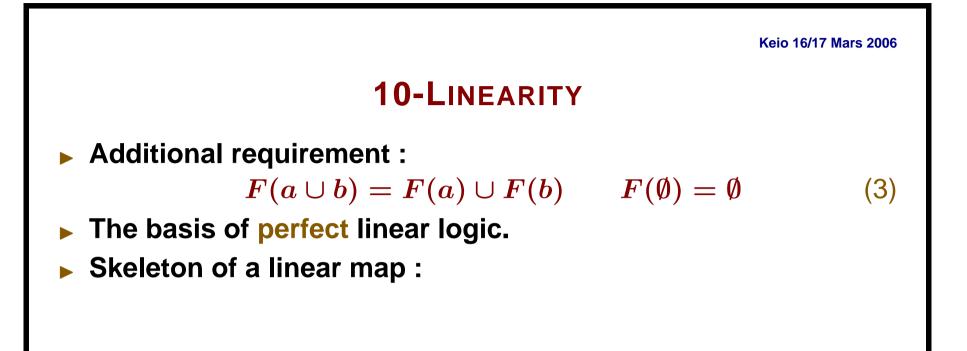
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Additional requirement :

 $F(a \cup b) = F(a) \cup F(b)$ $F(\emptyset) = \emptyset$ (3)

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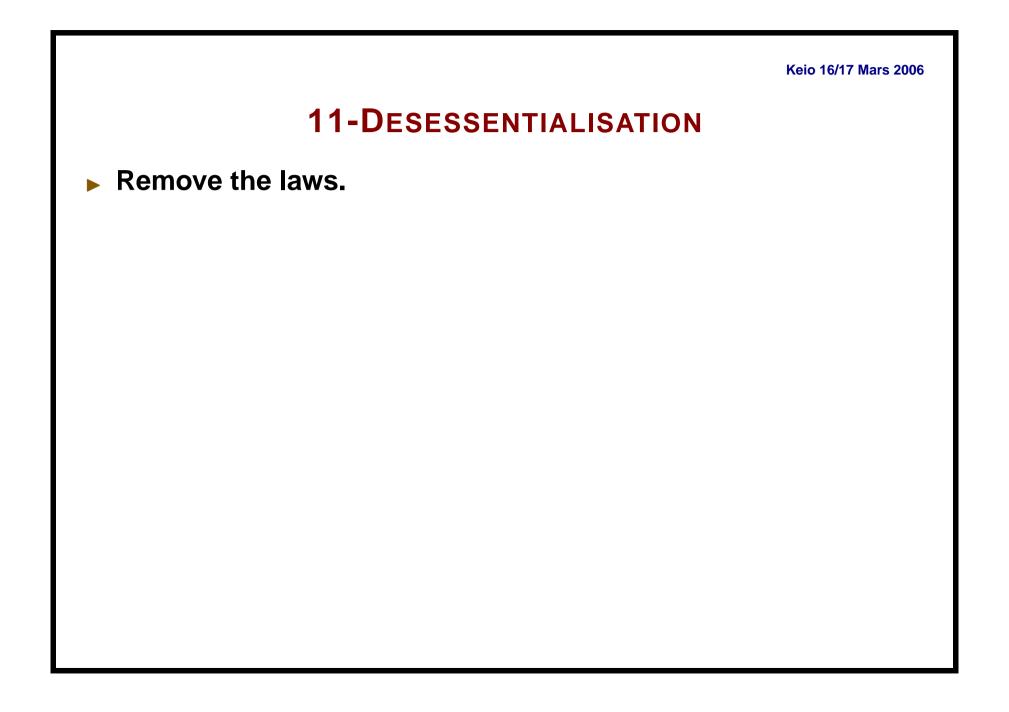
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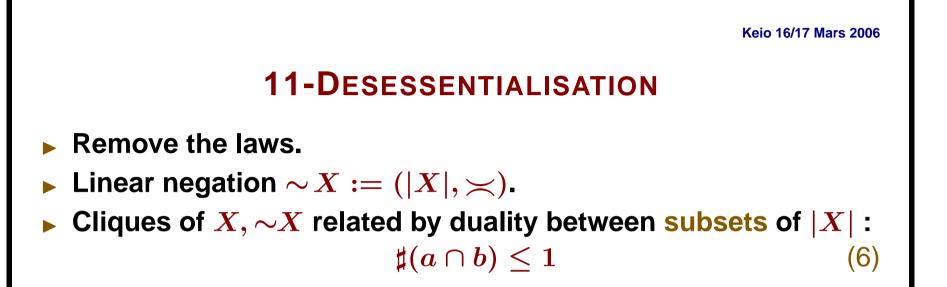


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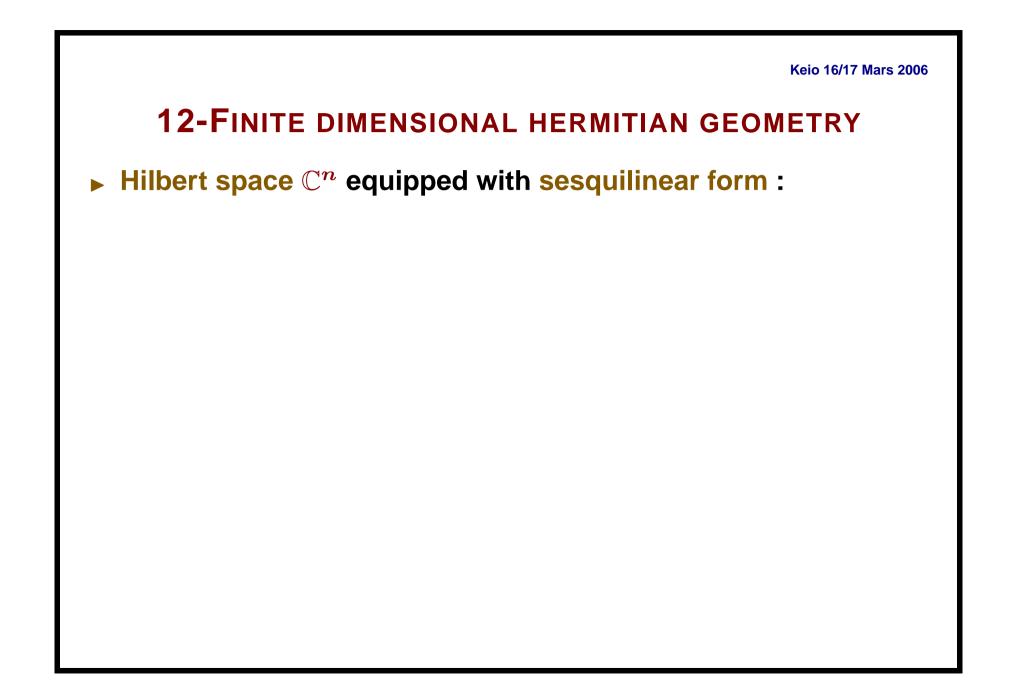
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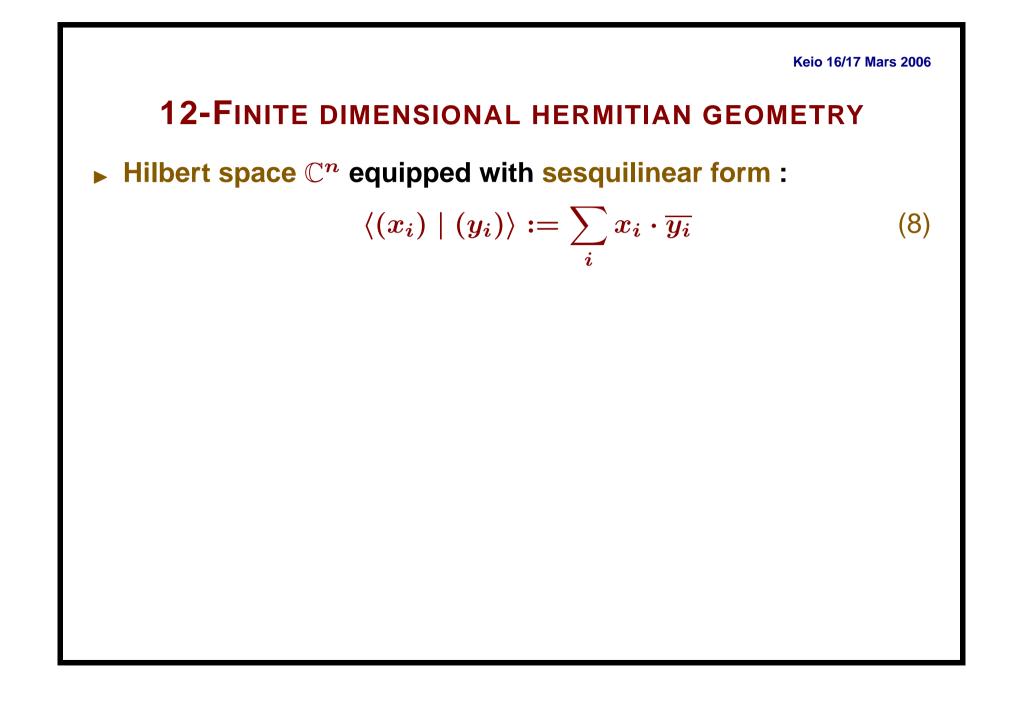
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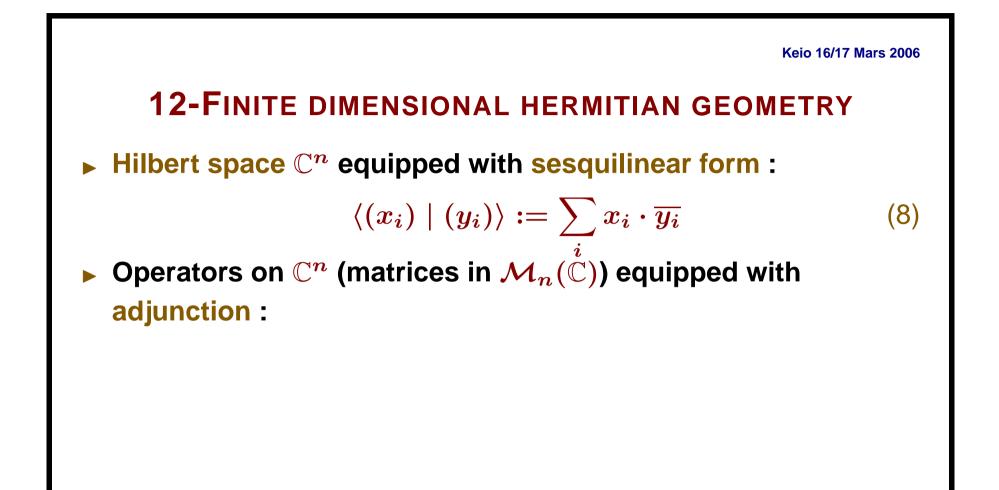
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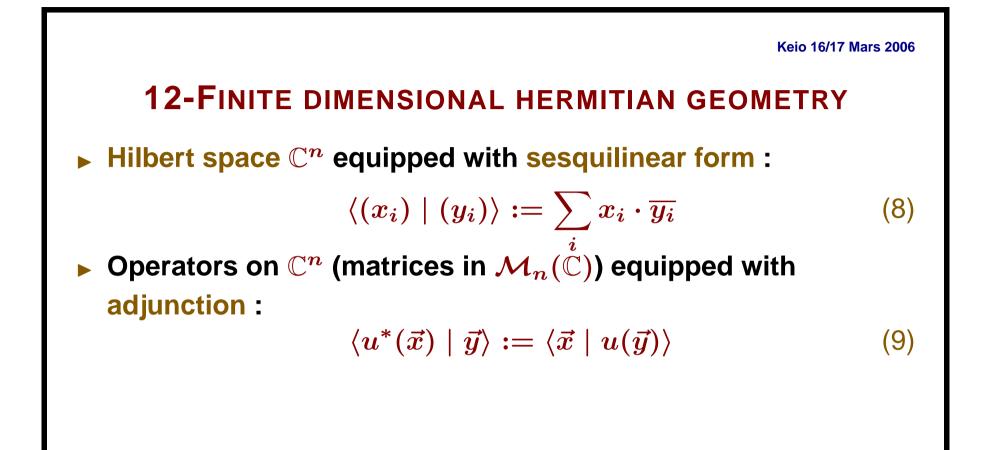
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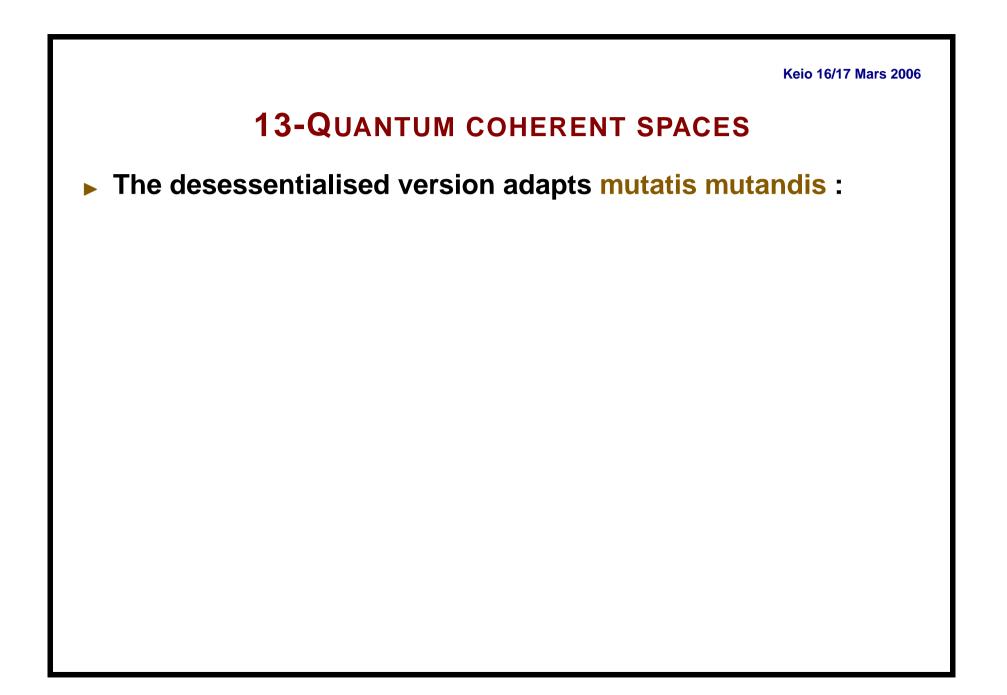
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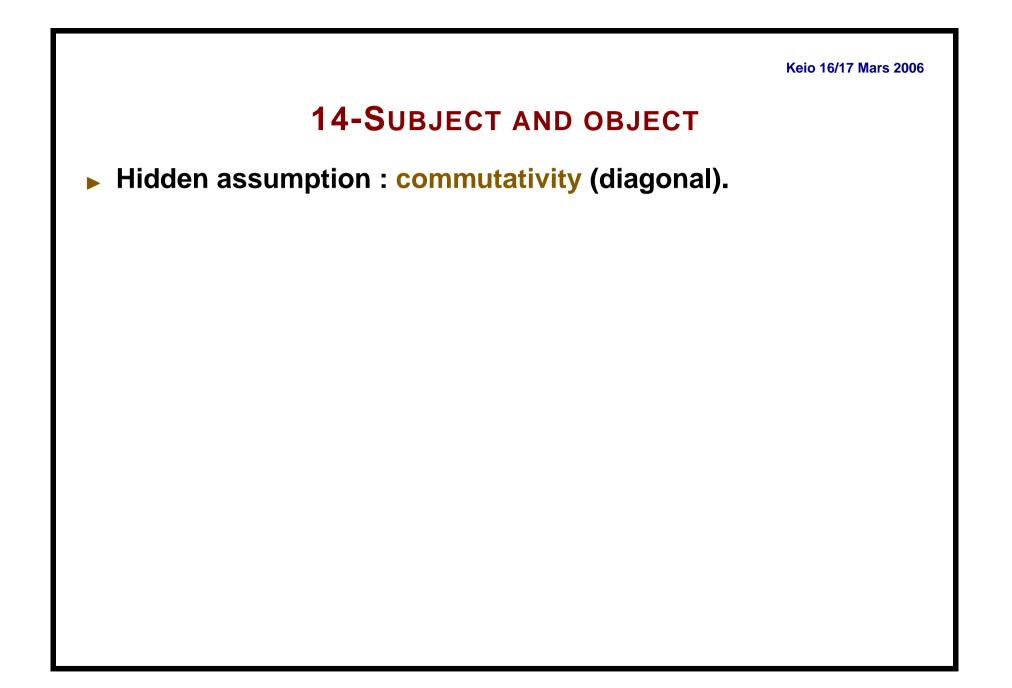
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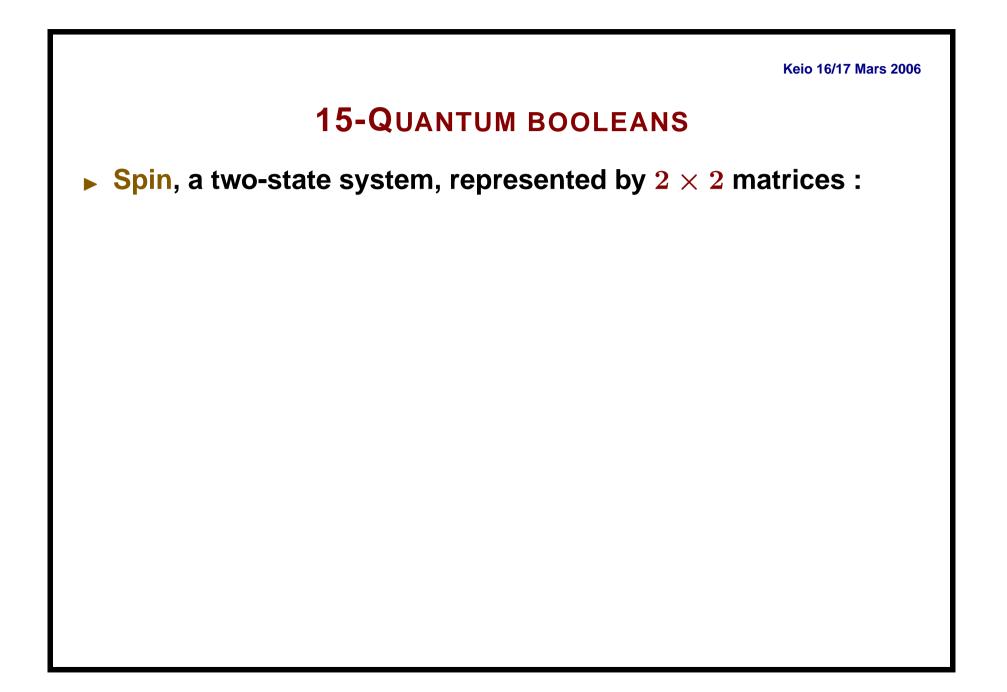
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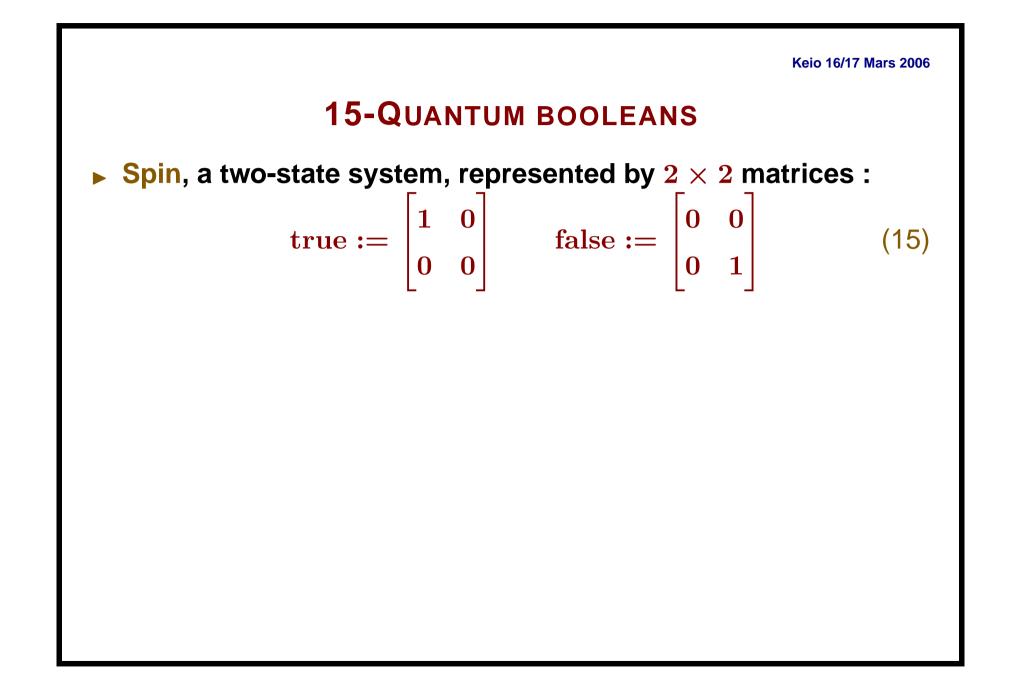
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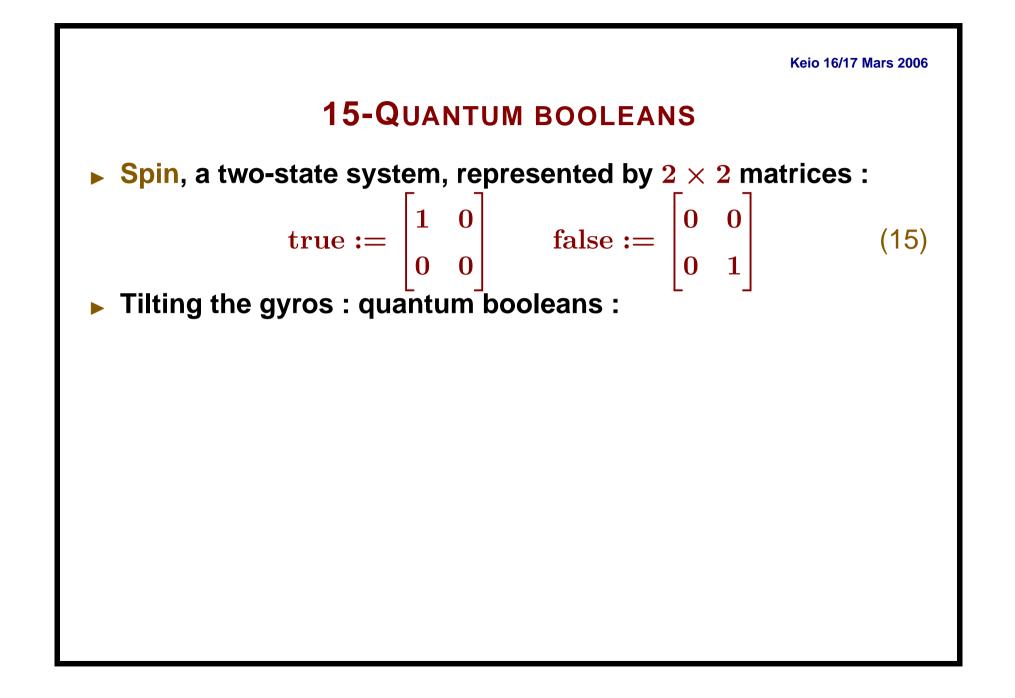
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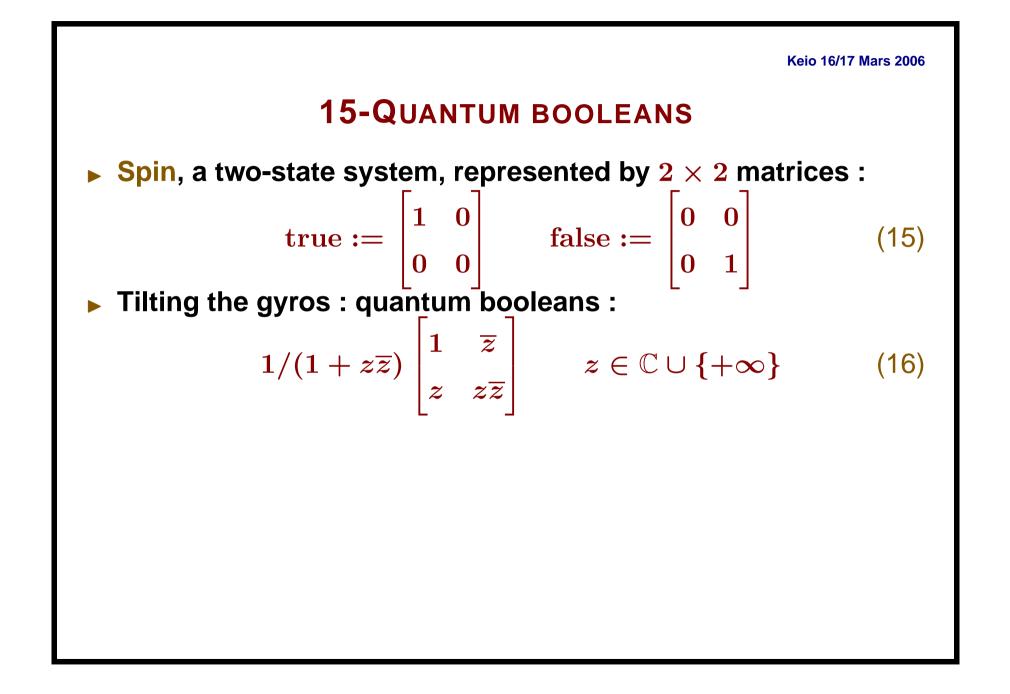
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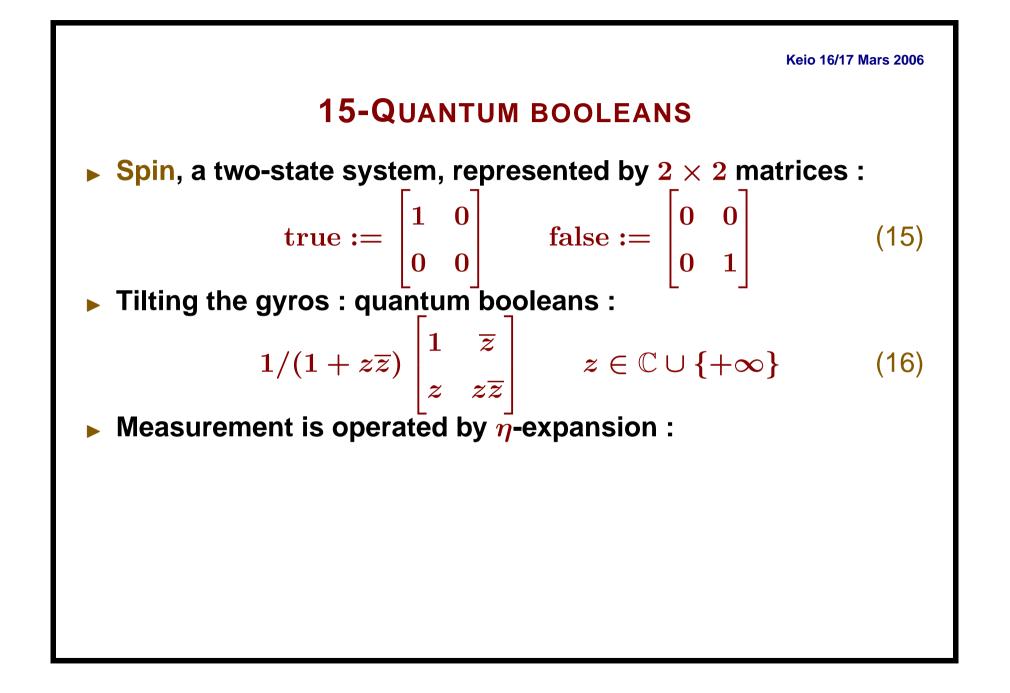
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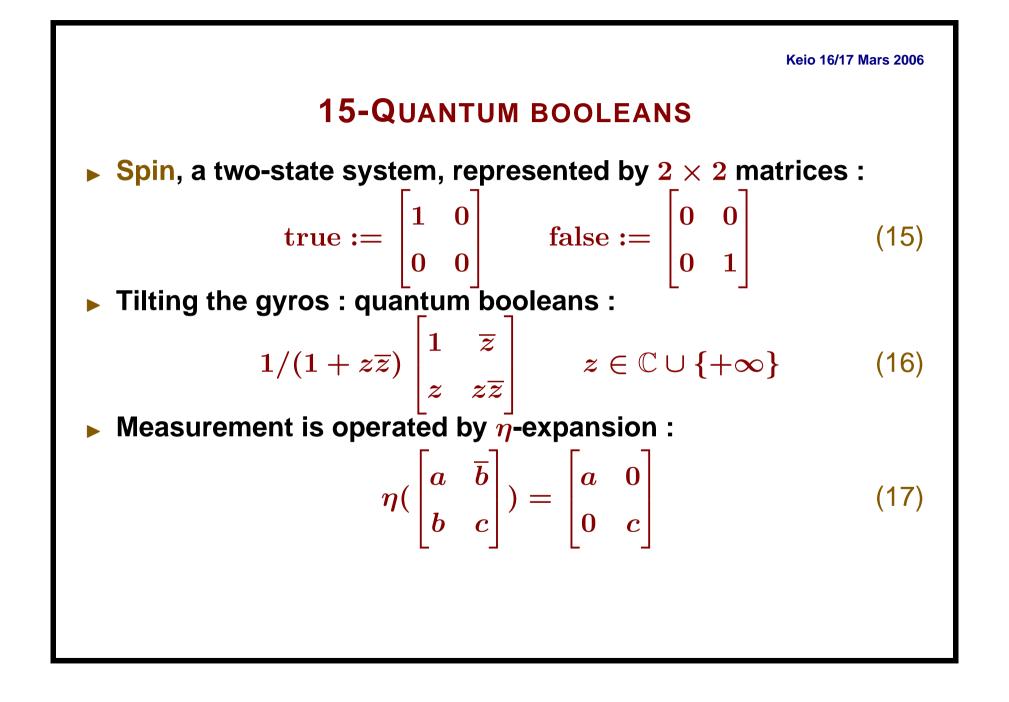


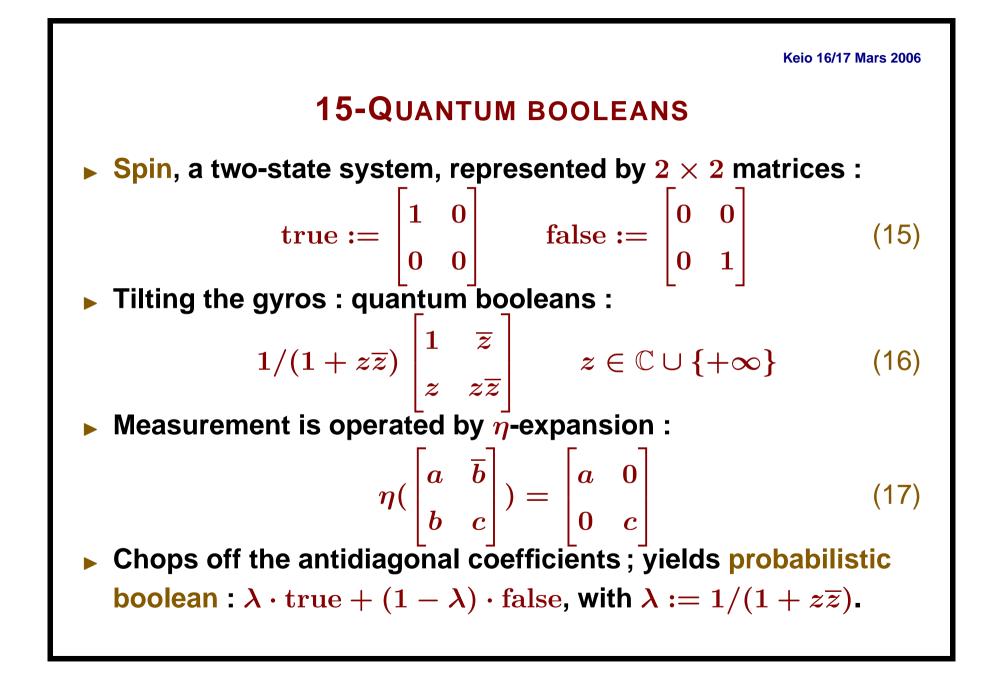






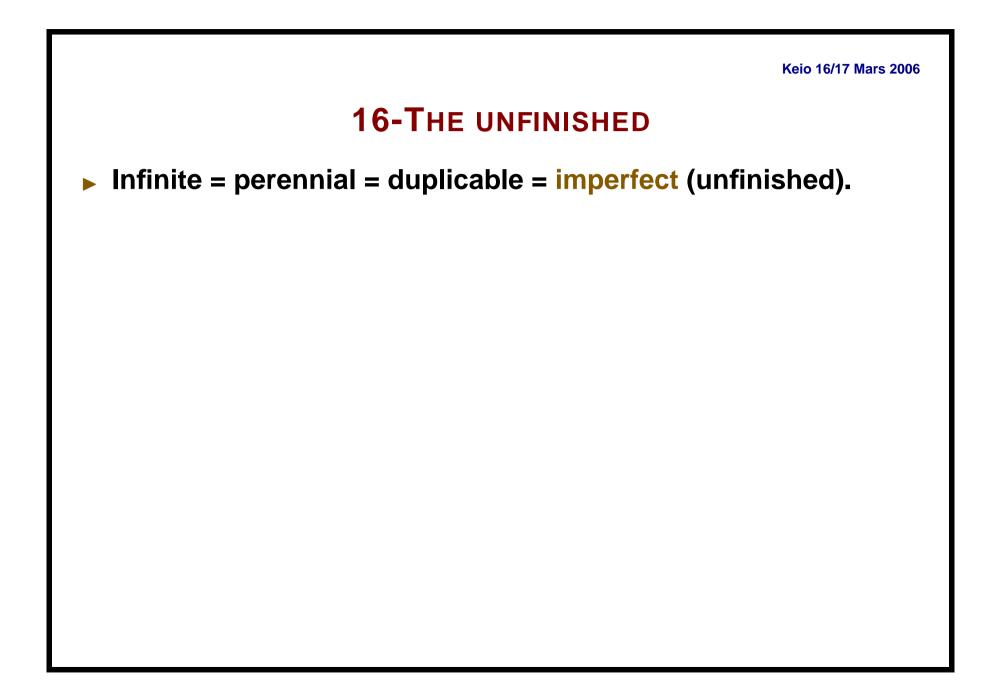






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III-PASSAGE TO INFINITY



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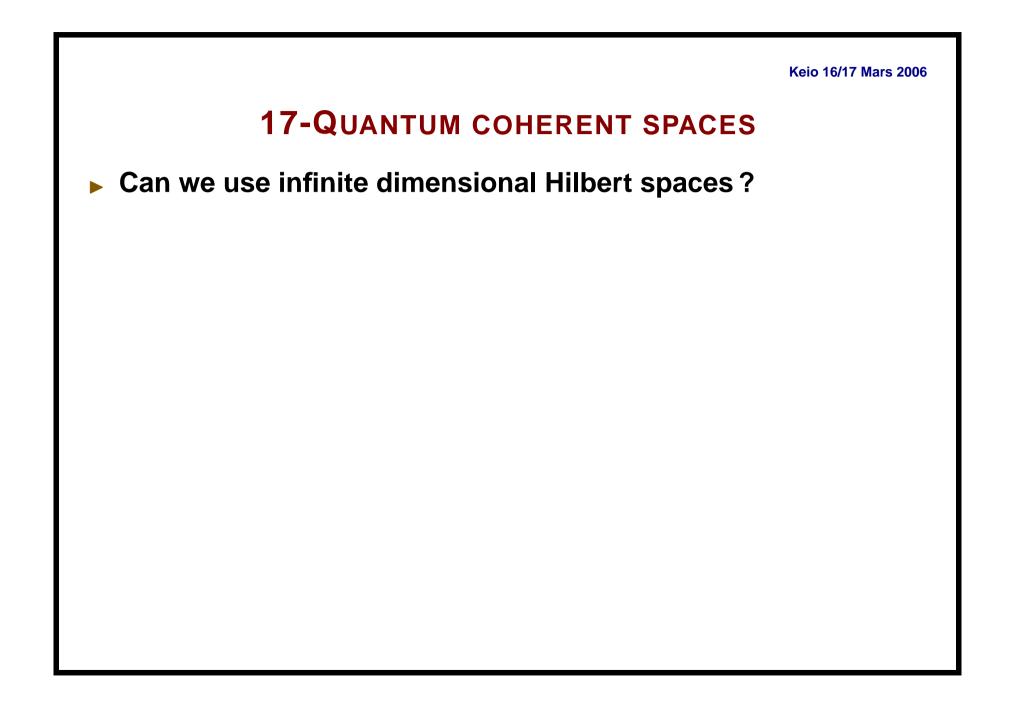
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- Light logics (LLL, ELL...); not grounded. But some hope!



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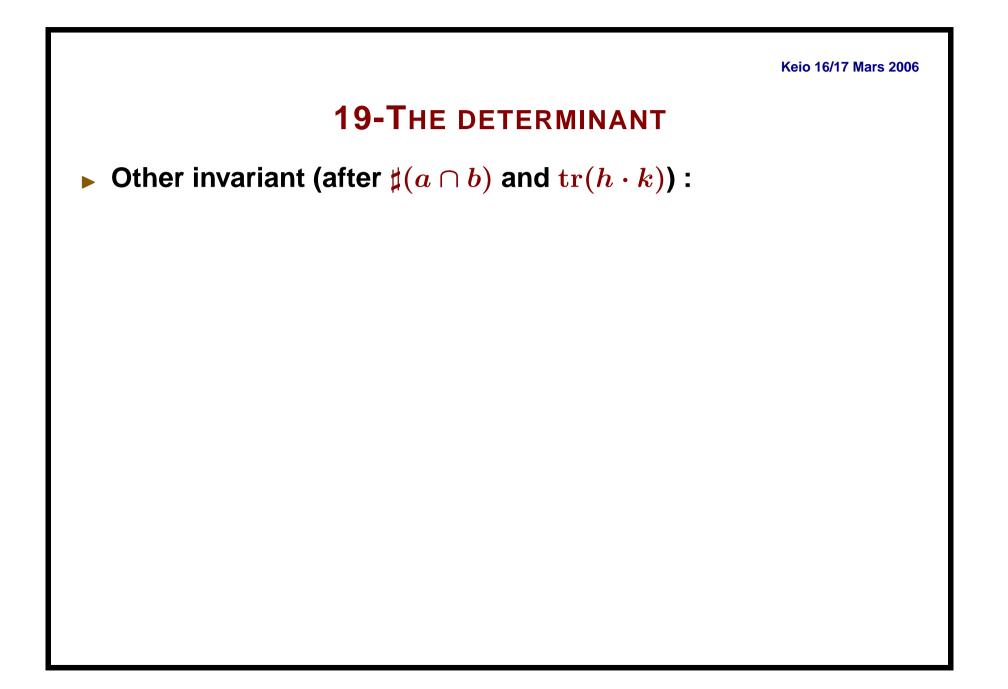
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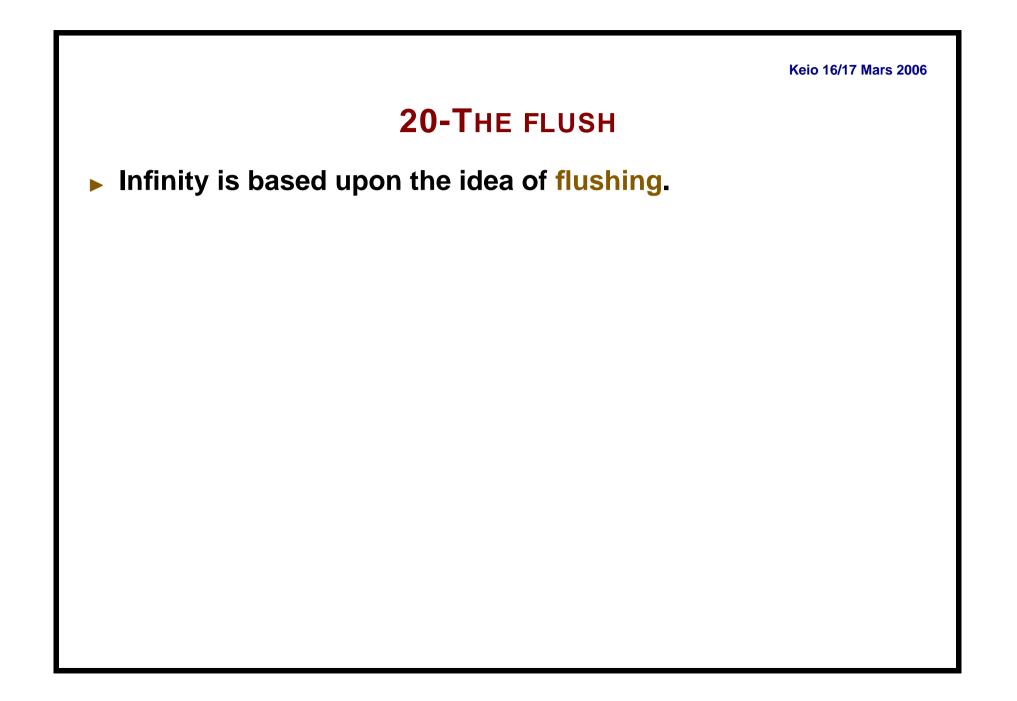
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Equation (22) does not pass infinite limits. Remains the determinant, i.e., Gol. One should remain potential.



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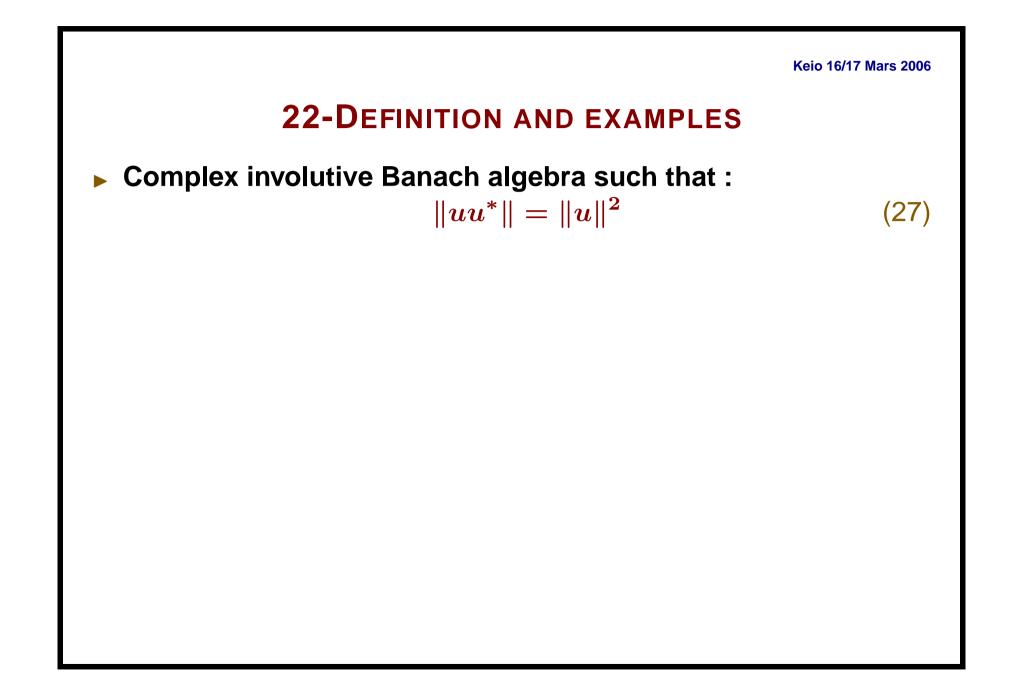
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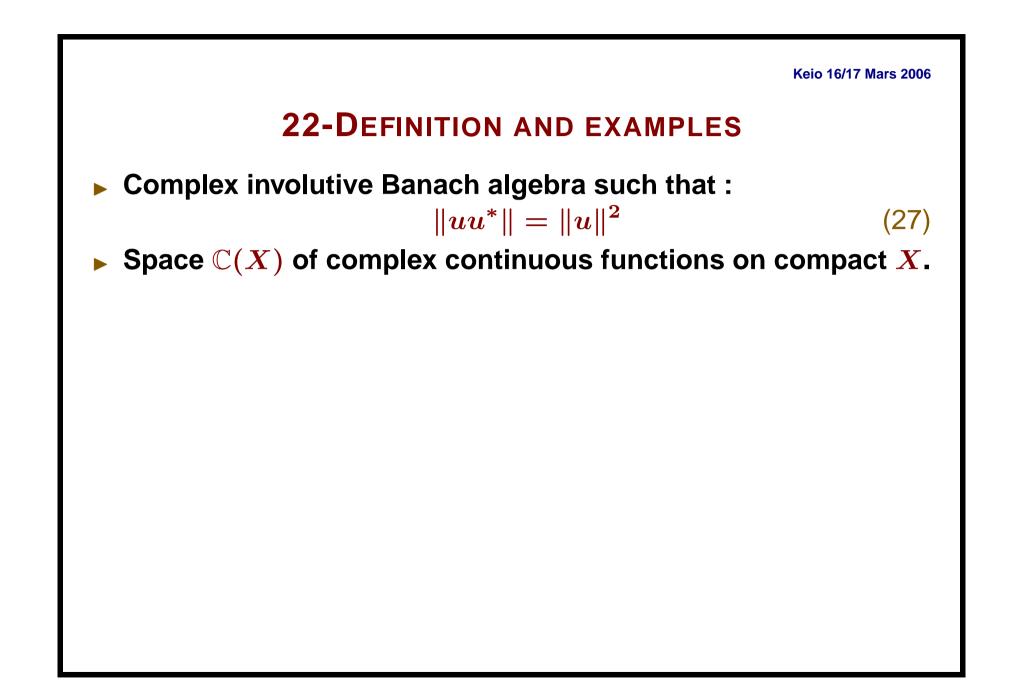
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IV-C*-ALGEBRAS



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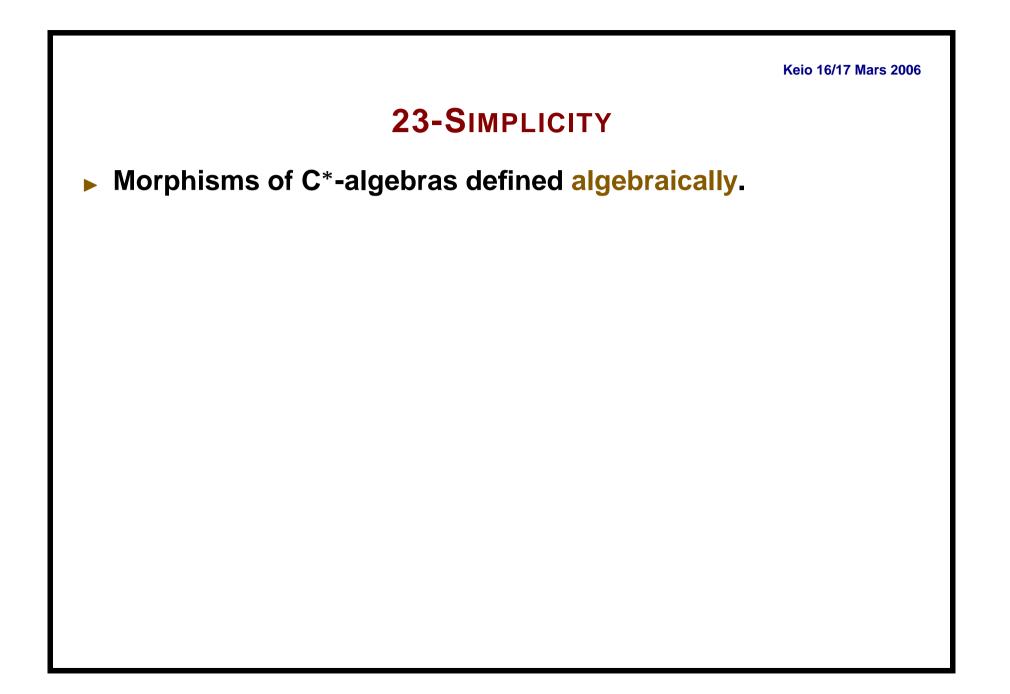
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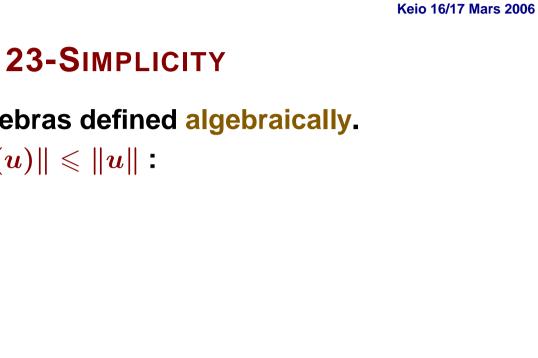
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- $\mathcal{B}(\mathbb{H})$ not simple (infinite dimension) : compact operators.

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• Canonical anticommutation relations, between creators $\kappa(a)$ and their adjoints, the annihilators $\zeta(b)$:

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 - The Clifford algebra : use $\kappa(a) + \zeta(a)$.

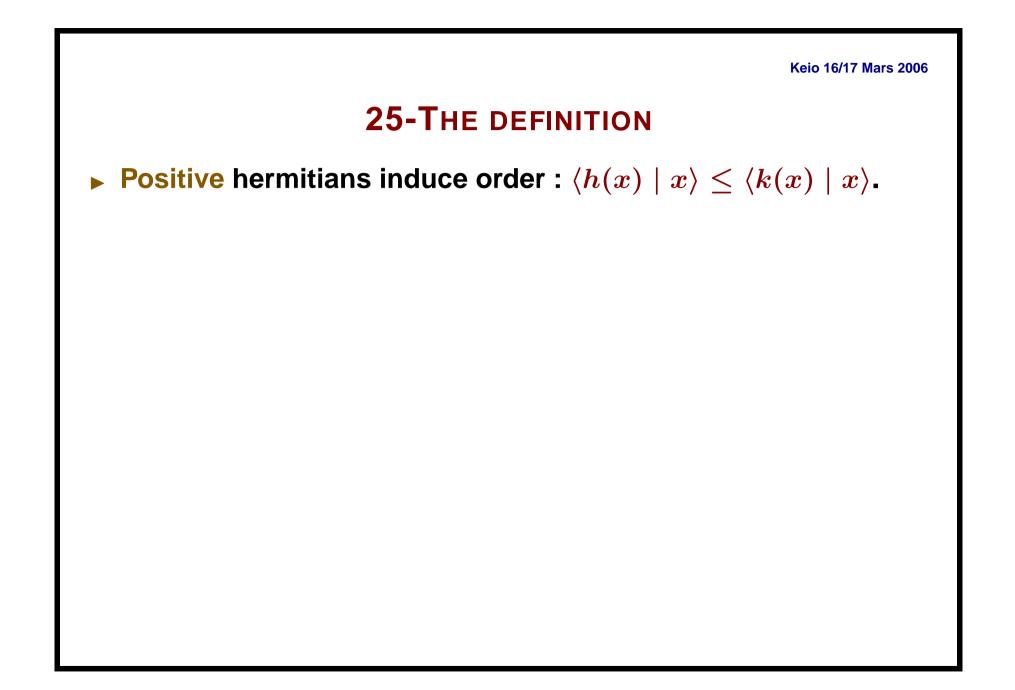
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- ▶ a, b range over a set A (or a Hilbert space $\delta_{ab} \rightsquigarrow \langle a \mid b \rangle$).
 - If A is finite, Car(A) algebraically isomorphic to matrices $n \times n$, with $n := 2^{\sharp(A)}$.
 - By simplicity, unique C*-norm on Car(A) for A finite.
 - The same holds in general : use inductive limits.
- Related topics :
 - The Clifford algebra : use $\kappa(a) + \zeta(a)$.
 - The (exterior) Fock space : represent $\kappa(a)(x) := a \wedge x$.

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V-vN ALGEBRAS



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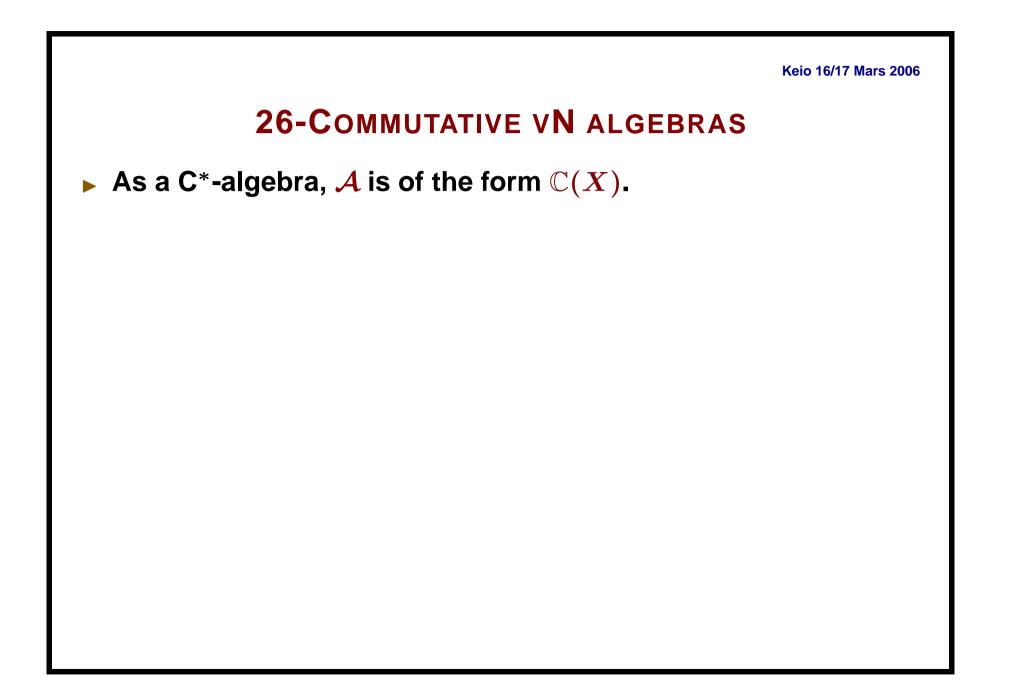
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- Also : the commutant of a self-adjoint subset of $\mathcal{B}(\mathbb{H})$.



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- ▶ In general : C*-algebra + faithful state ρ (i.e., $\rho(uu^*) = 0$ implies u = 0.) yields a vN completion.
- ▶ The CAR-algebra admits completions of all types I, II, III.



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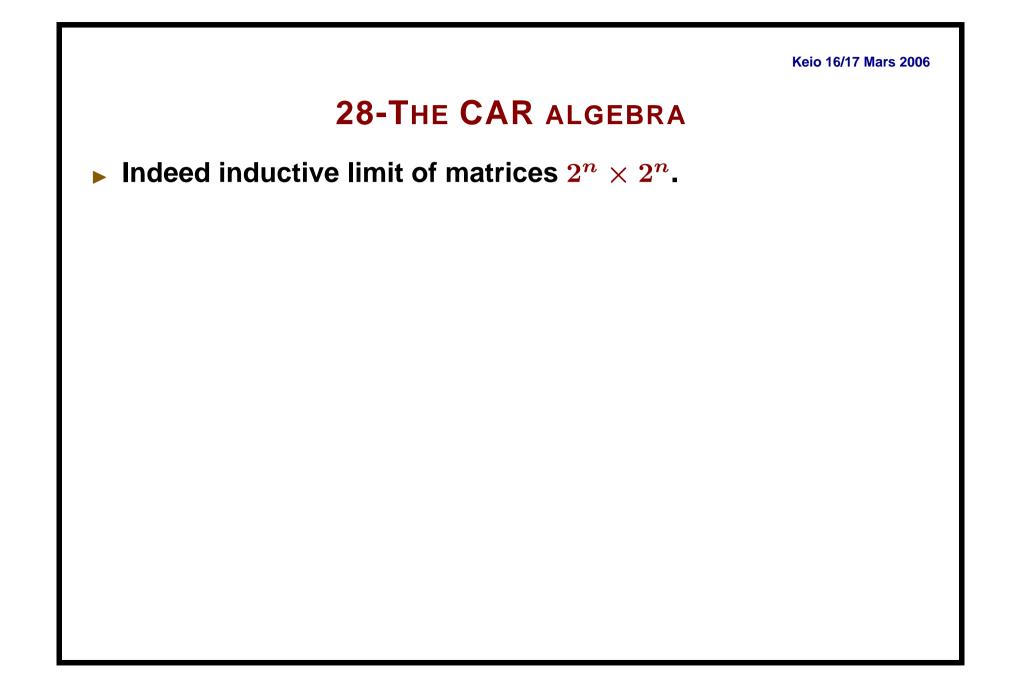
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- ► Typical case : simple algebras.



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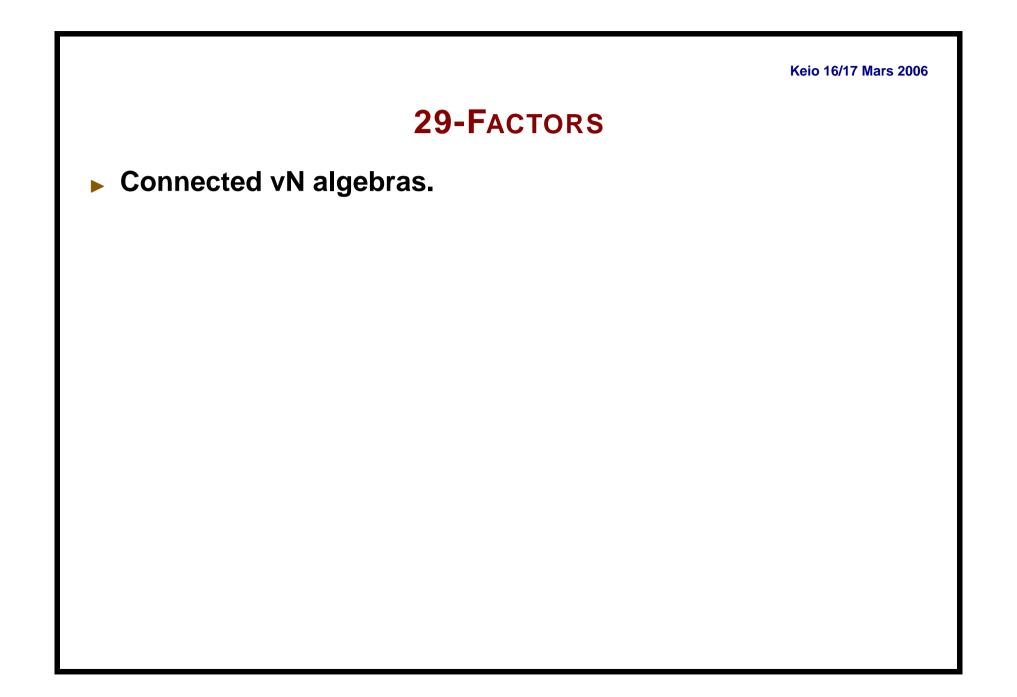
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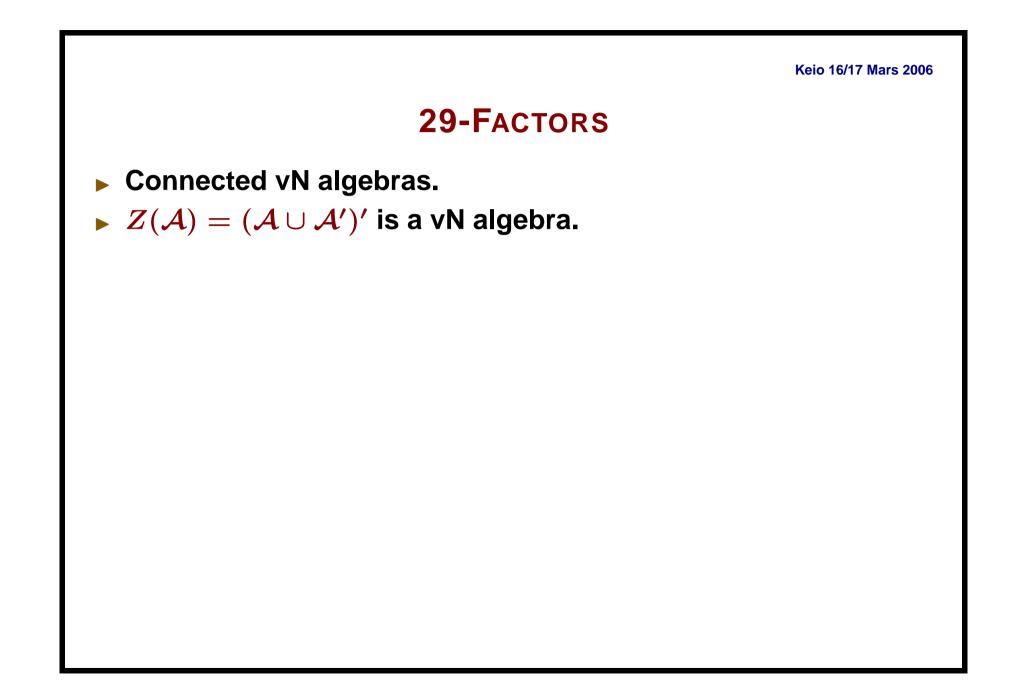
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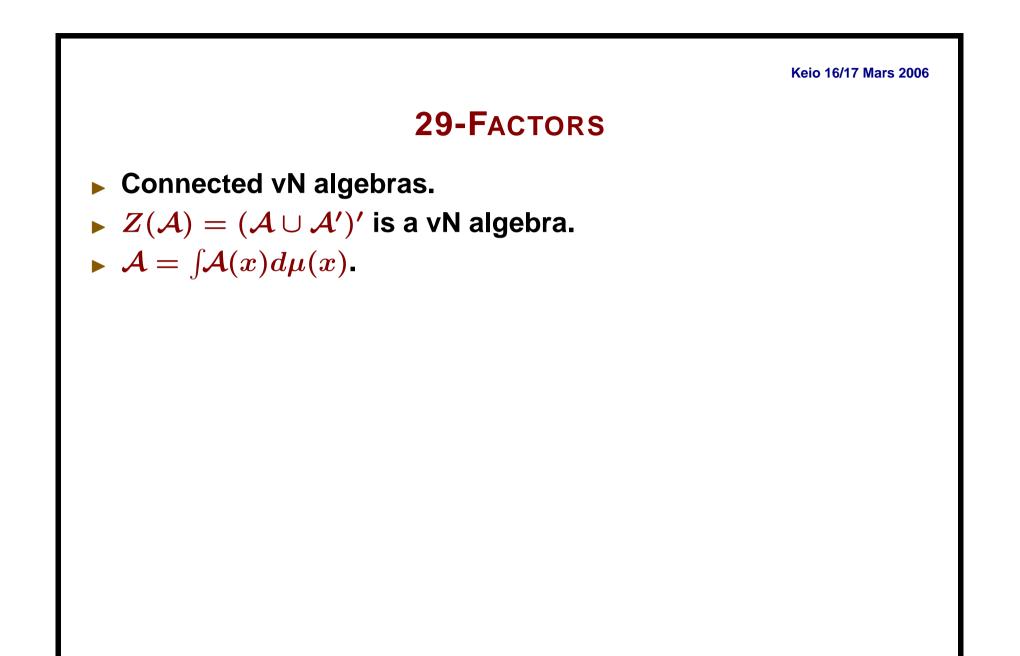
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Up to isomorphism, only one such vN algebra, the Murray-von Neumann factor *R*.

VI-THE FINITE/HYPERFINITE FACTOR







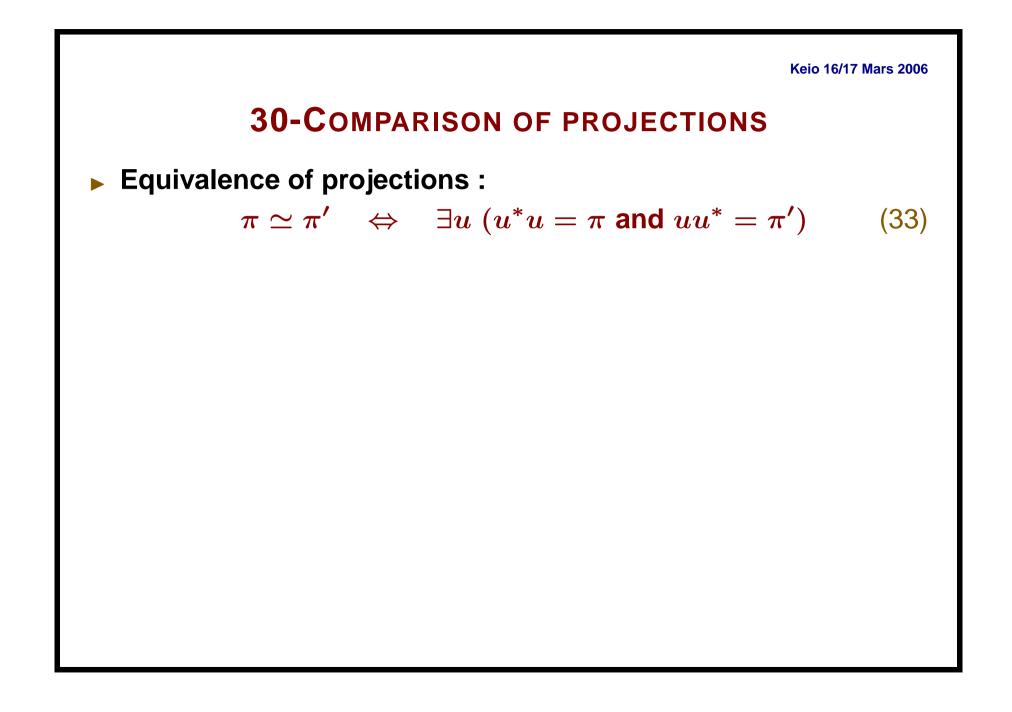


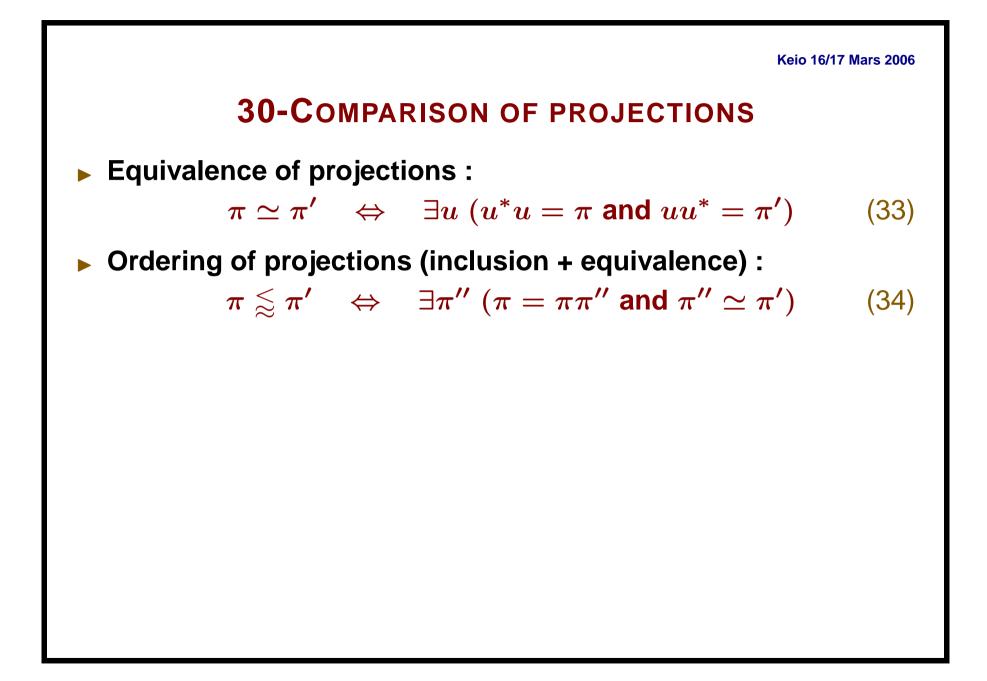
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- Connected vN algebras.
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- Classification of vN algebras thus reduces to classification of factors.







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Equivalence of projections :

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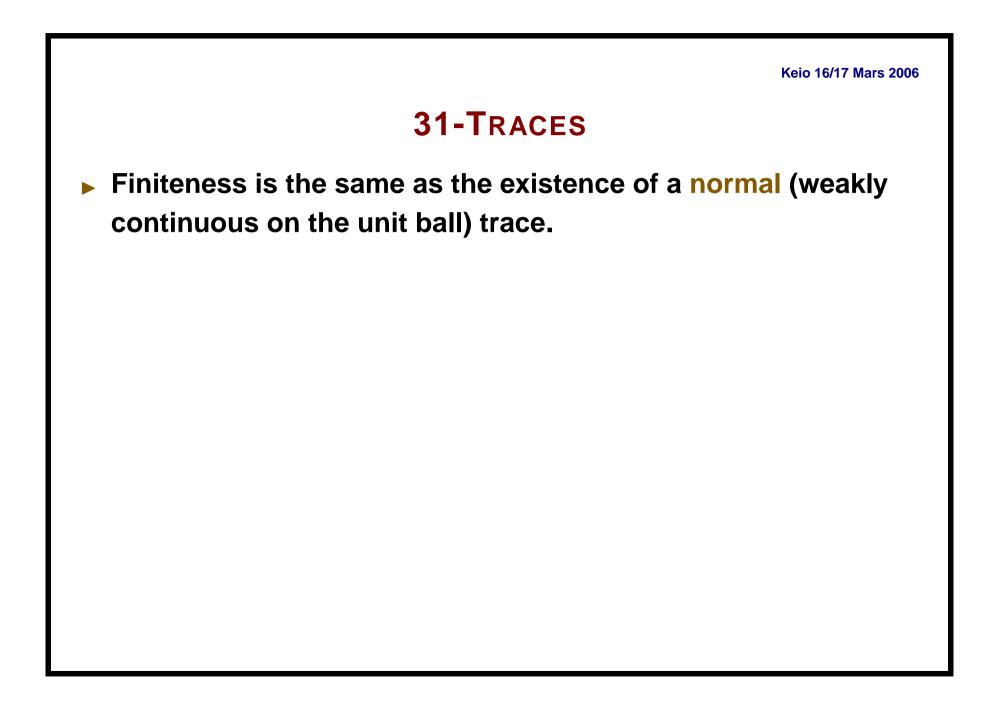
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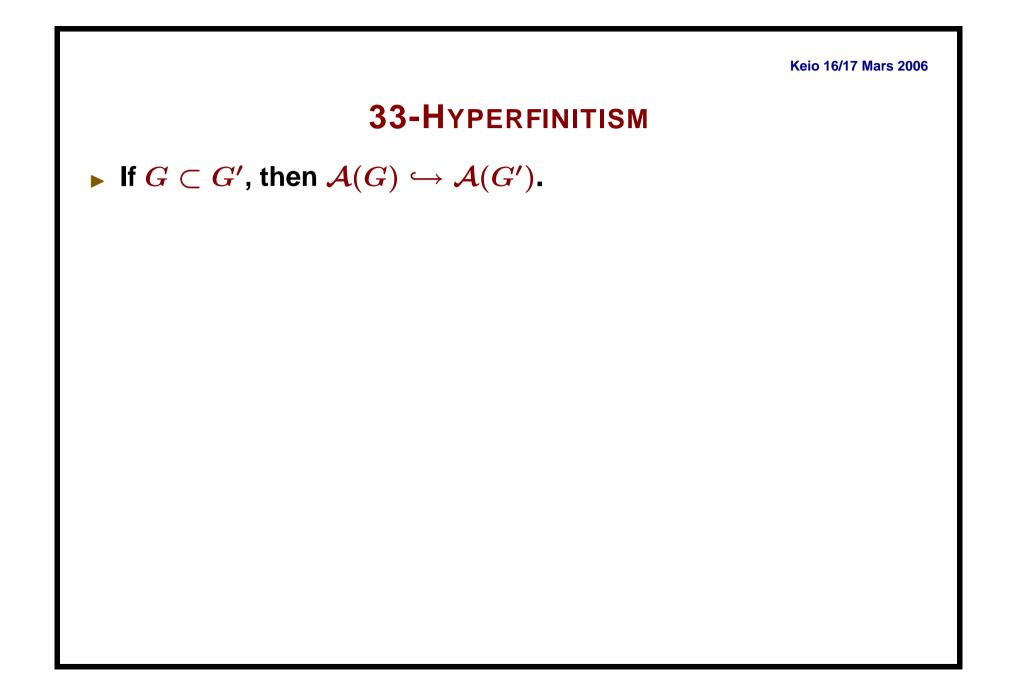
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▶ B.t.w.,
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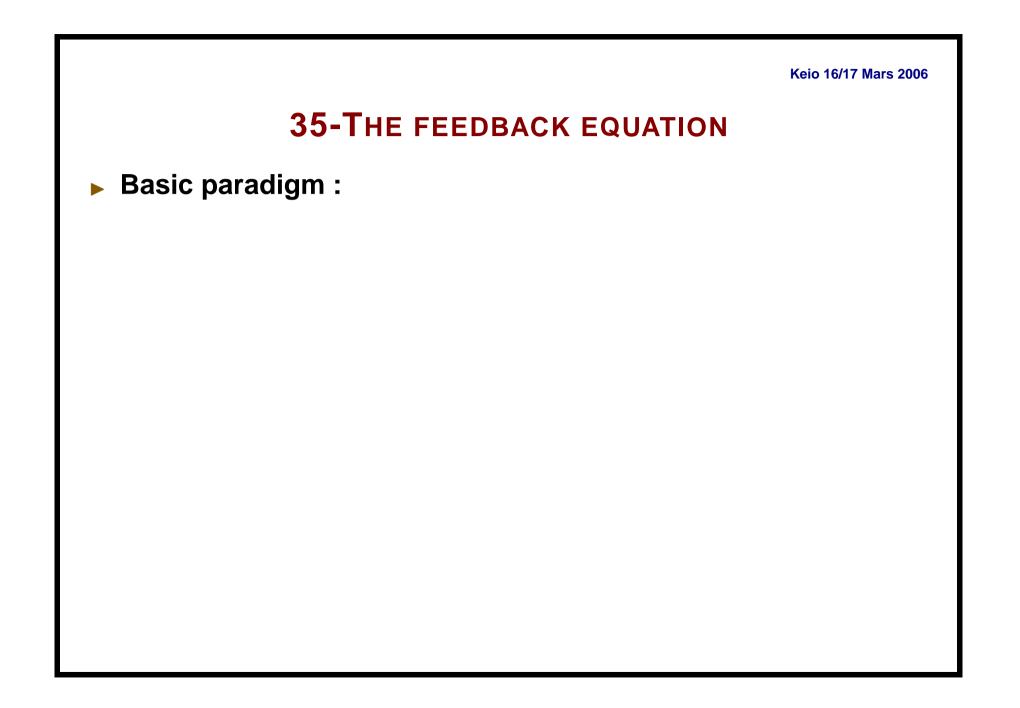
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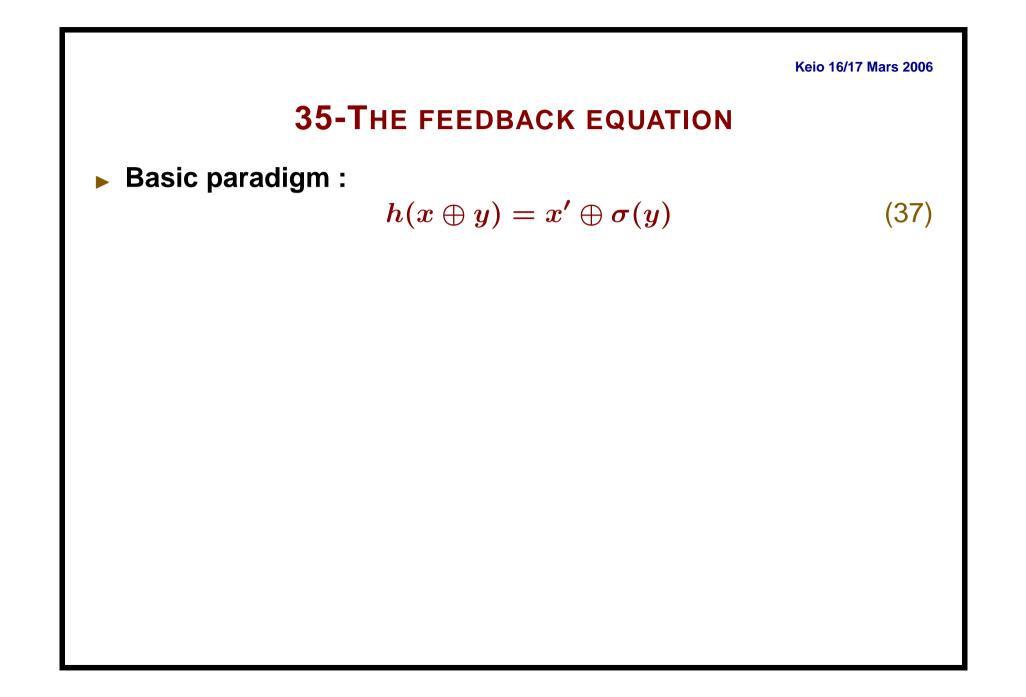
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VII-Gol





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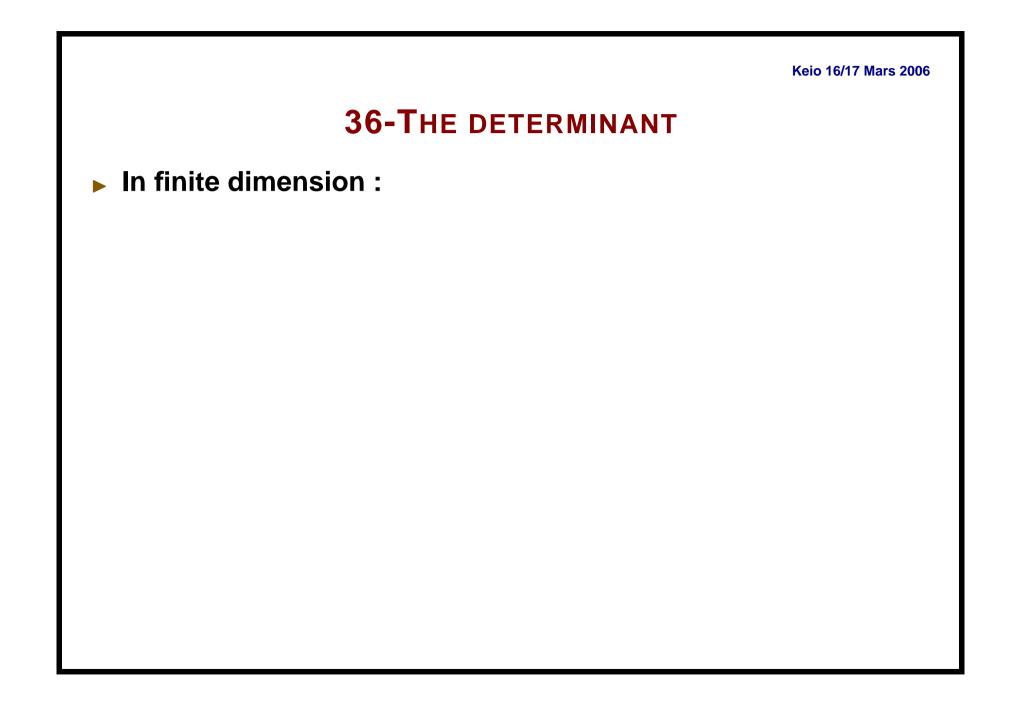
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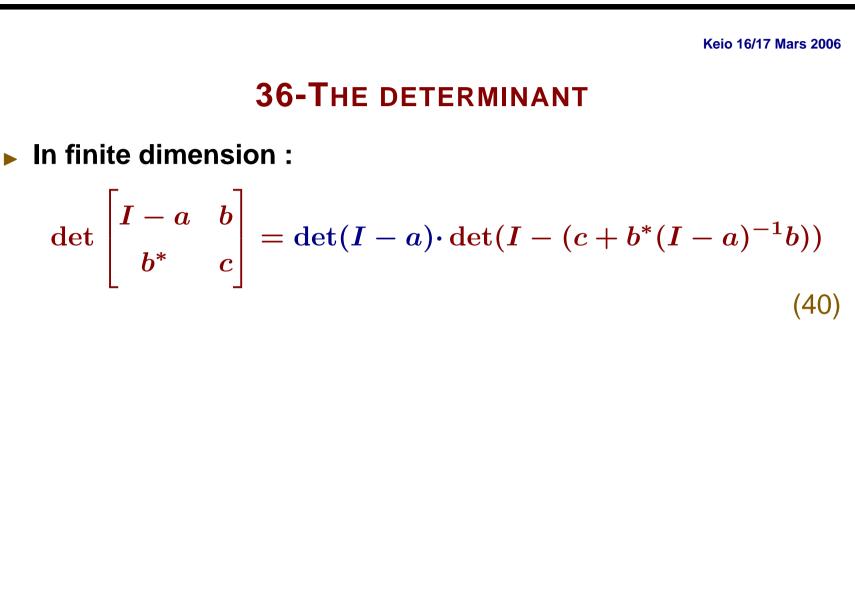
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▶ In finite dimension :

$$\det \begin{bmatrix} I-a & b \\ b^* & c \end{bmatrix} = \det(I-a) \cdot \det(I-(c+b^*(I-a)^{-1}b))$$
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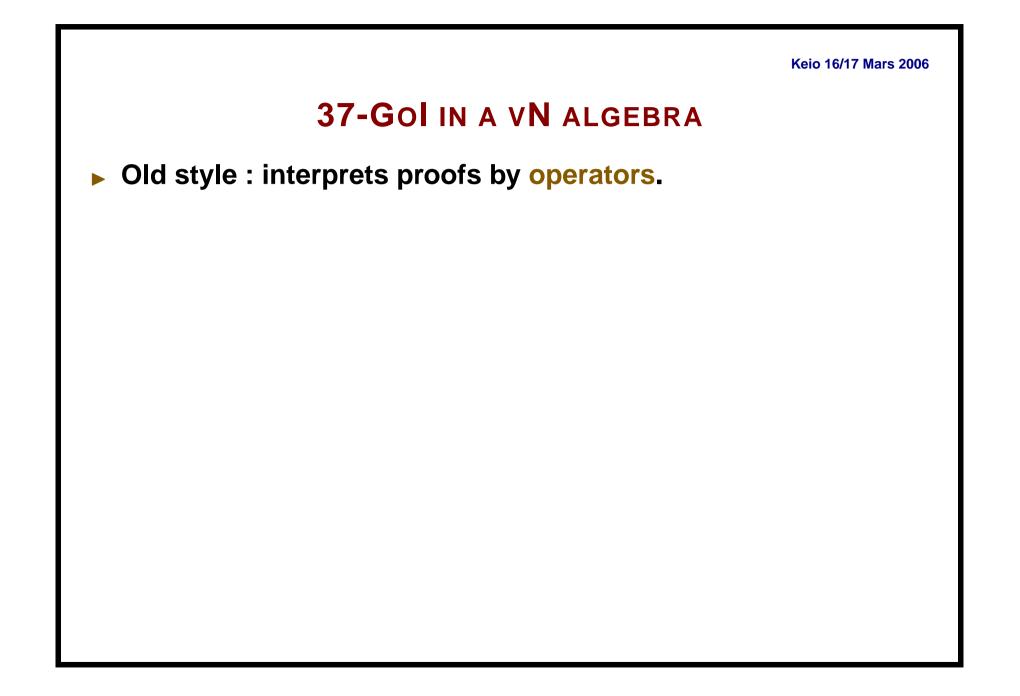
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VIII-FINITE GOI

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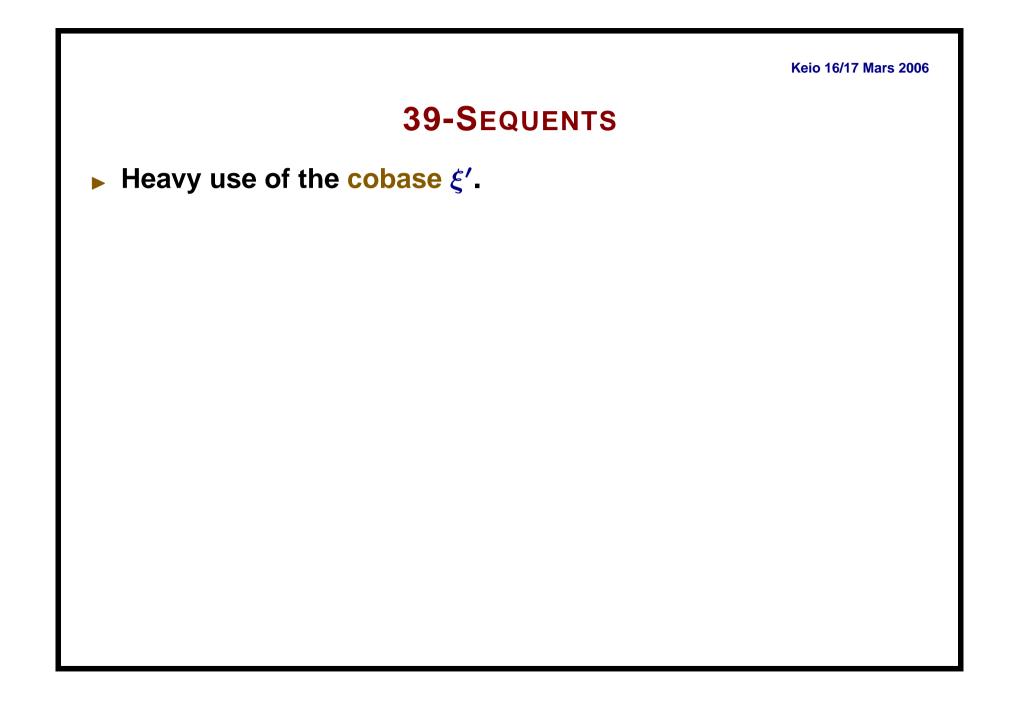
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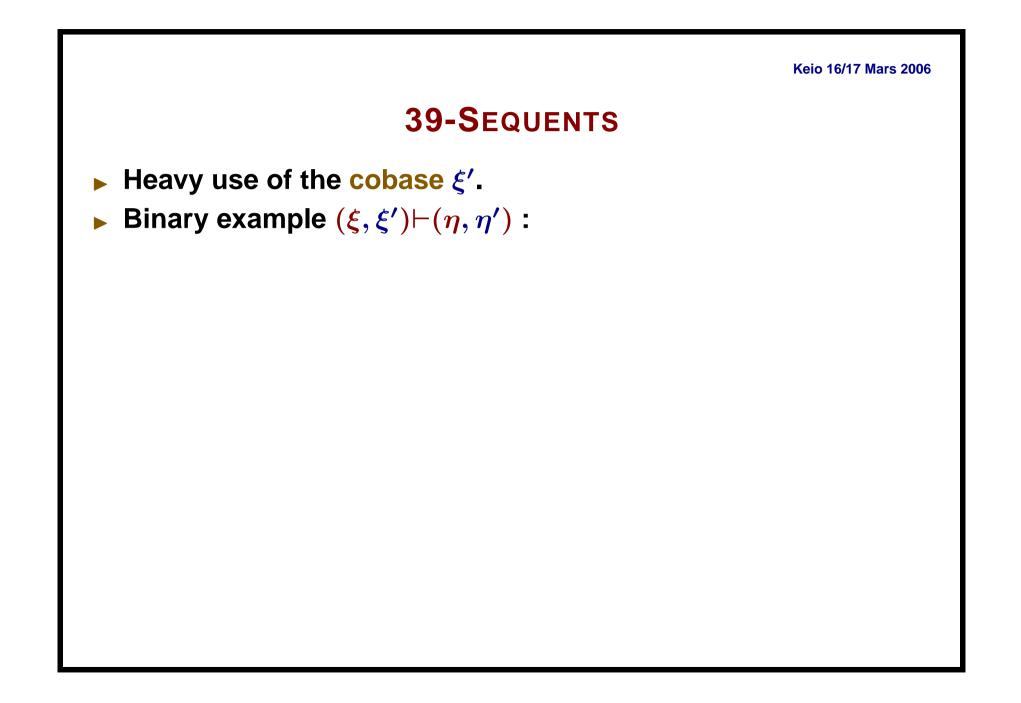
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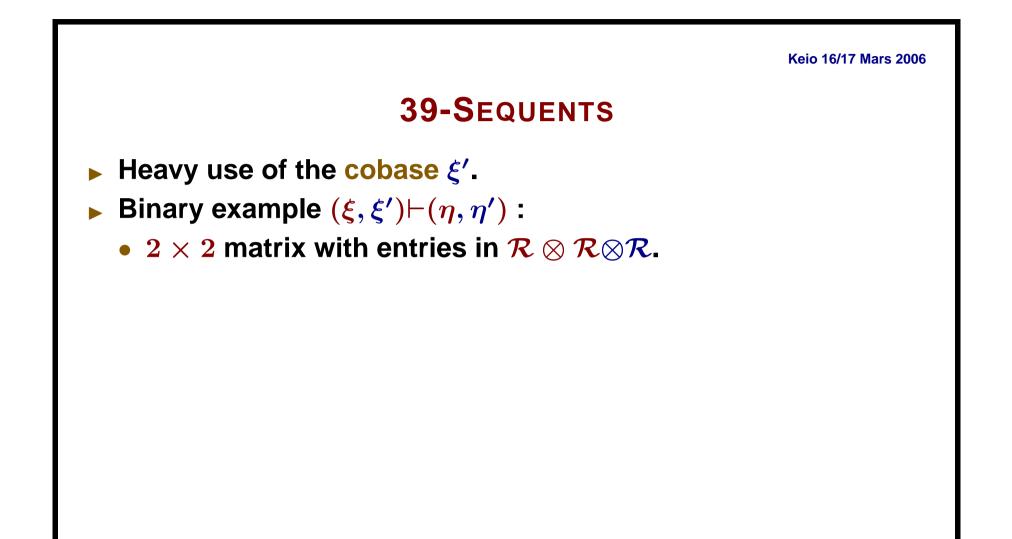
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• Behaviour : set B of designs of given base s.t. $B = \sim \sim B$.









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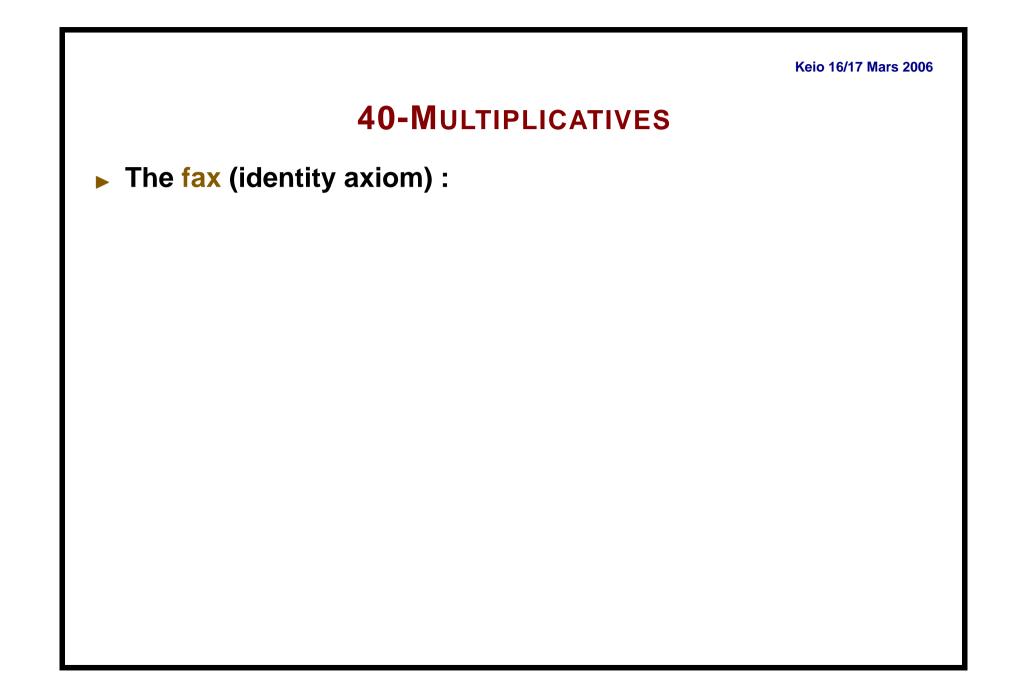


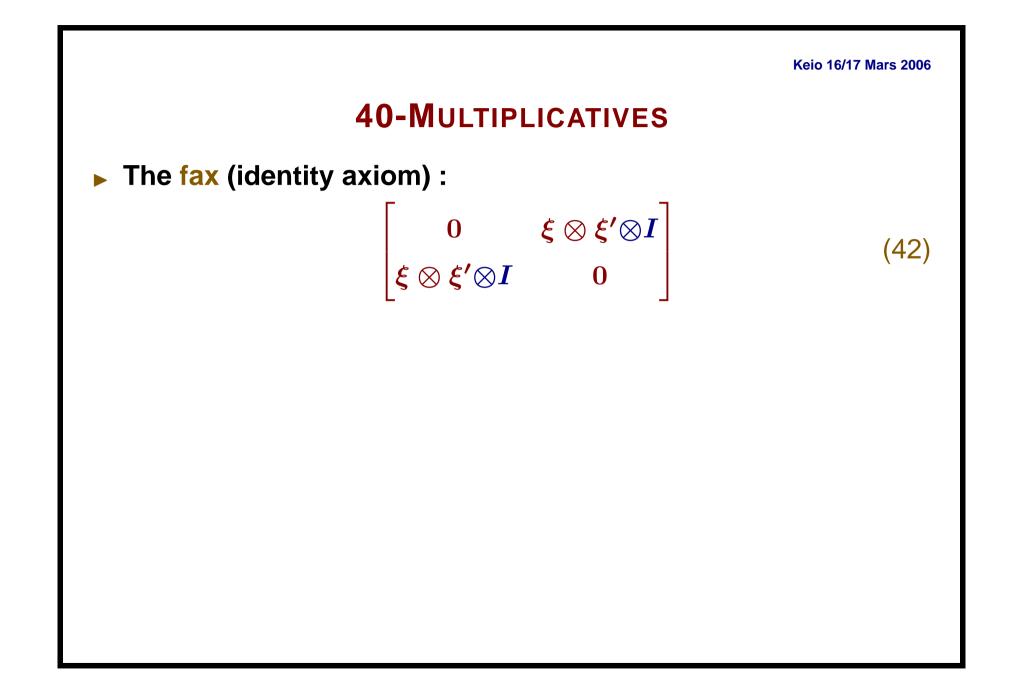
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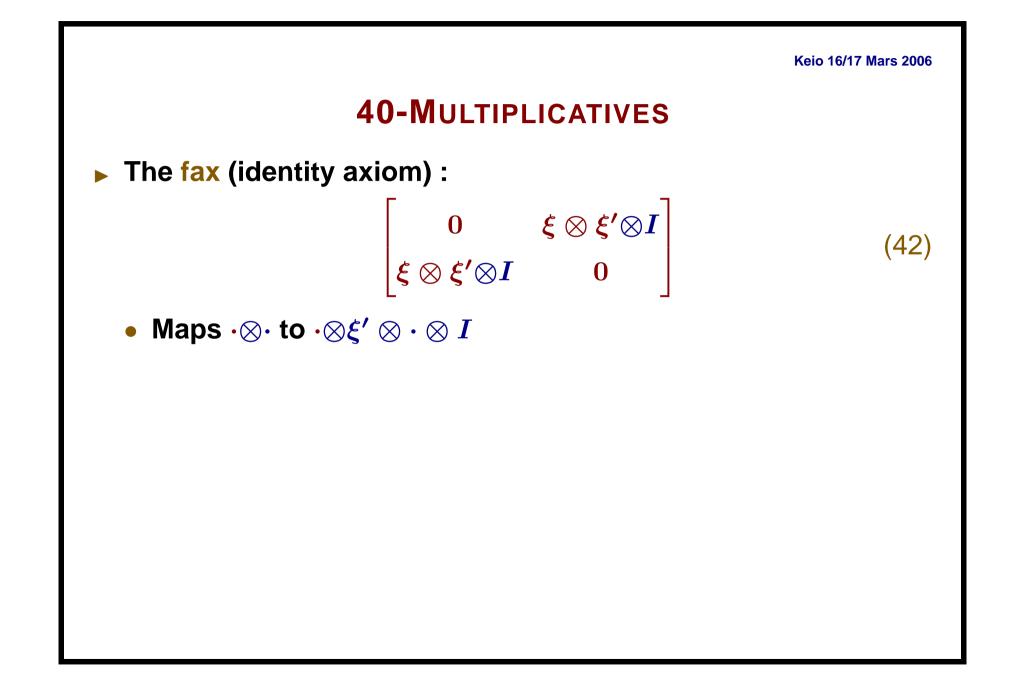
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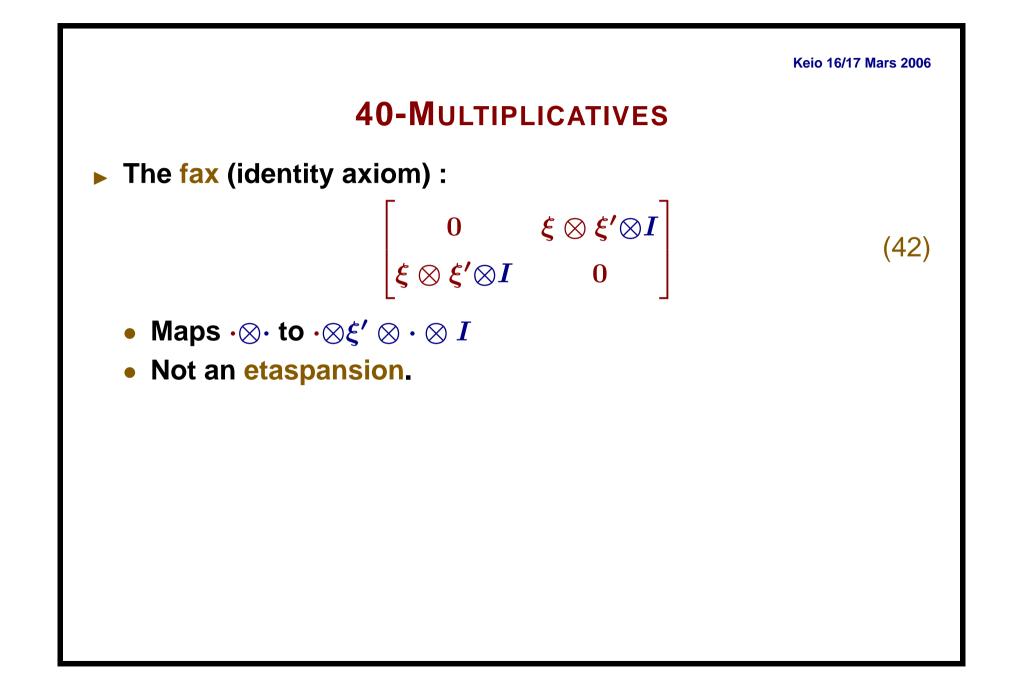
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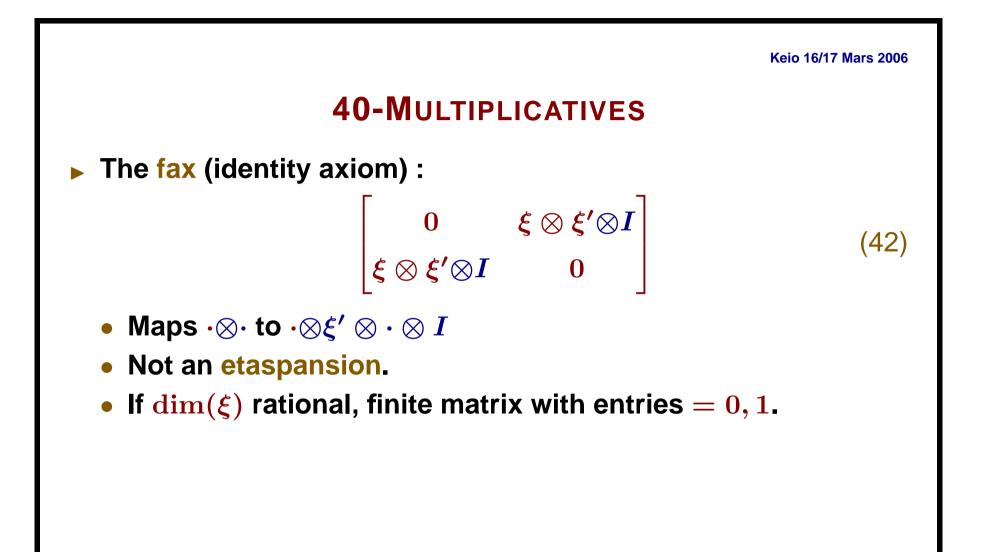
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- Output : $(\gamma^{\dim(\eta)} \cdot \delta \cdot \det(I h' \cdot k''), l)$

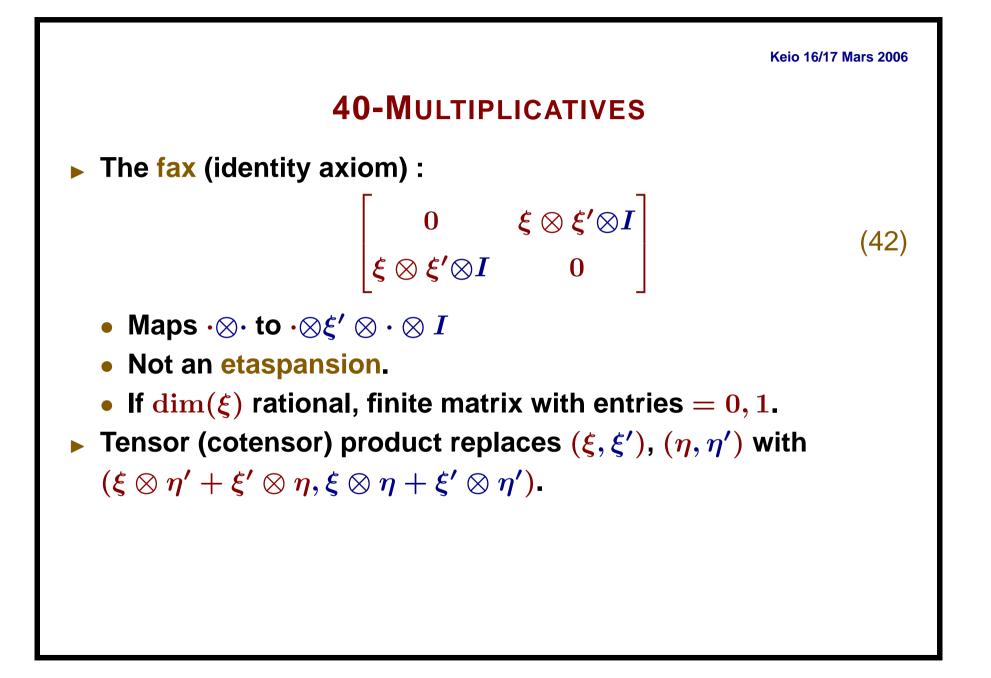














The fax (identity axiom) :

$$\begin{bmatrix} 0 & \xi \otimes \xi' \otimes I \\ \xi \otimes \xi' \otimes I & 0 \end{bmatrix}$$
(42)

Keio 16/17 Mars 2006

- Maps $\cdot \otimes \cdot$ to $\cdot \otimes \xi' \otimes \cdot \otimes I$
- Not an etaspansion.
- If $dim(\xi)$ rational, finite matrix with entries = 0, 1.
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The fax (identity axiom) :

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- ▶ $A \multimap A$ based on $(\xi \otimes \xi' + \xi' \otimes \xi, \xi \otimes \xi + \xi' \otimes \xi')$.

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41-THE ADDITIVE MIRACLE

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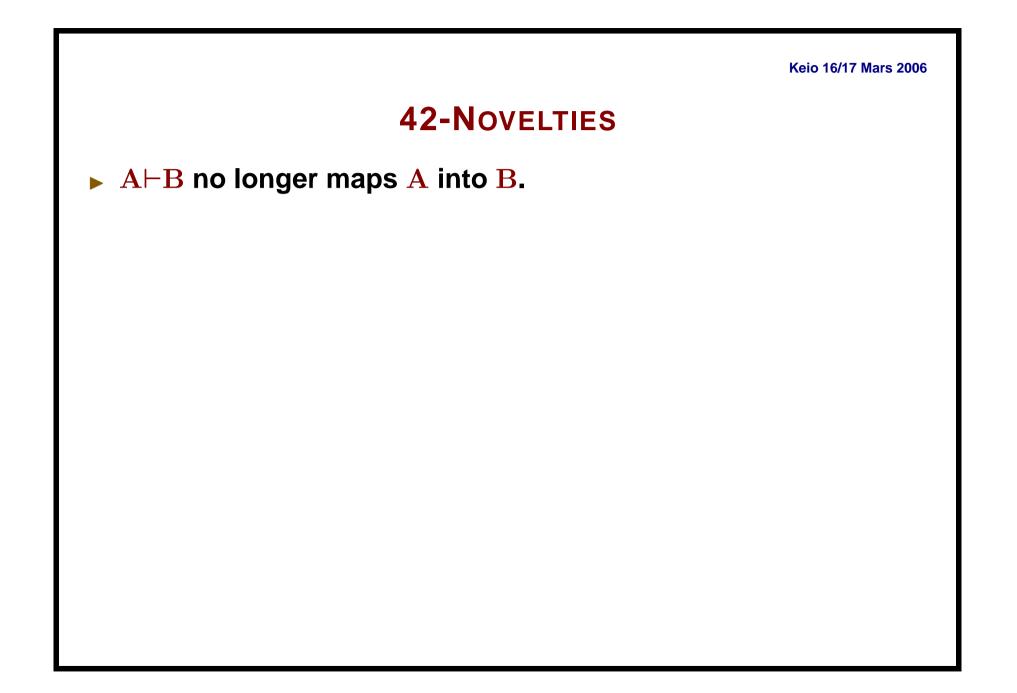
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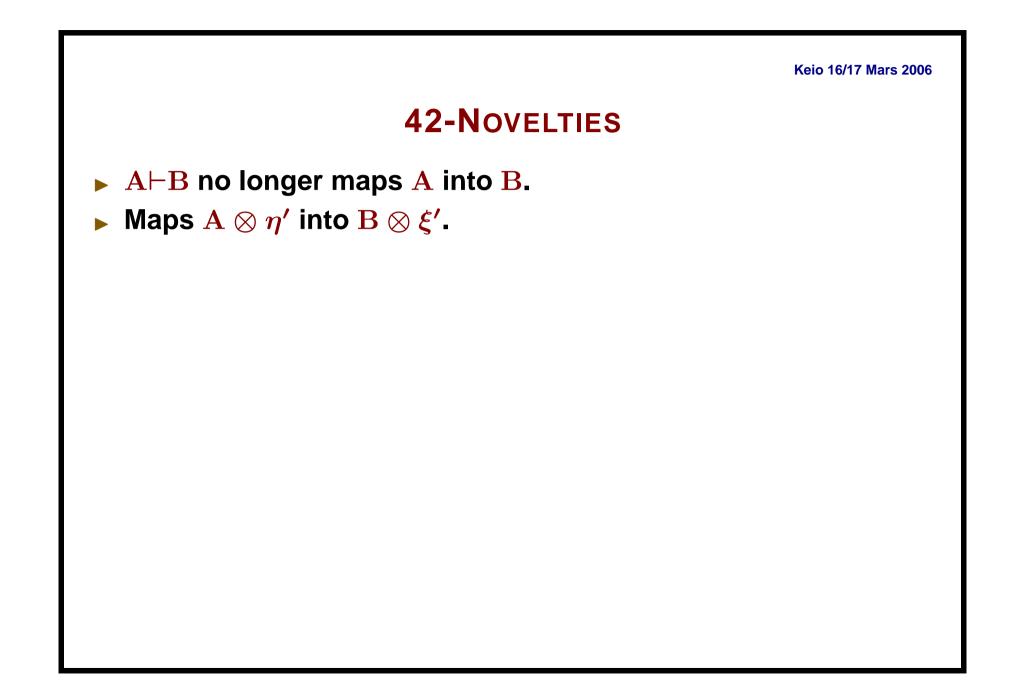
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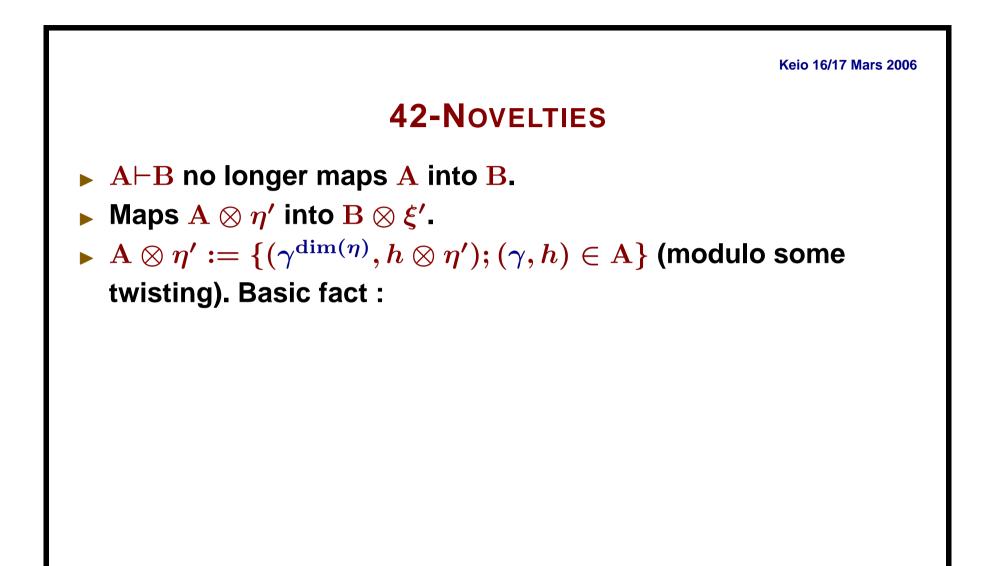
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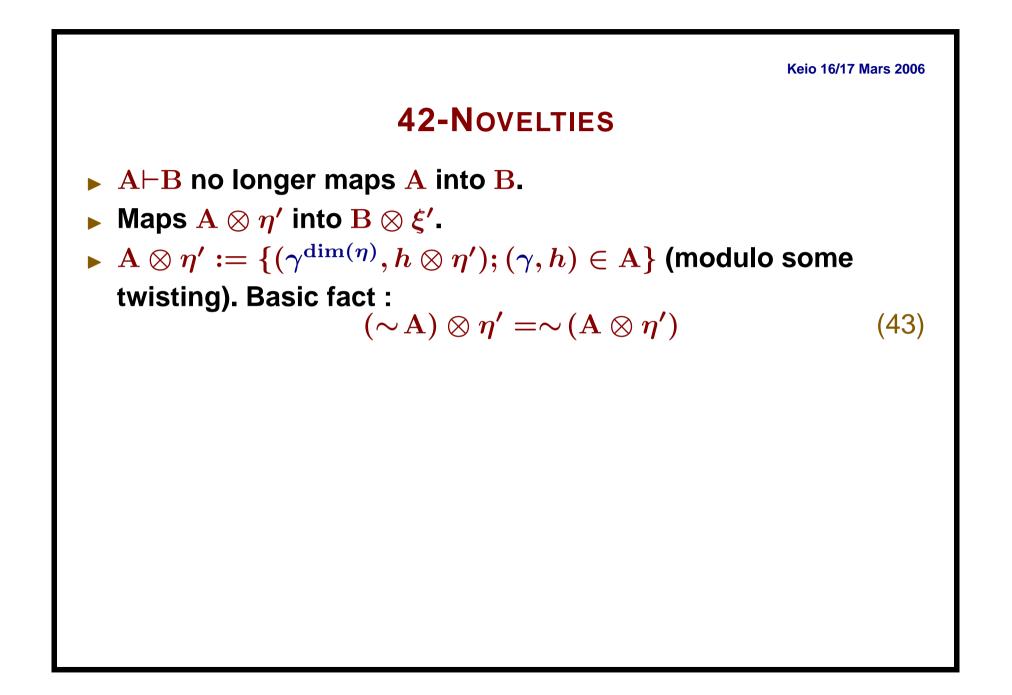
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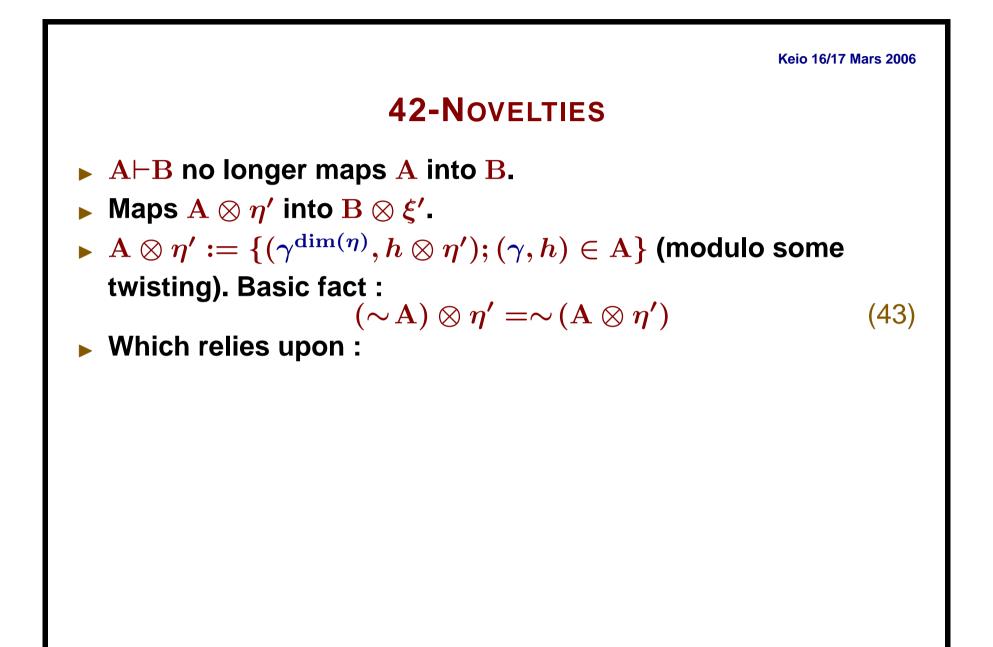
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- Summing up, perfect logic (in the linguistic sense) can be interpreted in the hyperfinite factor.

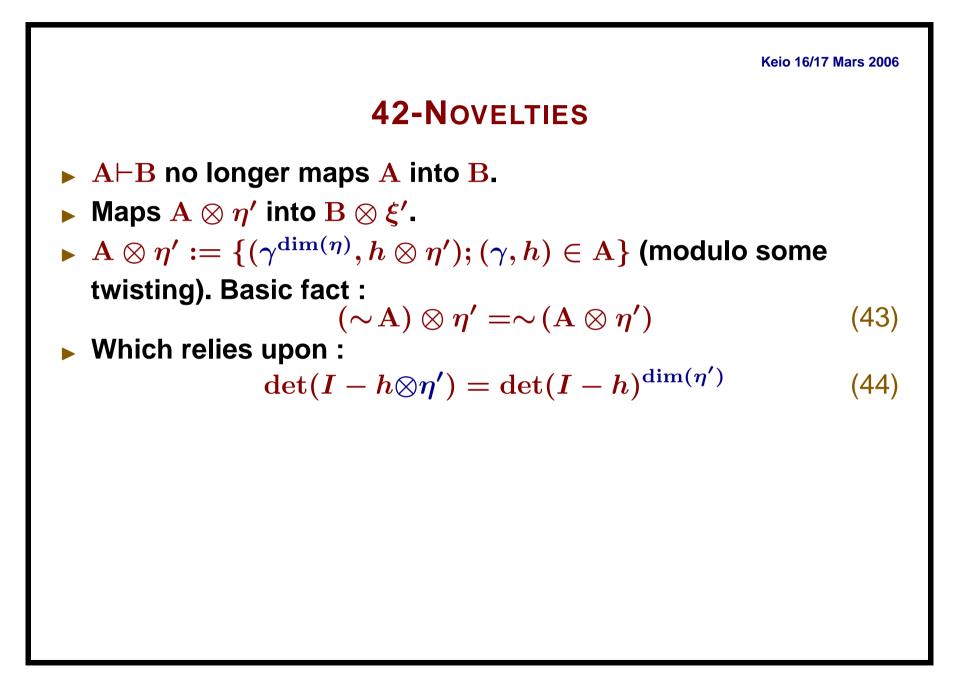


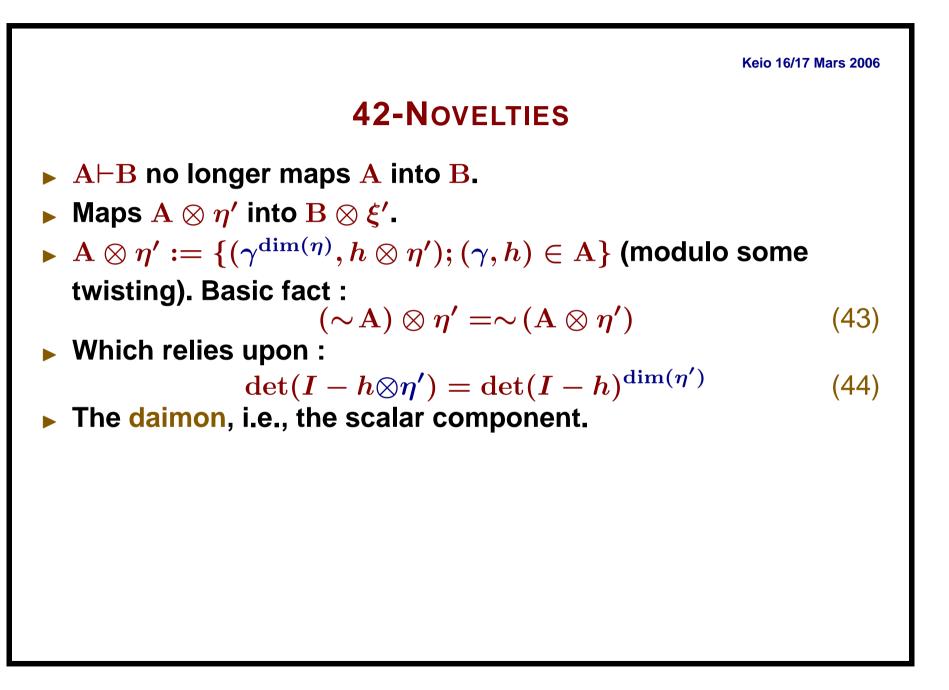














- ▶ $A \vdash B$ no longer maps A into B.
- Maps $\mathbf{A}\otimes \eta'$ into $\mathbf{B}\otimes \xi'$.
- $A \otimes \eta' := \{(\gamma^{\dim(\eta)}, h \otimes \eta'); (\gamma, h) \in A\}$ (modulo some twisting). Basic fact :

$$(\sim \mathbf{A}) \otimes \eta' = \sim (\mathbf{A} \otimes \eta')$$
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$$\det(I - h \otimes \eta') = \det(I - h)^{\dim(\eta')}$$
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- Truth (winning) not preserved by logical consequence.

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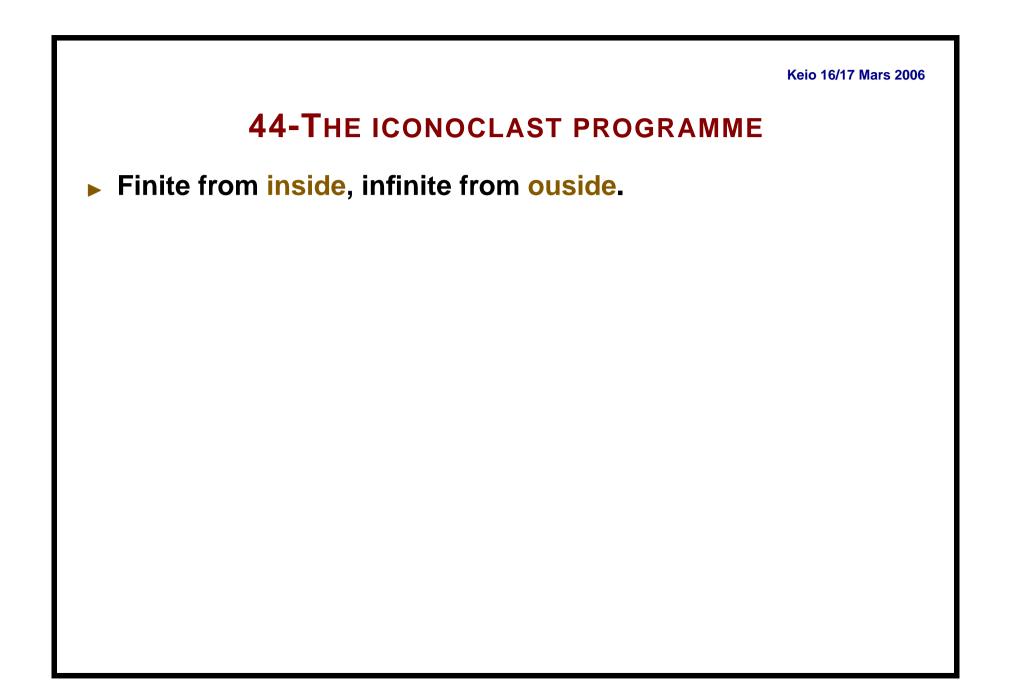
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- Subjectivity is the closest approximation to « h is graph-like ».
- Subjective winners are closed under logical consequence; indeed the feedback equation is of the nilpotent type and no daimon can be created.

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IX-AN ICONOCLAST LOGIC



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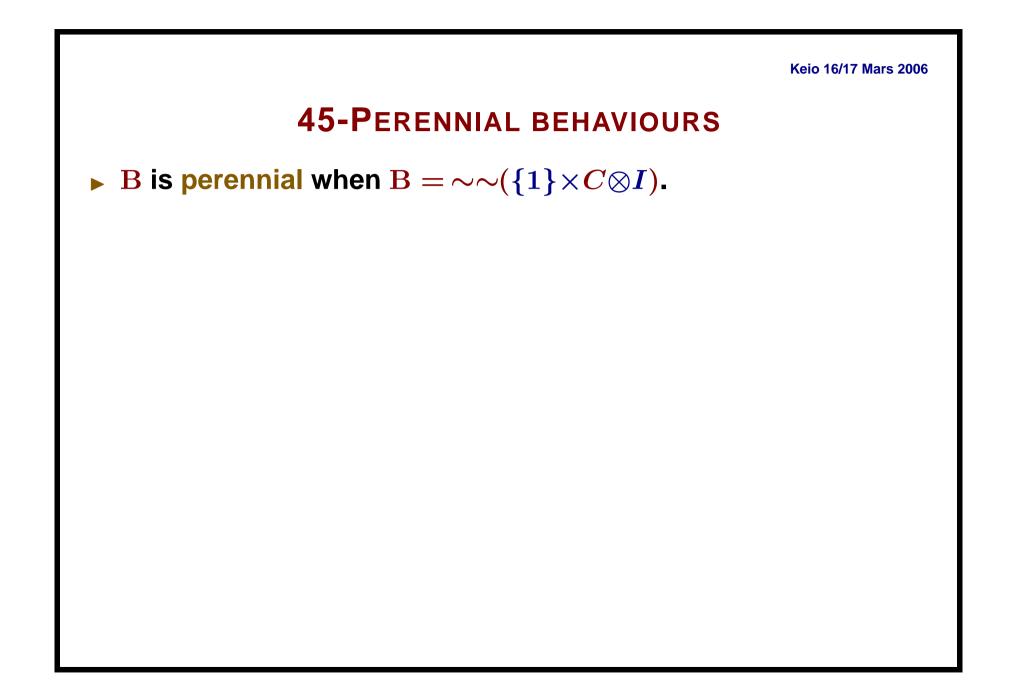
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- **B.t.w.**, logic in a factor of type II_1 should correspond to ELL.



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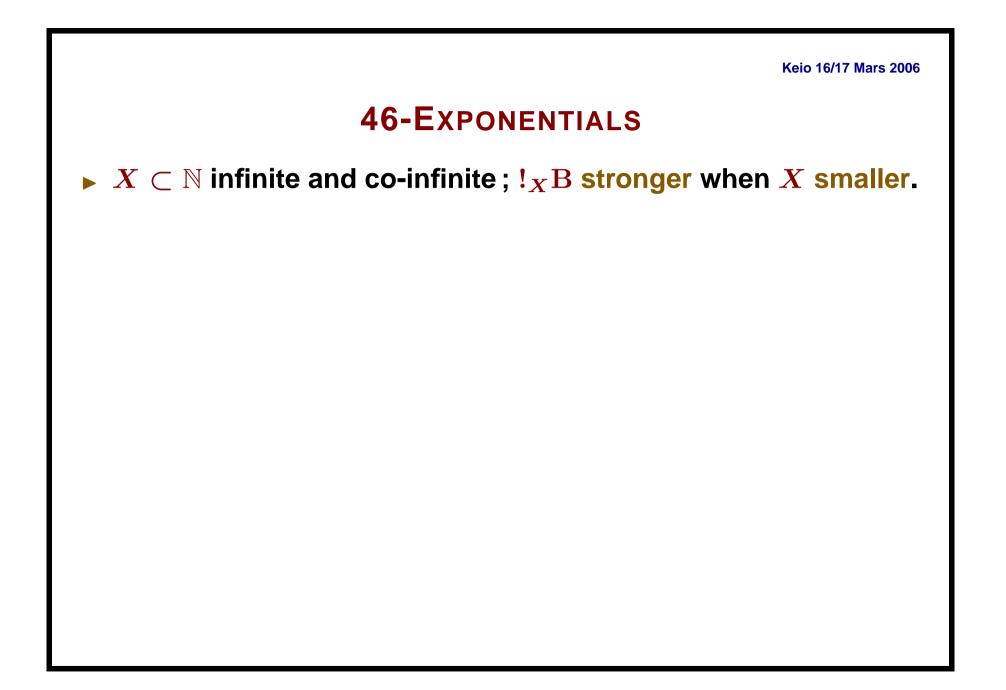
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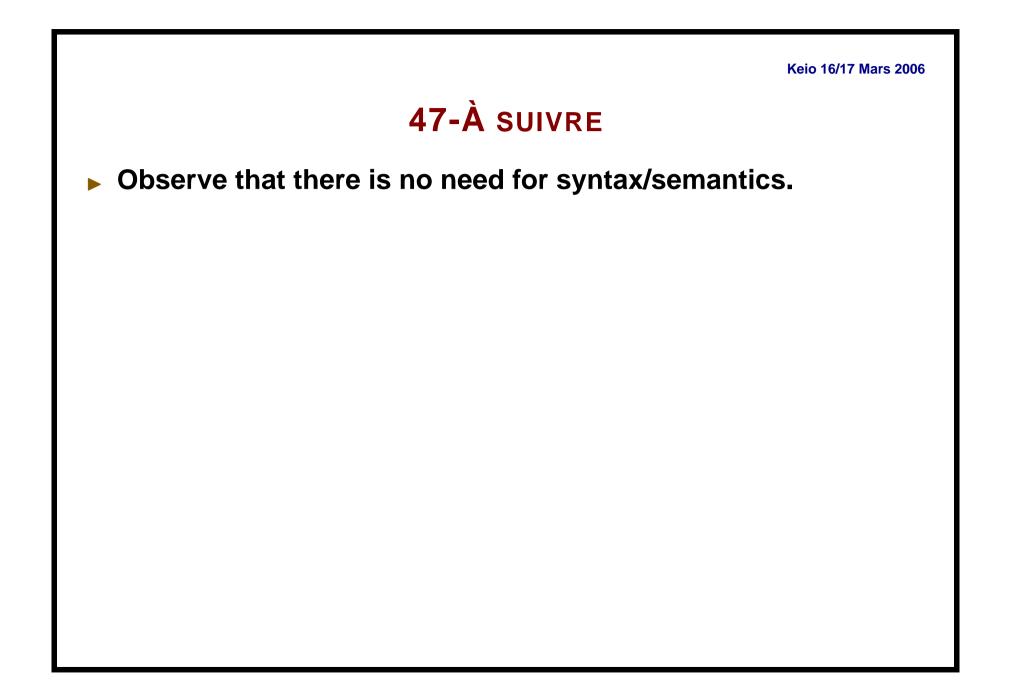
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Square : Type !_X \operatorname{nat}_{2Y} - \circ !_{X \sqcup X'} \operatorname{nat}_{2Y \sqcup 2Y + 1}.
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 - Height = depth of hereditary bits.

- Observe that there is no need for syntax/semantics.
- Don't bother with a sequent calculus :
 - Finite combinations in *G* will do everything.
- **b** Dynamics of G: a tower of exponentials.
 - Height = depth of hereditary bits.
- Which complexity classes can be expressed?